Subdivisions of K_5 in graphs containing $K_{2,3}$

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Abstract

Seymour conjectured that every 5-connected nonplanar graph contains a subdivision of K_5. We prove this conjecture for graphs containing $K_{2,3}$. As a consequence, Seymour's conjecture is true if the answer to the following question of Mader is affirmative: Does every simple graph on n vertices with at least $12(n-2)/5$ edges contain a K_4^-, a $K_{2,3}$, or a subdivision of K_5?

1 Introduction

We follow the notation and terminology used in [10, 11]. In particular, for a given graph K we use TK to denote a subdivision of K. The vertices of a TK corresponding to the vertices of K are called the *branch* vertices of this TK. Hence the degree 4 vertices in a TK_5 are its branch vertices. A *separation* in a graph G is a pair (G_1, G_2) of subgraphs of G such that $G = G_1 \cup G_2$, $E(G_1) \cap E(G_2) = \emptyset$, and $E(G_i) \cup (V(G_i) - V(G_{3-i})) \neq \emptyset$ for $i = 1, 2$. If, in addition, $|V(G_1 \cap G_2)| = k$ then (G_1, G_2) is said to be a *k-separation*. A collection of paths is said to be *independent* if no end of any path is internal to any other path in the collection.

Mader [12] proved that every simple graph on $n \geq 3$ vertices and with at least $3n-5$ edges contains TK_5, establishing a conjecture of Dirac [4]. In [8], Dirac's conjecture is reduced to the following conjecture of Seymour [15]: Every 5-connected nonplanar graph contains TK_5. (Kelmans [7] made the same conjecture two years later.) In [10, 11], Seymour's conjecture is established for graphs containing K_4^- (the graph obtained from K_4 by removing an edge).

Theorem 1.1 (*Ma and Yu [10, 11]*). *Every 5-connected nonplanar graph containing K_4^- contains TK_5.*

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One important step in [10] is to deal with the case when a 5-connected nonplanar graph G admits a 5-separation (G_1, G_2) such that $|G_2| \geq 7$ and G_2 has a plane representation in which all vertices in $V(G_1 \cap G_2)$ are incident with a common face. It is shown in [10] that in G_2 one can find a special collection of independent paths (used to construct a TK_5 in G). This result is also used in [5] by Kratovski, Stephens and Zha to show that Seymour's conjecture holds for graphs embedded in any surface (other than the sphere) with representativity at least 5.

It turns out to be very useful to exclude K_4^-. For example, by working with K_4^--free graphs, Kawarabayashi [6], Horev and Krakovski [1], and Ma, Thomas and Yu [9] independently proved Seymour's conjecture for apex graphs. (A graph is said to be *apex* if it has an *apex vertex*, i.e., a vertex whose deletion results in a planar graph.) In this paper we prove Seymour's conjecture for graphs containing $K_{2,3}$, and our proof makes heavy use of the fact that we can assume the graphs to be K_4^--free.

Theorem 1.2 *Every 5-connected nonplanar graph containing $K_{2,3}$ contains TK_5.*

Theorems 1.1 and 1.2 imply that Seymour's conjecture holds if the answer to the following question of Mader [12] is affirmative: Does every simple graph on $n \geq 4$ vertices with at least $12(n-2)/5$ edges contain a K_4^-, a $K_{2,3}$, or a subdivision of K_5?

In order to give a high level description of our proof of Theorem 1.2, we need some notation and terminology. Let H be a graph H and $A \subseteq V(H)$. We use $H[A]$ to denote the subgraph of H induced by A, and use $N_H(A)$ to denote the neighborhood of A. For any subgraph K of H, we write $H[K] := H[V(K)]$ and $N_H(K) := N_H(V(K))$. When understood, the subscript H may be omitted.

For any positive interger k, we say that H is (k, A)-*connected* if, for any cut set T of H with $|T| \leq k-1$, each component of $H - T$ contains a vertex in A.

We now introduce a concept that is closely related to existence of disjoint paths. A *3-planar graph* $(G, \mathcal{A})$ consists of a graph G and a set $\mathcal{A} = \{A_1, \ldots, A_k\}$ of pairwise disjoint subsets of $V(G)$ (possibly $\mathcal{A} = \emptyset$) such that

- (a) for $i \neq j$, $N(A_i) \cap A_j = \emptyset$,
- (b) for $1 \leq i \leq k$, $|N(A_i)| \leq 3$, and
- (c) if $p(G, \mathcal{A})$ denotes the graph obtained from G by (for each i) deleting A_i and adding new edges joining every pair of distinct vertices in $N(A_i)$, then $p(G, \mathcal{A})$ can be drawn in a closed disc with no edge crossings.

If, in addition, $b_0, b_1, \ldots, b_n$ are vertices in G such that $b_i \notin A$ for all $A \in \mathcal{A}$ and $0 \leq i \leq n$, $p(G, A)$ can be drawn in a closed disc with no edge crossings, and $b_0, b_1, \ldots, b_n$ occur on the boundary of the disc in this cyclic order, then we say that $(G, \mathcal{A}, b_0, b_1, \ldots, b_n)$ is *3-planar*. If there is no need to specify $\mathcal{A}$, we will simply say that $(G, b_0, b_1, \ldots, b_n)$ is 3-planar.

We make a simple, but useful, observation. If P is a path in $p(G, \mathcal{A})$ then we may produce a path P^* in G with the same ends of P as follows: For each edge uv of P with $\{u, v\} \subseteq N(A_i)$ for some i, replace uv with a path in $G[A_i \cup \{u, v\}]$ between u and v. As a consequence, any set of independent paths in $p(G, \mathcal{A})$ gives a set of independent paths in G with the same ends.

Given a graph G and $S \subseteq V(G)$, we say that (G, S) is *planar* if G has a drawing in the closed disc without edge crossings such that the vertices in S all appear on the bouddary of the disc. We say that (G,S) is *3-planar* the vertices in S can be ordered as $b_0,\dots,b_n$ such that $(G,b_0,\dots,b_n)$ is 3-planar.

Another concept we need is from [3]. A *block* of a graph G is either a maximal 2-connected subgraph of G or a subgraph of G induced by a cut edge. A block is *nontrivial* if it is 2-connected, and it is *trivial* otherwise. A connected graph C is a *chain* if its blocks can be labeled as $B_1,\dots,B_k$, where $k \ge 1$ is an integer, and its cut vertices can be labeled as $v_1,\dots,v_{k-1}$ such that

(i) $V(B_i)\cap V(B_{i+1}) = \{v_i\}$ for $1 \le i \le k-1$ and

(ii) $V(B_i)\cap V(B_j) = \emptyset$ if $|i-j| \ge 2$ and $1 \le i,j \le k$.

We write $C := B_1v_1B_2v_2\dots v_{k-1}B_k$ for this situation, and also view C as $\bigcup_{i=1}^k B_i$. If $k \ge 2$, $v_0 \in V(B_1) - \{v_1\}$ and $v_k \in V(B_k) - \{v_{k-1}\}$, or, if $k = 1$, $v_0, v_k \in V(B_1)$ and $v_0 \ne v_k$, then we say that C is a v_0-v_k *chain* or a chain *from* v_0 *to* v_k, and we denote this by $C := v_0B_1v_1\dots v_{k-1}B_kv_k$.

Let G be a graph and let $C := v_0B_1v_1\dots v_{k-1}B_kv_k$ be a chain. If C is an induced subgraph of G, then we say that C is a *chain in* G. We say that C is a *planar chain in* G if, for each $1 \le i \le k$ with $|V(B_i)| \ge 3$ (or equivalently, B_i is 2-connected), there exist distinct vertices $x_i, y_i \in V(G) - V(C)$ such that

- $(G[V(B_i)\cup\{x_i,y_i\}] - x_iy_i, x_i, v_{i-1}, y_i, v_i)$ is planar, and
- $B_i - \{v_{i-1}, v_i\}$ is a component of $G - \{x_i, y_i, v_{i-1}, v_i\}$.

We also say that C is a *planar* v_0-v_k *chain*. We say that C is a *3-planar* chain if in the definition of a planar chain we allow $x_i = y_i$ and when $x_i \ne y_i$ only require that $(G[V(B_i)\cup\{x_i,y_i\}] - x_iy_i, x_i, v_{i-1}, y_i, v_i)$ be 3-planar.

We are now ready to give a high level description of our proof of Theorem 1.2. Let G be a 5-connected graph and $\{x_1,x_2,y_1,y_2,y_3\} \subseteq V(G)$ such that $G[x_1,x_2,y_1,y_2,y_3] \cong K_{2,3}$ in which x_1,x_2 have degree 3. We will force a K_4^- in G and invoke Theorem 1.1, or force a 5-separation (G_1,G_2) such that G_2 is apex with apex vertex a and $(G_2-a, V(G_1\cap G_2)-\{a\})$ is planar, and then invoke Corollary 2.9 proved in Section 2.

Step 1. We show that either G contains TK_5 or $H := G - \{y_1,y_2,y_3\}$ contains a 3-planar chain from x_1 to x_2, say C, such that $H - C$ is 2-connected. This is done by first producing a nonseparating induced path X in H between x_1 and x_2, then augment a given 2-connected block in $H - X$. In the case the given block cannot be augmented we find a TK_5 or are left with the desired 3-planar chain. This is dealt with in Section 3.

Step 2. There are two types of blocks in a 3-planar chain. In Section 4, we show that if there is a block, say D, with two neighbors in $H - C$, say b_D, c_D, then G contains TK_5. This is done roughly as follows. Let D^* be obtained from $G[D + \{b_D, c_D, y_1, y_2, y_3\}]$ by identifying y_1, y_2, y_3 to a signle vertex y, and let u_D, v_D be the ends of D. Then D^* is an apex graph with apex vertex y, and $(D^* - y, b_D, u_D, c_D, v_D)$ is 3-planar. We first show that G contains TK_5 or D^* is $(5, \{b_D, c_D, u_D, v_D, y\})$-connected. We then prove two results in Section 2 which in turn allow us to find a special collection of independent paths in D^*. Finally, we use these paths to force a 5-separation (G_1,G_2) in G such that G_2 is apex with apex vertex a and $(G_2 - a, V(G_1\cap G_2) - \{a\})$ is planar, and invoke Corollary 2.9.

STEP 3. We may thus assume that each nontrivial block of C has only one neighbor in $H-C$. We show that at least two of $\{y_1, y_2, y_3\}$ have neighbors in $H-C$. This makes it easier to find a TK_5. Again in this process, whenever we are stuck we are rescued by a K_4^- or a 5-separation (G_1, G_2) such that G_2 is apex with apex vertex a and $(G_2-a, V(G_1 \cap G_2)-\{a\})$ is planar. This is done in Section 5.

STEP 4. Finally, we arrive at the case when C is simply an induced path X. It is then easy to show that G contains TK_5 or none of $\{y_1, y_2, y_3\}$ has a neighbor in $X-\{x_1, x_2\}$. So $G-X$ is 2-connected. If in $G-X$ there is a cycle containing $\{y_1, y_2, y_3\}$ then such a cycle, together with $G[\{x_1, x_2, y_1, y_2, y_3\}] \cup X$, gives a TK_5 in G. So we may assume that such a cycle does not exist in $G-X$. Then we know the structure of $G-X$, which is given by a result of Watkins and Mesner in [21]. A case analysis similar to that in [10] finds TK_5 in G.

2 Previous results and lemmas

In this section we list some known results and prove a few lemmas that are needed in our proof of Theorem 1.2. We begin with a result of Tutte [20].

Lemma 2.1 (*Tutte [20]*). *Let G be a 3-connected graph, $e \in E(G)$ and $v \in V(G)$ such that v is not incident with e. Then $G-v$ contains an induced cycle C such that $e \in C$ and $G-C$ is connected.*

We will need the following result of Seymour [16] about the existence of disjoint paths; equivalent versions can be found in [14, 17, 19].

Lemma 2.2 (*Seymour [16]*). *Let G be a graph and s_1, s_2, t_1, t_2 be distinct vertices of G. Then either G contains disjoint paths from s_1 to t_1 and from s_2 to t_2, or (G, s_1, s_2, t_1, t_2) is 3-planar.*

We state a simpler version for graphs with higher connectivity.

Corollary 2.3 *Let G be a connected graph and s_1, s_2, t_1, t_2 be distinct vertices of G such that G is $(4, \{s_1, s_2, t_1, t_2\})$-connected. Then either G contains disjoint paths from s_1 to t_1 and from s_2 to t_2, or (G, s_1, s_2, t_1, t_2) is planar.*

We will heavily use the $k=3$ case of the following result of Perfect [13].

Lemma 2.4 *Let G be a graph, $u \in V(G)$, and $A \subseteq V(G-u)$. Suppose there exist k independent paths from u to distinct $a_1, \ldots, a_k \in A$, respectively, and otherwise disjoint from A. Then for any $n \geq k$, if there exist n independent paths $P_1, \ldots, P_n$ in G from u to n distinct vertices in A and otherwise disjoint from A then $P_1, \ldots, P_n$ may be chosen so that $a_i \in P_i$ for $i = 1, \ldots, k$.*

We also need a result of Watkins and Mesner [21] on cycles through three vertices.

Lemma 2.5 (*Watkins and Mesner [21]*). *Let R be a 2-connected graph and let y_1, y_2, y_3 be three distinct vertices of R. Then there is no cycle through y_1, y_2 and y_3 in R if, and only if, one of the following statements holds.*

(*i*) *There exists a 2-cut S in R and, for $u \in \{y_1, y_2, y_3\}$, there exist pairwise disjoint subgraphs D_u of $R - S$ such that $u \in D_u$ and each D_u is a union of components of $R - S$.*

(*ii*) *For $u \in \{y_1, y_2, y_3\}$, there exist 2-cuts S_u in R and pairwise disjoint subgraphs D_u of R, such that $u \in D_u$, each D_u is a union of components of $R - S_u$, $S_{y_1} \cap S_{y_2} \cap S_{y_3} = \{z\}$, and $S_{y_1} - \{z\}, S_{y_2} - \{z\}, S_{y_3} - \{z\}$ are pairwise disjoint.*

(*iii*) *For $u \in \{y_1, y_2, y_3\}$, there exist pairwise disjoint 2-cuts S_u in R and pairwise disjoint subgraphs D_u of $R - S_u$ such that $u \in D_u$, D_u is a union of components of $R - S_u$, and $R - V(D_{y_1} \cup D_{y_2} \cup D_{y_3})$ has precisely two components, each containing exactly one vertex from S_u.*

The lemmas above are used in [10,11] to prove Theorem 1.1, which turns out to be useful here as well. The following lemma is proved in [10] and will be needed here.

Lemma 2.6 *Let G be a 5-connected nonplanar graph, and let (G_1, G_2) be a 5-separation of G such that $|V(G_2)| \geq 7$ and $(G_2, V(G_1) \cap V(G_2))$ is planar. Then G contains TK_5.*

In order to prove Theorem 1.2, we need to generalize Lemma 2.6 by allowing G_2 to be apex. Our original work on this generalization is quite complex, which is simplified by the following lemma (and its proof) due to Thomas [18].

Lemma 2.7 *Let G be a connected graph with $|V(G)| \geq 7$, let $A \subseteq V(G)$ with $|A| = 5$, and let $a \in A$ such that G is $(5, A)$-connected, $(G - a, A - \{a\})$ is planar, and either (1) $A - \{a\}$ is independent and $d_{G-a}(v) \geq 2$ for all $v \in A - \{a\}$ or (2) $d_{G-a}(v) \geq 4$ for all $v \in A - \{a\}$. Then G contains K_4^-, or G has a 5-separation (G_1, G_2) such that $a \in V(G_1 \cap G_2)$, $A \subseteq V(G_1)$, and $|V(G_2)| \geq 7$.*

Proof. Let $A = \{a, a_1, a_2, a_3, a_4\}$, and assume that $G - a$ is drawn in a closed disc in the plane without edge crossings such that a_1, a_2, a_3, a_4 occur on the boundary of the disc in clockwise order. Since $|V(G)| \geq 7$ and G is $(5, A)$-connected, $a_1a_3, a_2a_4 \notin E(G)$.

Let $H = (G - a) + \{a_1a_2, a_2a_3, a_3a_4, a_4a_1\}$ if (1) holds, and let $H = G - a$ if (2) holds; so that when (1) occurs H is a plane graph with outer cycle $a_1a_2a_3a_4a_1$. Note that the minimum degree of H satisfies $\delta(H) \geq 4$. Since G is $(5, A)$-connected, for $v \in V(H) - \{a_1, a_2, a_3, a_4\}$, if $d_H(v) = 4$ then $va \in E(G)$.

Let $uvwu$ be a facial triangle in H. We say that $uvwu$ (and the face it bounds) is *bad* if $|\{u, v, w\} \cap A| = 2$, or $\{u, v, w\} \cap A = \{a_i\}$ and $d_H(a_i) = 4$ for some $1 \leq i \leq 4$. Clearly, there are at most 8 bad facial triangles in H. In fact, it is easy to show that if there are 8 bad facial triangles in H then the outer cycle of $H - \{a_1, a_2, a_3, a_4\}$ is a 4-cycle $b_1b_2b_3b_4$, and we may choose the notation so that $a_1b_1a_2b_2a_3b_3a_4b_4a_1$ is a cycle in H. If $|V(G)| \geq 11$, then G has a 5-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{a, b_1, b_2, b_3, b_4\}$, $A \subseteq V(G_1)$, and $|V(G_2)| \geq 7$. If $|V(G)| = 10$ then, since G is $(5, A)$-connected, the vertex in $V(G) - \{a, a_i, b_i : i = 1, 2, 3, 4\}$ is adjacent to all of $\{b_1, b_2, b_3, b_4\}$, forcing a K_4^- in G. So $|V(G)| = 9$. Then, since G is $(5, A)$-connected, $\{b_1, b_2, b_3, b_4\} \subseteq N_G(a)$, or $b_1b_3 \in E(G)$, or $b_2b_4 \in E(G)$; so G contains K_4^-. Thus, we may assume that H has at most 7 bad facial triangles.

We may assume that if $uvwu$ is a facial triangle and is not bad, then two of $\{u, v, w\}$ must have degree at least 5 in H. Clearly $\{u, v, w\} \not\subseteq A$ because $a_1a_3, a_2a_4 \notin E(G)$. Now let

$v, w \notin A$. If $d_H(v) \geq 5$ and $d_H(w) \geq 5$ then we are done. So we may assume that $d_H(v) = 4$; hence $va \in E(G)$. If $d_H(w) = 4$ then $wa \in E(G)$ and $G[\{a, u, v, w\}]$ contains K_4^-. So we may assume that $d_H(w) \geq 5$. Similar argument shows that if $u \notin A$ then $d_H(u) \geq 5$. So assume $u \in A$. Then $d_H(u) \geq 5$ as $uvwu$ is not bad.

Suppose G contains no K_4^-; we will derive a contradiction by applying a simple discharging to H. Let $F(H)$ denote the set of faces of H, and for any $f \in F(H)$ let $d_H(f)$ denote the number of vertices incident with f. Let $\sigma : V(H) \cup F(H) \to \mathbb{Z}$ (the set of integers) such that $\sigma(x) = 4 - d_H(x)$ for all $x \in V(H) \cup F(H)$. Then by Euler's formula, the total charge is

$$\sigma(H) = \sum_{v \in V(H)} \sigma(v) + \sum_{f \in F(H)} \sigma(f) = 8.$$

Note that for any $x \in V(H) \cup F(H)$, if $\sigma(x) > 0$ then $x \in F(H)$, $d_H(x) = 3$, and $\sigma(x) = 1$. We now redistribute charges as follows, such that the total charge remains unchaged. For each $f \in F(H)$ with $d_H(f) = 3$ and f not bad, pick two of its incident vertices with degree at least 5 in H, and send a charge 1/2 from f to each of these two vertices. Let τ denote the resulting charge function. Then $\tau(f) \leq 0$ for all $f \in F(H)$ that is not bounded by a triangle or is not bad, and $\tau(x) = 0$ if $x \in V(H)$ and $d_H(x) = 4$. Now suppose $x \in V(H)$ and $d_H(x) \geq 5$. Since we assume $K_4^- \not\subseteq G$, x is contained in at most $\lfloor d_H(x)/2 \rfloor$ facial triangles. Hence $\tau(x) \leq \sigma(x) + \lfloor d_H(x)/2 \rfloor/2 = 4 - d_H(x) + \lfloor d_H(x)/2 \rfloor/2$. Note that

$$4 - d_H(x) + \lfloor d_H(x)/2 \rfloor/2 = \begin{cases} 4 - 3k, & \text{if } d_H(x) = 4k; \\ 3 - 3k, & \text{if } d_H(x) = 4k + 1; \\ 5/2 - 3k, & \text{if } d_H(x) = 4k + 2; \\ 3/2 - 3k, & \text{if } d_H(x) = 4k + 3. \end{cases}$$

Since $d_H(x) \geq 5$, $k \geq 1$, and $k \geq 2$ if $d_H(x) = 4k$. Hence, $\tau(x) \leq 4 - d_H(x) + \lfloor d_H(x)/2 \rfloor/2 \leq 0$. Thus the total new charge is $\tau(H) \leq 7$ because there are at most 7 bad facial triangles. This is a contradiction. ∎

The following is an easy consequence of Lemma 2.7. It was proved independently by Kawarabayashi [6], by Aigner-Horev and Krakovski [1], by Ma, Thomas and Yu [9].

Corollary 2.8 *Every 5-connected nonplanar apex graph contains TK_5.*

Proof. Let G be a 5-connected nonplanar apex graph and a be its apex vertex. By Theorem 1.1, we may assume that $K_4^- \not\subseteq G$. So $G - a$ has a plane representation in which the outer cycle is not a triangle. Let a_1, a_2, a_3, a_4 be four arbitrary vertices in the outer cycle of $G - a$, and let $A = \{a, a_1, a_2, a_3, a_4\}$. Then G, A, a satisfy the conditions of Lemma 2.7 (in particular, (2)). Hence, since $K_4^- \not\subseteq G$, G has a 5-separation (G_1, G_2) such that $a \in V(G_1 \cap G_2)$, $A \subseteq V(G_1)$, and $|V(G_2)| \geq 7$. We choose such (G_1, G_2) so that G_2 is minimal, and let $A' = V(G_1 \cap G_2)$. If $|V(G_2)| = 7$ then, since G_2 is $(5, A')$-connected and $(G_2 - a, A' - \{a\})$ is planar, $K_4^- \subseteq G_2$, a contradiction. So $|V(G_2)| \geq 8$. Hence, by the minimality of G_2, A' is independent in G_2 and $d_{G_2}(v) \geq 2$ for all $v \in A' - \{a\}$. So G_2, A', a satisfies the conditions of Lemma 2.7 (in particular, (1)). As a consequence, $K_4^- \subseteq G_2$, a contradiction. ∎

As mentioned before, we need an apex version of Lemma 2.6, which is also an easy consequence of Lemma 2.7.

Corollary 2.9 *Let G be a 5-connected nonplanar graph, (G_1, G_2) a 5-separation of G, and $a \in A := V(G_1) \cap V(G_2)$ such that $|V(G_2)| \geq 7$ and $(G_2 - a, A - \{a\})$ is planar. Then G contains TK_5.*

Proof. We choose such separation (G_1, G_2) so that G_2 is minimal. Then $A - \{a\}$ is independent in G_2. If $|V(G_2)| = 7$ then, since G_2 is $(5, A)$-connected and $(G_2 - a, A - \{a\})$ is planar, $K_4^- \subseteq G_2$. If $|V(G_2)| \geq 8$ then by the minimality of G_2, A is independent in G and $d_{G_2 - a}(v) \geq 2$ for all $v \in A - \{a\}$; so $K_4^- \subseteq G_2$ by Lemma 2.7. Therefore, the assertion of this corollary follows from Theorem 1.1. ∎

In the proof of Lemma 2.6 in [10], an important step is to find a collection of independent paths in G_2, the planar part. For the purpose of this paper, we need to extend this to the apex side of a 5-separation. The following result is due to Thomas [18] which significantly simplifies our proofs of such results (see Corollaries 2.11 and 2.12).

Lemma 2.10 *Let G be a connected graph with $|V(G)| \geq 7$, $A \subseteq V(G)$ with $|A| = 5$, and $a \in A$ such that G is $(5, A)$-connected, $(G - a, A - \{a\})$ is planar, and G has no 5-separation (G_1, G_2) such that $A \subseteq G_1$ and $|V(G_2)| \geq 7$. Let $w \in V(G) - A$ and assume that the vertices in $G - a$ cofacial with w induce a cycle C in $G - a$. Then there exist paths P_1, P_2, P_3, P_4 in G from w to A such that $V(P_i \cap P_j) = \{w\}$ for $i \neq j$, and $|V(P_i \cap C)| \leq 1$ and $|V(P_i) \cap A| = 1$ for $i = 1, 2, 3, 4$.*

Proof. Since G has no 5-separation (G_1, G_2) with $A \subseteq G_1$ and $|V(G_2)| \geq 7$, A must be independent in G. Let $H := G - (C - N(w))$.

Suppose H has four paths P_1, P_2, P_3, P_4 from w to A such that $V(P_i \cap P_j) = \{w\}$ and $|V(P_i) \cap A| = 1$. We may assume that these paths are induced paths. Hence $|V(P_i \cap C)| \leq 1$ for $1 \leq i \leq 4$. (Note that $|V(P_i) \cap C| = 0$ occurs when $P_i = wa$.) So P_i, $i = 1, 2, 3, 4$, are the desired paths.

Thus we may assume that such paths in H do not exist. By Menger's theorem, there is a cut T, $|T| \leq 3$, in H separating w from A. For convenience, assume that $G - a$ is drawn in a closed disc in the plane with no edge crossings such that $A - \{a\}$ is contained in the boundary of the disc. Thus there is a simple closed curve γ in the plane intersecting $G - a$ only in $(T - \{a\}) \cup (V(C) - N(w))$ such that w is inside γ and $A - \{a\}$ is outside of or on γ. The elements of $T - \{a\}$ divide γ into $|T - \{a\}|$ simple curves (including the points in $T - \{a\}$), called the *segments* of γ. For two distinct points u, v on γ we use $u\gamma v$ to denote the simple curve in γ from u to v in clockwise order; and if $u = v$ then $u\gamma v$ consists of the single point $u = v$. We claim that

(1) if $u, v \in V(C) - N(w)$ and $u\gamma v$ is contained in a segment of γ, then $uCv - \{u, v\}$ contains no neighbor of w.

For, otherwise, we may choose such u, v that u and v are consecutive on γ. Then $\{a, u, v, w\}$ is a 4-cut in G separating $uCv - \{u, v\}$ from A, contradicting the $(5, A)$-connectedness of G.

Note that $\gamma \cap V(C) \cap N(w) = \emptyset$ and $T \cap (V(C) - N(w)) = \emptyset$. Also note that since G is $(5, A)$-connected,

(2) $|T| + |\gamma \cap (V(C) - N(w))| \geq 5$.

We consider cases based on $|T - \{a\}|$.

Case 1. $|T - \{a\}| \leq 1$.

First, suppose $T - \{a\} = \emptyset$. Then $|\gamma \cap (V(C) - N(w))| \geq 4$ by (2). Let $u, v \in \gamma \cap (V(C) - N(w))$. By (1), neither $uCv - \{u,v\}$ nor $vCu - \{u,v\}$ contains a neighbor of w. Hence, $\{a, u, v\}$ is a 3-cut in G separating w from A, a contradiction.

Now, suppose $|T - \{a\}| = 1$. Then $|\gamma \cap (V(C) - N(w))| \geq 3$ by (2). Let $u, v \in \gamma \cap (V(C) - N(w))$ such that $T - \{a\} \subseteq v\gamma u$ and, subject to this, $v\gamma u$ is minimal. Then by (1), $uCv - \{u,v\}$ contains no neighbor of w. So $\{a, u, v\} \cup (T - \{a\})$ is a 4-cut in G separating w from A, a contradiction.

Case 2. $|T - \{a\}| = 2$.

Let $T - \{a\} = \{t_1, t_2\}$. Then $|\gamma \cap (V(C) - N(w))| \geq 2$ by (2).

First, assume $(t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C) = \emptyset$. Then for $i = 1, 2$, let $u_i \in (t_2\gamma t_1 - \{t_1, t_2\}) \cap V(C)$ with u_i closest to t_i. By (1), $N(w) \cap u_1Cu_2$. Hence $\{a, t_1, t_2, u_1, u_2\}$ is a 5-cut in G separating w and $N(w)$ from A, a contradiction (to the nonexistence of such a separation).

Thus $(t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C) = \emptyset$. Similarly, $(t_2\gamma t_1 - \{t_1, t_2\}) \cap V(C) = \emptyset$.

For $i = 1, 2$, let $u_i \in (t_2\gamma t_1 - \{t_1, t_2\}) \cap V(C)$ with u_i closest to t_i, and $v_i \in (t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C)$ with v_i closest to t_i. Then by (1), $N(w) \subseteq (u_1Cv_1 - \{u_1, v_1\}) \cup (v_2Cu_2 - \{u_2, v_2\})$. As $|N(w) \cap V(C)| \geq 4$, we may assume by symmetry that $|N(w) \cap V(u_1Cv_1 - \{u_1, v_1\})| \geq 2$. Hence $\{a, t_1, u_1, v_1, w\}$ is a 5-cut in G separating A from at least two vertices, a contradiction.

Case 3. $|T - \{a\}| = 3$.

Let $T - \{a\} = \{t_1, t_2, t_3\}$. In this case, $a \notin T$ and a has no neighbors strictly inside γ. By (2), $|\gamma \cap (V(C) - N(w))| \geq 2$.

First, assume $\gamma \cap (V(C) - N(w))$ is contained in some segement of γ, say $\subseteq t_1\gamma t_2$. For $i = 1, 2$, let $u_i \in (t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C)$ with u_i closest to t_i. By (1), $N(w) \cap u_2Cu_1$. Hence $\{t_1, t_2, t_3, u_1, u_2\}$ is a 5-cut in G separating w and $N(w)$ from A, a contradiction.

Therefore, $\gamma \cap (V(C) - N(w))$ is not contained in any segment of γ.

Next, assume that the interior of some segment of γ, say $t_3\gamma t_2 - \{t_2, t_3\}$, is disjoint from $V(C)$. For $i = 1, 2$, let $u_i \in (t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C)$ with u_i closest to t_i; and for $i = 2, 3$, let $v_i \in (t_2\gamma t_3 - \{t_2, t_3\}) \cap V(C)$ with v_i closest to t_i. Then by (1), $N(w) \subseteq (u_2Cv_2 - \{u_2, v_2\}) \cup (v_3Cu_1 - \{u_1, v_3\})$. Since $|N(w) \cap V(C)| \geq 4$, $|N(w) \cap V(u_2Cv_2 - \{u_2, v_2\})| \geq 2$ or $|N(w) \cap (v_3Cu_1 - \{u_1, v_3\})| \geq 2$. In the first case, $\{t_2, u_2, v_2, w\}$ is 4-cut in G separating A from some neighbor of w, a contradiction; and in the second case, $\{t_1, t_3, u_1, v_3, w\}$ is a 5-cut in G separating A from at least two vertices, a contradiction.

Thus, $(t_i\gamma t_{i+1} - \{t_i, t_{i+1}\}) \cap (V(C) - N(w)) \neq \emptyset$ for $i = 1, 2, 3$, where $t_4 = t_1$. For $i = 1, 2$, let $u_i \in (t_1\gamma t_2 - \{t_1, t_2\}) \cap V(C)$ with u_i closest to t_i; for $i = 2, 3$, let $v_i \in (t_2\gamma t_3 - \{t_2, t_3\}) \cap V(C)$ with v_i closest to t_i; and for $i = 1, 3$, let $w_i \in (t_3\gamma t_1 - \{t_1, t_3\}) \cap V(C)$ with w_i closest to t_i. Then by (1), $N(w) \subseteq (u_2Cv_2 - \{u_2, v_2\}) \cup (v_3Cw_3 - \{v_3, w_3\}) \cup (w_1Cu_1 - \{u_1, w_1\})$. Since $|N(w) \cap V(C)| \geq 4$, $|N(w) \cap V(u_2Cv_2 - \{u_2, v_2\})| \geq 2$ or $|N(w) \cap (v_3Cw_3 - \{v_3, w_3\})| \geq 2$ or $|N(w) \cap V(w_1Cu_1 - \{u_1, w_1\})| \geq 2$. In the first case, $\{t_2, u_2, v_2, w\}$ is 4-cut in G separating A from some neighbor of w, a contradiction; in the second case, $\{t_3, v_3, w_3, w\}$ is a 4-cut in G separating A from some neighbor of w, a contradiction; and in the third case, $\{t_1, u_1, w_1, w\}$ is a 4-cut in G separating A from some neighbor of w, a contradiction. ∎

As consequences of Lemma 2.10, we derive the following two results about independent paths.

Corollary 2.11 *Let G be a connected graph, $A \subseteq V(G)$ with $|A| = 5$, and $a \in A$ such that $(G - a, A - a)$ is planar. Suppose G is $(5, A)$-connected and $|V(G)| \geq 7$, and G has no 5-separation (G_1, G_2) with $A \subseteq G_1$ and $|V(G_2)| \geq 7$. Let $w \in N(a)$ such that w does not belong to the outer walk of $G - a$. Then*

(i) the vertices of $G - a$ cofacial with w induce a cycle C in $G - a$,

(ii) $G - a$ contains paths P_1, P_2, P_3 from w to $A - \{a\}$ such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 3$, and $|V(P_i \cap C)| = |V(P_i) \cap A| = 1$ for $1 \leq i \leq 3$.

Proof. Let D denote the outer walk of $G - a$ which contains $A - \{a\}$. Then $w \notin D$. Since G is $(5, A)$-connected and by planarity of $G - a$, the vertices of G cofacial with w induce a cycle in $G - a$, denoted by C. Applying Lemma 2.10, we obtain four paths P_1, P_2, P_3, P_4 with one of them, say P_4, being wa. Now P_1, P_2, P_3 are the desired paths. ∎

The next consequence of Lemma 2.10 is more technical. We require that $G - a$ be K_4^--free instead of G. This is because in certain applications of this corollary, the vertex a is the result of identifying several vertices and therefore may be contained in some K_4^-.

Corollary 2.12 *Let G be a connected graph, $A \subseteq V(G)$ with $|A| = 5$, and $a \in A$ such that $(G - a, (A - a) \cup N(a))$ is planar and $K_4^- \not\subseteq G - a$. Suppose G is $(5, A)$-connected and $|V(G)| \geq 7$, and assume that G has no 5-separation (G_1, G_2) with $A \subseteq G_1$ and $|V(G_2)| \geq 7$. Then $G - a$ is 2-connected. Moreover, either G is the graph obtained from the edge-disjoint union of an 8-cycle $x_1x_2x_3x_4x_5x_6x_7x_8x_1$ and a 4-cycle $x_2x_4x_6x_8x_2$ by adding a and the edges ax_i, $i = 2, 4, 6, 8$, with $A = \{a, x_1, x_3, x_5, x_7\}$, or there exists $w \in V(G) - A$ such that*

(i) the vertices of $G - a$ cofacial with w induce a cycle C in $G - a$,

(ii) there exist paths P_1, P_2, P_3, P_4 in G from w to A such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C)| = |V(P_i) \cap A| = 1$ for $1 \leq i \leq 4$, and

(iii) $C \cap D = \emptyset$, where D denotes the outer cycle of $G - a$, and either (a) $a \in \bigcup_{i=1}^4 P_i$ or (b) $a \in \bigcup_{i=1}^4 P_i$ and we may write $A - \{a\} = \{a_1, a_2, a_3, a_4\}$ such that $a \in P_1$ and $a_i \in P_i$ for $i = 2, 3, 4$, and $a_1, a_2, a_3, P_1 \cap D, a_4$ occur D in cyclic order.

Proof. Since G has no 5-separation (G_1, G_2) such that $A \subseteq G_1$ and $|V(G_2)| \geq 7$,

(1) A is independent in G and every vertex in A has degree at least 2 in G.

We claim that

(2) $G - a$ is 2-connected.

Otherwise, we may write $G - a = H_1 \cup H_2$ such that $|V(H_i)| \geq 2$ and $|V(H_1) \cap V(H_2)| \leq 1$. Then $|V(H_i) \cap A| \leq 2$ for some i. Hence G has a separation (G_1, G_2) such that $G_2 - (V(G_1) \cap V(G_2)) = G[(H_i - H_{3-i}) \cup \{a\}]$ and $V(G_1 \cap G_2) = (V(H_i) \cap A) \cup V(H_1 \cap H_2) \cup \{a\}$ (which has size at most 4). Clearly, $A \subseteq G_1$. Since A is independent in G and every vertex in A has degree at least 2 in G, $V(G_i) - V(G_{3-i}) \neq \emptyset$ for $i = 1, 2$. This contradicts the assumption that G is $(5, A)$-connected.

By (2), let D denote the outer cycle of $G - a$; so $A - \{a\} \subseteq D$.

(3) every edge in $(G-a)-E(D)$ must join two neighbors of a vertex in $A-\{a\}$.

Let $uv \in E(G-a)-E(D)$. Then $G-a$ has a 2-separation (H_1, H_2) such that $V(H_1) \cap V(H_2) = \{u, v\}$ and $V(H_i) - V(H_{3-i}) \neq \emptyset$ for $i = 1, 2$. By symmetry, we may assume that $|V(H_1 - \{u, v\}) \cap A| \leq |V(H_2 - \{u, v\}) \cap A|$.

First, suppose $|V(H_1 - \{u, v\}) \cap A| = 2$. Then, since A is independent and G is $(5, A)$-connected, $\{a, u, v\} \cup (V(H_1 - \{u, v\}) \cap A)$ is a 5-cut in G separating A from just one vertex, say x, and x is adajcent to all of $\{a, u, v\} \cup (V(H_1 - \{u, v\}) \cap A)$. Then it is easy to see that $K_4^- \subseteq H_1$, a contradiction.

Thus, $|V(H_1 - \{u, v\}) \cap A| \leq 1$. Since G is $(5, A)$-connected, $\{a, u, v\} \cup (V(H_1 - \{u, v\}) \cap A)$ cannot be a cut in G separating A from some vertex; so $|V(H_1)| = 3$ and the vertex in $V(H_1) - \{u, v\}$ must belong to A.

Suppose $V(G-a) = V(D)$. By (3) and because $(G-a, A-\{a\})$ is planar and G is $(5, A)$-connected, we see that must be the graph obtained from the edge-disjoint union of an 8-cycle $x_1x_2x_3x_4x_5x_6x_7x_8x_1$ and a 4-cycle $x_2x_4x_6x_8x_2$ by adding a and the edges ax_i, $i = 2, 4, 6, 8$, with $A = \{a, x_1, x_3, x_5, x_7\}$.

So we may assume that $V(G-a) \neq V(D)$. Furthermore,

(4) there exists $w \in V(G-a) - V(D)$ such that w is not cofacial with any vertex of D.

For, suppose every vertex of $V(G-a) - V(D)$ is cofacial with some vertex of D. Then $G - a - V(D)$ is outerplanar. So there exists $w \in V(G-a) - V(D)$ such that w has degree at most 2 in $G - a - V(D)$.

Since G is $(5, A)$-connected and $N(a) \subseteq V(D)$, w has at least three neighbors in D. Let $w_1, \ldots, w_k$ be the neighbors of w on D (so $k \geq 3$), and assume that they occur on D in this clockwise order. Moreover, by planarity, we may choose w so that there is no vertex inside the cycle ww_1Dw_kw. Since $K_4^- \not\subseteq G-a$, $|V(w_1Dw_k)| \geq 4$. So by (1), $V(w_1Dw_k - \{w_1, w_k\}) \not\subseteq A$.

Suppose for some $v \in V(w_1Dw_k - \{w_1, w_k\}) - A$, $v \notin N(w)$. Then since G is $(5, A)$-connected and by (3), there exist $vv_1, vv_2 \in E(G-a) - E(D)$ such that $\{v, v_i\} = N(a_i)$ for $a_i \in A$ $(i = 1, 2)$, and $N(v) = \{a, a_1, a_2, v_1, v_2\}$. Assume $v_1 \in w_1Dv_2$. Now by (1), $\{a, v_1, v_2\} \cup (A \cap V(v_2Dv_1))$ is a 5-cut of G separating A from at least two vertices, a contradiction.

So $V(w_1Dw_k - \{w_1, w_k\}) - A \subseteq N(w)$. Let $v \in V(w_1Dw_k - \{w_1, w_k\}) - A$. Since G is $(5, A)$-connected, there exist $vv_1 \in E(G-a) - E(D)$. By (3), $\{v, v_1\} = N(a_i)$ for some $a_i \in A$. By (1), $v' \notin A$; so $v, v' \in N(w)$. Now $G[\{a_i, v, v', w\}] \cong K_4^-$, a contradiction.

Since G is $(5, A)$-connected and by planarity of $G-a$, we see that the vertices of $G-a$ cofacial with w induce a cycle in $G-a$, denoted by C. Then $C \cap D = \emptyset$ by (4).

By applying Lemma 2.10, there exist paths P_1, P_2, P_3, P_4 in G from w to A such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, and $|V(P_i \cap C)| = |V(P_i) \cap A| = 1$ for $1 \leq i \leq 4$. If $a \notin \bigcup_{i=1}^4 P_i$, we are done. So we may assume without loss of generality that $a \in P_1$.

Let $A - \{a\} = \{a_1, a_2, a_3, a_4\}$ such that $a_i \in P_i$ for $i = 2, 3, 4$, let w_i denote the neighbor of w in P_i for $i = 1, 2, 3, 4$, and let a' dneote the neighbor of a in P_1. If there exists a permutaion ijk of $\{2, 3, 4\}$ such that a_1, a_i, a_j, a', a_k occur D in cyclic order then (b) of (iii) holds. So we may assume, without loss of generality, that a_1, a', a_2, a_3, a_4 occur on D in clockwise order. Since $C \cap D = \emptyset$, $a_1Da' \cup a'P_1w_1$ contains a path P_1' such that $V(P_1' \cap C) = \{w_1\}$. Now P_1', P_2, P_3, P_4 show that (iii) holds. ∎

3 Planar chains

Throughout the rest of this paper, let G be a 5-connected nonplanar graph and $x_1, x_2, y_1, y_2, y_3 \in V(G)$ be distinct such that $K := G[x_1, x_2, y_1, y_2, y_3] \cong K_{2,3}$ in which x_1, x_2 have degree 3. Let $H := G - \{y_1, y_2, y_3\}$.

In this section we will show that G contains TK_5 or H contains a 3-planar chain C from x_1 to x_2 such that $H - C$ is 2-connected. We need the concept of a bridge. Let K be a graph and $L \subseteq G$. An *L-bridge* of K is a subgraph of K induced by the edges of a component of $K - L$ and all edges from that component to L.

First, we prove a very useful lemma that G contains TK_5 or no vertex other than x_1 and x_2 may be adjacent to two of $\{y_1, y_2, y_3\}$.

Lemma 3.1 *Suppose $x_3 \in V(G)$ and $|N(x_3) \cap \{y_1, y_2, y_3\}| \geq 2$. Then G contains TK_5.*

Proof. Without loss of generality, we may assume that $x_3y_1, x_3y_2 \in E(G)$. Note the symmetry among x_1, x_2, y_1, y_2 and between x_3 and y_3.

If $G - \{x_3, y_3\}$ contains four independent paths from some $u \in V(G - \{x_3, y_3\}) - \{x_1, x_2, y_1, y_2\}$ to x_1, x_2, y_1, y_2, respectively, then these paths and $K \cup y_1x_3y_2$ form a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that such paths do not exist. Then

(1) G has a 5-separation (H_1, H_2) such that $\{x_3, y_3\} \subseteq V(H_1) \cap V(H_2)$, $u \in H_1 - H_2$, and $\{x_1, x_2, y_1, y_2\} \subseteq H_2$.

We choose (H_1, H_2) in (1) so that H_2 is minimal. Let $S := V(H_1 \cap H_2) - \{x_3, y_3\} = \{s_1, s_2, s_3\}$. We may assume that

(2) $S \not\subseteq \{x_1, x_2, y_1, y_2\}$.

For, suppose $S \subseteq \{x_1, x_2, y_1, y_2\}$. By symmetry we may assume that $x_1 \notin S$. By Menger's theorem, $H_2 - \{y_1, y_2, y_3\}$ contains two indpendent paths P_2, P_3 from x_1 to x_2, x_3, respectively. If $H_1 - y_3$ contains disjoint paths from x_2 to x_3 and from y_1 to y_2 then these paths and $(K - y_3) \cup y_1x_3y_2 \cup P_2 \cup P_3$ form a TK_5 in G with branch vertices x_1, x_2, x_3, y_1, y_2. So we may assume that such disjoint paths do not exist. Then by Corollary 2.3, $(H_1 - y_3, x_2, y_1, x_3, y_2)$ is planar. If $|V(H_1) - V(H_2)| \geq 2$ then, by Corollary 2.9, G contains TK_5. So we may assume that $|V(H_1) - V(H_2)| = 1$. Thus, since G is $(5, A)$-connected, the unique vertex in $V(H_1) - V(H_2)$ is adjacent to x_2, y_1, y_2; so G contains K_4^- and hence TK_5 by Theorem 1.1.

By (2) we may assume $s_1 \notin \{x_1, x_2, y_1, y_2\}$. We claim that

(3) H_2 contains four paths S_i, $i = 0, 1, 2, 3$, from $\{x_1, x_2, y_1, y_2\}$ to s_i, respectively, where $s_0 = s_1$, such that $S_0 \cap S_1 = \{s_1\}$, and $S_i \cap S_j = \emptyset$ whenever $i \neq j$ and $\{i, j\} \neq \{0, 1\}$.

Let H_2' be obtained from $H_2 - \{x_3, y_3\}$ by duplicating s_1, and use s_0 to denote the duplicate of s_1. (Hence, s_0 and s_1 have the same neighborhood in H_2'.) By the minimality of H_2 and by Menger's theorem, H_2' contains four disjoint paths S_i from $\{x_1, x_2, y_1, y_2\}$ to s_i, $i = 0, 1, 2, 3$, respectively. Note that S_1, S_2, S_3 are paths in $H_2 - \{x_3, y_3\}$. By identifying s_0 with s_1, we view S_0 as a path in $H_2 - \{x_3, y_3\}$ from s_1.

(4) We may assume that s_1 has a unique neighbor in H_1, and denote it by u.

If $H_1 - \{x_3, y_3\}$ contains independent paths P_2, P_3 from s_1 to s_2, s_3, then $S_0 \cup S_1 \cup (P_2 \cup S_2) \cup (P_3 \cup S_3) \cup K \cup y_1x_3y_2$ is a TK_5 in G with branch vertices s_1, x_1, x_2, y_1, y_2. So we may assume that such paths do not exist. Then $H_1 - \{x_3, y_3\}$ has a cut vertex v separating s_1 from $\{s_2, s_3\}$. Since G is 5-connected, the v-bridge of $H_1 - \{x_3, y_3\}$ containing s_1 is induced by the edye s_1v. Hence (4) holds.

(5) We may assume that there exist $b_0 \in S_0$ and $b_1 \in S_1$ such that in $H_2 - \{x_3, y_3\}$, $\{b_0, b_1, s_2, s_3\}$ separates s_1 from $\{x_1, x_2, y_1, y_2\}$.

To see this let H_2'' be obtained from $H_2 - \{x_3, y_3\}$ by duplicating s_1 twice and identifying s_2 and s_3 (also denote it by s_2), and let s_1', s_1'' denote the duplicates of s_1.

Suppose H_2'' contains four disjoint paths from $\{s_1, s_1', s_1'', s_2\}$ to $\{x_1, x_2, y_1, y_2\}$. Then $H_2 - \{x_3, y_3\}$ has four independent paths to $\{x_1, x_2, y_1, y_2\}$, three from s_1 and one from s_2 or s_3, say s_2. Thus, these four paths, $K \cup y_1x_3y_2$, and a path in $H_1 - \{x_3, y_3, s_3\}$ from s_1 to s_2 form a TK_5 in G with branch vertices s_1, x_1, x_2, y_1, y_2.

So we may assume that such four paths in H_2'' do not exist. Then H_2'' has a separation (R, R') such that $|V(R) \cap V(R')| \leq 3$, $\{s_1, s_1', s_1'', s_2\} \subseteq R$, and $\{x_1, x_2, y_1, y_2\} \subseteq R'$. Choose (R, R') so that $V(R) \cap V(R')$ is minimal. By minimality of $V(R) \cap V(R')$ and since s_1, s_1', s_1'' have the same neighborhood in H_2'', $s_1, s_1', s_1'' \notin V(R) \cap V(R')$. By minimality of H_2, we must have $s_2 = s_3 \in V(R) \cap V(R')$.

Thus, $(H_2 - \{x_3, y_3\}) - \{s_2, s_3\}$ has a cut $T := V(R \cap R') - \{s_2 = s_3\}$ separating s_1 from $\{x_1, x_2, y_1, y_2\}$, and $s_1 \notin T$ and $|T| \leq 2$. Since $s_1 \notin T$ and because of S_0 and S_1, $|T| = 2$; so letting $T = \{b_0, b_1\}$, $b_0 \in S_0$, and $b_1 \in S_1$ we complete the proof of (5).

Let R^* denote the component of $(H_2 - \{x_2, x_3\}) - \{b_0, b_1, s_2, s_3\}$ containing s_1. Choose $\{b_0, b_1\}$ so that R^* is minimal.

(6) We may assume that $s_2, s_3 \notin N(R^*)$, and for any $w \in \{x_3, y_3\}$, $G[R^* + \{b_0, b_1\}w]$ contains independent paths from s_1 to w, b_0, b_1, respectively.

First, assume that s_2 or s_3, say s_2, has a neighbor in R^*. Then by the minimality of R^*, $G[R^* + \{b_0, b_1, s_2\}]$ contains three independent paths from s_1 to b_0, b_1, s_2, respectively; and we may assume that $s_1S_0b_0$ and $s_1S_1b_1$ are two of them. Now these three paths, $S_0 \cup S_1 \cup S_2 \cup S_3 \cup K \cup y_1x_3y_2$, and a path in $H_1 - \{s_2, x_3, y_3\}$ from s_1 to s_3 form a TK_5 in G with branch vertices s_1, x_1, x_2, y_1, y_2.

So we may assume that R^* contains no neighbor of $\{s_2, s_3\}$. If $R^* = \{s_1\}$ then by (4), $s_1x_3, s_1y_3 \in E(G)$; so (6) holds. Hence we may assume that $|V(R^*)| \geq 2$. Thus, since G is 5-connected and by (4), R^* has neighbors of both x_3 and y_3. By the minimality of R^*, we see that for any $w \in \{x_3, y_3\}$, $G[R^* + \{b_0, b_1, w\}]$ contains independent paths from s_1 to w, b_0, b_1, respectively. Again, we have (6).

Let $R_1 = G[R^* + \{b_0, b_1, x_3, y_3\}]$. Note that when $R^* \neq \{s_1\}$ we have symmetry between R_1 and H_1.

(7) We may assume that $|V(H_1)| \geq 7$.

For, suppose $|V(H_1)| = 6$. Then u (see (4)) is adjacent to all of $\{s_1, s_2, s_3, x_3, y_3\}$. If $s_1x_3, s_1y_3 \in E(G)$ then $G[s_1, u, x_3, y_3] \cong K_4^-$, so G contains TK_5 by Theorem 1.1. Thus we may assume $s_1x_3 \notin E(G)$ or $s_1y_3 \notin E(G)$. This implies $|V(R^*)| \geq 2$ (as s_1 has degree at least 5 in G). If $|V(R^*)| \geq 3$ then $|V(R_1)| \geq 7$; so by the symmetry between R_1 and H_1, we may assume $|V(H_1)| \geq 7$. Thus, we may assume $R^* = \{s_1, v\}$. Clearly v is adjacent to all of $\{b_0, b_1, s_1, x_3, y_3\}$. If $s_1b_0 \notin E(G)$ or $s_1b_1 \notin E(G)$ then $s_1x_3, s_1y_3 \in E(G)$ by (4), and so $G[\{s_1, v, x_3, y_3\}]$ contains K_4^-; if $s_1b_0, s_1b_1 \in E(G)$ then $G[b_0, b_1, s_1, v]$ contains K_4^-. Hence G contains TK_5 by Theorem 1.1, completing the proof of (7).

We may assume by symmetry that S_0, S_1, S_2, S_3 end at x_1, y_1, y_2, x_2, respectively. If $H_1 - s_3$ contains no disjoint paths from x_3 to y_3 and from s_1 to s_2 then by Corollary 2.3, $(H_1 - s_3, x_3, s_1, y_3, s_3)$ is planar, and G contains TK_5 by (7) and Corollary 2.9. So we may assume such disjoint paths exist in $H_1 - s_3$. These disjoint paths, $(K - x_2y_3) \cup y_1x_3y_2 \cup b_0S_0x_1 \cup b_1S_1y_1 \cup S_2$, and three independent paths in $G[R^* + x_3]$ from s_1 to x_3, b_0, b_1, respectively (by (6)) form a TK_5 in G with branch vertices s_1, x_1, x_3, y_1, y_2. ∎

The next result will allow us to modify an existing x_1-x_2 path in H.

Lemma 3.2 *Let Q be an x_1-x_2 path in H and let $B(Q)$ be a 2-connected block in $H - Q$. Then G has a TK_5, or H has an induced x_1-x_2 path Q' such that $H - Q'$ is connected and $B(Q) \subseteq H - Q'$, or H has an induced x_1-x_2 path Q' such that $H - Q'$ is connected and $\{y_1, y_2, y_3\} \in N(B(Q'))$ for some 2-connected block $B(Q')$ of $H - Q'$.*

Proof. Suppose for any induced x_1-x_2 path Z in H with $B(Q) \subseteq H - Z$, $H - Z$ has at least two components. We choose Z so that

(1) $\beta(Z)$ is minimum.

Let C denote a component of $H - Z$ such that $B(Q) \cap C = \emptyset$. Let $u_1, u_2 \in N(C) \cap V(Z)$ such that u_1Zu_2 is maximal, and we may assume x_1, u_1, u_2, x_2 occur on Z in order.

Then

(2) $N(C \cup (u_1Zu_2 - \{u_1, u_2\})) = \{u_1, u_2, y_1, y_2, y_3\}$.

For, otherwise, since G is 5-connected, $u_1Zu_2 - \{u_1, u_2\}$ contains a neighbor of some component of $H - Z$ other than C. We now use Lemma 2.1 to find a path P in $G[C + \{u_1, u_2\}]$ from u_1 to u_2. Let $B_1 \dots B_k$ denote the chain of blocks in $G[C + \{u_1, u_2\}]$ from u_1 to u_2, with $u_1 \in B_1$ and $u_2 \in B_k$. Let C' be obtained from $G[C \cup u_1Zu_2]$ by contracting $G[C \cup u_1Zu_2] - \bigcup_{i=1}^k B_i$ to a single vertex u. Then $C' + u_1u_2$ is 3-connected. So by Lemma 2.1, $C' + u_1u_2$ contains an induced cycle T such that $u_1u_2 \in E(T)$, $u \notin V(T)$ and $C' - T$ is connected. Let $P := T - u_1u_2$. Then $G[C \cup u_1Zu_2] - P$ is connected. Let $Q' := u_1Zx_1 \cup P \cup u_2Zx_2$. Then Q' is an induced x_1-x_2 path in H. Since $(u_1Zu_2 - \{u_1, u_2\}) \cap P = \emptyset$ and $u_1Zu_2 - \{u_1, u_2\}$ contains a neighbor of some component of $H - Z$ other than C, we have $\beta(Q') < \beta(X)$, contradicting (1).

We may assume that

(3) $H - Z$ has just two components, namely C and the component D containing $B(Q)$, and if $w_1, w_2 \in N(D) \cap V(Z)$ such that $N(D) \cap V(Z) \subseteq V(w_1Zw_2)$ then $u_1Zu_2 \subseteq w_1Zw_2$ and $\{u_1, u_2\} \neq \{w_1, w_2\}$.

Let D be an arbitrary component D of $H - X$ with $D \neq C$.

First, suppose $D \cap B(Q) = \emptyset$. If $u_1 Z u_2 \subseteq w_1 Z w_2$ then by (2) we have $N(D) \cap V(Z) = \{w_1, w_2\} = \{u_1, u_2\} = N(C) \cap V(Z)$. In $G[C + \{u_1, u_2, y_1, y_2, y_3\}]$ we apply Menger's theorem to find five independent paths P_1, P_2, P_3, P_4, P_5 from some $x \in V(C)$ to u_1, u_2, y_1, y_2, y_3, respectively. In $G[D + \{y_1, y_2\}]$ we find a path P between y_1 and y_2. Now $(P_1 \cup u_1 Z x_1) \cup (P_2 \cup u_2 Z x_2) \cup P_1 \cup P_2 \cup P \cup K$ is a TK_5 in G with branch vertices x, x_1, x_2, y_1, y_2. Thus we may assume that $u_1 Z u_2 \not\subseteq w_1 Z w_2$. Then by (2) and by symmetry we may assume that $x_1, w_1, w_2, u_1, u_2, x_2$ occut on Z in this order. By (2), we may use Menger's theorem to find in $G[C \cup u_1 Z u_2 + \{y_1, y_2, y_3\}]$ independent paths P_1, P_2, P_3, P_4, P_5 from some $x \in V(C)$ to u_1, u_2, y_1, y_2, y_3, respectively. If $G[D \cup w_1 Z w_2 + \{y_1, y_2\}]$ contains disjoint paths Q_1, Q_2 from y_1, w_1 to y_2, w_2, respectively, then $(P_1 \cup u_1 Z w_2 \cup Q_2 \cup w_1 Z x_1) \cup (P_2 \cup u_2 Z x_2) \cup P_1 \cup P_2 \cup Q_1 \cup K$ is a TK_5 in G with branch vertices x, x_1, x_2, y_1, y_2. So assume that Q_1, Q_2 do not exist. Then by (2) and by Corollary 2.3, $(G[D \cup w_1 Z w_2 + \{y_1, y_2\}], y_1, w_1, y_2, w_2)$ is planar. By Lemma 3.1, $|V(D) \cup V(u_1 Z x_2 - \{u_1, u_2\})| \geq 2$. So it follows from Corollary 2.9 that G contains TK_5.

Therefore, we may assume that $H - Z$ has only two components, namely C and D, and $B \subseteq D$. If $\{w_1, w_2\} = \{u_1, u_2\}$ then the argument in the first half of the above paragraph shows that G contains TK_5. Now suppose $u_1 Z u_2 \not\subseteq w_1 Z w_2$. Then by (2), we may assume that x_1, w_1, w_2, u_1, u_2 occur on Z in order. The argument in the second half of the above paragraph shows that G contains TK_5, completing the proof of (3).

By (2) and (3), we may assume $x_1, w_1, u_1, u_2, w_2, x_2$ occur on Z in this order. Note by (2) that $\{u_1, u_2, y_1, y_2, y_3\}$ is a cut in G separating $C \cup u_1 Z u_2$ from D. By (3) and by symmetry, we may assume that $u_1 \neq w_1$. We now apply Lemma 2.1 as in the proof of (2) to find an induced w_1-w_2 path P in $G[D + \{w_1, w_2\}]$ such that $G[D \cup w_1 X w_2] - P$ is connected. Now let Z' be obtained from Z by replacing $w_1 Z w_2$ with P. Clearly Z' is induced, and $H - Z'$ is connected. If $G[C \cup (u_1 Z u_2 - u_2)]$ is 2-connected, then it is the desired $B(Q')$. So suppose $G[C \cup (u_1 Z u_2 - u_2)]$ is not 2-connected. By Lemma 3.1, every vertex in $u_1 Z u_1 - \{u_1, u_2\}$ has at east two nighbors in C. So $G[C \cup (u_1 Z u_2 - u_2)]$ has an endblock, say C', disjoint from $u_1 X u_2 - u_2$. Let v be the cut vertex of $G[C \cup (u_1 Z u_2 - u_2)]$ contained in C'. Since G is 5-connected, $y_1, y_2, y_3 \in N(C')$. By Lemma 3.1, C' is 2-connected. So C' is the desired $B(Q')$. ∎

The next lemma says that we can choose X so that the minimum degree of $H - X$ is at least 2. In particular, $H - X$ has a 2-connected block.

Lemma 3.3 *Let X ne an induced x_1-x_2 path in H such that $H - X$ is connected. Then $K_4^- \subseteq G$, or H contains an induced x_1-x_2 path X' such that $H - X'$ is connected, contains all 2-connected blocks of $H - X$, and has minimum degree at least 2.*

Proof. For an arbitrary induced x_1-x_2 path Z in H for which $H - Z$ is connected and contains all 2-connected blocks of $H - X$, let $\alpha_1(Z)$ denote the number of vertices of $H - Z$ with degree at most 1 in $H - Z$, and let $\alpha_2(Z)$ denote the number of vertices of $H - Z$ with degree at least 2 in $H - Z$. We choose such Z that $\alpha_1(Z)$ is minimum and, subject to this, $\alpha_2(Z)$ is maximum. If $\alpha_1(Z) = 0$, then $X' := Z$ is the desired path. So assume $\alpha_1(Z) \geq 1$, and let u be a vertex of degree at most 1 in $H - Z$.

Since G is 5-connected, Lemma 3.1 implies that u has at least three neighbors on Z. Let $u_1, u_2 \in N(u) \cap V(Z)$ with $u_1 Z u_2$ maximal, and we may assume that x_1, u_1, u_2, x_2 occur on Z

in order. Let $X' = x_1Zu_1uu_2Zx_2$. Clearly, X' is an induced path in G, and all 2-connected blocks of $H - Z$ (hence those of $H - Z$) are contained in $H - X'$.

By Lemma 3.1, each vertex of $u_1Zu_2 - \{u_1, u_2\}$ has at least 1 neighbor in $H - Z - u$. If $|u_1Zu_2| = 3$ then $G[u_1Zu_2 + u] \cong K_4^-$. So we may assume $|u_1Zu_2| \geq 4$. Then $\alpha_1(X') \leq \alpha_1(Z)$ and $\alpha_2(X') > \alpha_2(Z)$, a contradiction. ∎

Recall that we wish to find an induced path X in H from x_1 to x_2 such that $H - X$ 2-connected, which will be the work of the next two sections. But first we show that we can find a 3-planar chain C in H from x_1 to x_2 such that $H - C$ is 2-connected, and we also need $H - C$ to have neighbors of as many y_i as possible. This leads to the following notation:

$$\gamma(X) := \max\{|N(B) \cap \{y_1, y_2, y_3\}| : B \text{ is a 2-connected block of } H - X\},$$

and let $B(X)$ denote a 2-connected block of $H - X$ with $|N(B(X)) \cap \{y_1, y_2, y_3\}| = \gamma(X)$.

By Lemma 3.3, we see that there exists induced x_1-x_2 path X in H such that $H - X$ has 2-connected blocks. So $\gamma(X)$ and $B(X)$ are defined for such X. Throughout the rest of this paper, we choose X and $B(X)$ so that the following are satisfied in order listed:

(1) $\gamma(X)$ is maximum,

(2) $|\{y_i : |N(y_i) \cap V(B(X))| \geq 2\}|, 1 \leq i \leq 3\}|$ is maximum, and

(3) $B(X)$ is maximal.

When understood, we will simply refer to $B(X)$ as B.

One lemma we need before proceeding is that if a $(B \cup X)$-bridge of H is not an edge then it has at least two attachments on X.

Lemma 3.4 *We may assume that H contains no 2-cut separating $B \cup X$ from some vertex.*

Proof. Suppose that $\{u, v\}$ is a 2-cut in H separating $B \cup X$ from some vertex. Let D denote a $\{u, v\}$-bridge containing neither B nor X. Since $H - X$ is connected and B is a 2-connected block of H, we may assume that H has disjoint paths P_u, P_v from v, u to $x \in V(X), b \in V(B)$, respectively, and internally disjoint from $B \cup D \cup X$ and $u \notin B$. Since G is 5-connected, $\{y_1, y_2, y_3\} \subseteq N(D - \{u, v\})$.

We claim that $\{y_1, y_2, y_3\} \subseteq N(B)$. If $D - u$ is 2-connected then this follows from Lemma 3.2 and the choice of X (as $D - u \subseteq H - X$). So we may assume that $D - u$ is not 2-connected, and let C denote an endblock of $D - u$. Since G is 5-connected, $\{y_1, y_2, y_3\} \subseteq N(C)$. By Lemma 3.1, we may assume that C is 2-connected. Hence, since $C \subseteq H - X$, it follows from Lemma 3.2 and the choice of X that $\{y_1, y_2, y_3\} \subseteq N(B)$.

By Lemma 3.1 we may assume that no two of $\{y_1, y_2, y_3\}$ share a common neighbor. Thus, since B is 2-connected, $G[B + \{y_1, y_2, y_3\}]$ has two disjoint paths Q_1, Q_2 with ends in $\{b, y_1, y_2, y_3\}$. Without loss of generality, we may assume that Q_1 is between y_1 and y_2 and Q_2 is between y_3 and b.

If $G[D+\{y_1, y_2, y_3\}]-u$ contains disjoint paths R_1, R_2 from y_1, y_2 to v, y_3, respectively, then $Q_1 \cup Q_2 \cup (R_1 \cup P_1) \cup R_2 \cup X \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, y_3. So we may assume that such R_1, R_2 do not exist. Then by Corollar 2.3, $(G[D+\{y_1, y_2, y_3\}]-u, y_1, y_2, v, y_3)$

is planar. By Lemma 3.1 we may assume that $|V(D) - \{u, v\}| \geq 3$. Hence G contains TK_5 by Corollary 2.9. ∎

In [3], it is shown that 4-connected graphs contain non-separating planar chains between any two specific vertices. We now use a similar argument to show that $H - B$ is a 3-planar chain. We proceed by proving three lemmas.

Lemma 3.5 *Suppose H has two connected subgraphs C, D such that $|V(C \cap B)| \leq 1$ and $|V(D \cap B)| \leq 1$, $V(C \cap X) = \{u, v\}$ and $V(D \cap X) = \{u, v\}$ or $V(D \cap X) = V(uXv)$, $\{u, v\} \cup V(C \cap B)$ is cut in H separating C from $B \cup D \cup (X - uXv)$, and $\{u, v\} \cup (V(D \cap B)$ is a cut in H separating D from $B \cup C \cup (X - uXv)$. Then G contains TK_5.*

Proof. Without loss of generality assume that x_1, u, v, x_2 occur on X in order. Let

$$S_C := \{u, v\} \cup V(C \cap B) \cup (N(C - \{u, v\} - V(C \cap B)) \cap \{y_1, y_2, y_3\})$$

and

$$S_D := \{u, v\} \cup V(D \cap B) \cup (N(D - \{u, v\} - V(D \cap B)) \cap \{y_1, y_2, y_3\}).$$

Since G is 5-connected, $|S_C| \geq 5$ and $|S_D| \geq 5$.

We claim that $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$. Let A denote an endblock of $C - \{u, v\}$ and let $a \in V(A)$ such that if $A = C - \{u, v\}$ and $C \cap B \neq \emptyset$ then $a \in C \cap B$, if $A = C - \{u, v\}$ and $C \cap B = \emptyset$ let $a \in V(A)$ be arbitrary, and if $A \neq C - \{u, v\}$ then let a be the cut vertex of $C - \{u, v\}$ contained in A. Since G is 5-connected, we see that $|N(A - a) \cap \{y_1, y_2, y_3\}| \geq 2$. By Lemma 3.1, A is 2-connected. Hence the claim follows from the choice of X and Lemma 3.2.

By Lemma 2.4, $G[C + S_C]$ contains five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex $w \in V(C)$ to S_C such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i \neq j \leq 5$, $V(P_i) \cap S_C| = 1$ for $1 \leq i \leq 5$, P_1 ends at u, and P_2 ends at v. By symmetry, we may assume that $y_1 \in P_3$ and $y_2 \in P_4$.

If $y_1, y_2 \in S_D$ then $G[D + \{y_1, y_2\}] - \{u, v\}$ contains a path Q between y_1 and y_2; and $(P_1 \cup uXx_1) \cup (P_2 \cup vXx_2) \cup P_3 \cup P_4 \cup Q \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_1, y_2. Similary, if $y_1, y_2 \in N(B)$ then $G[B + \{y_1, y_2\}]$ contains a path Q between y_1 and y_2; again $(P_1 \cup uXx_1) \cup (P_2 \cup vXx_2) \cup P_3 \cup P_4 \cup Q \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_1, y_2.

Thus we may assume that $y_1 \notin S_D$ and $\{y_1, y_2\} \not\subseteq N(B)$. Hence $y_2, y_3 \in S_D$ and $|V(D \cap B)| = 1$. Let $d \in V(D \cap B)$. By Menger's theorem, $G[D \cup S_D]$ contains five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some $x \in V(D)$ to u, v, y_2, y_3, d, respectively. If $y_2, y_3 \in N(B)$ then $G[B + \{y_2, y_3\}]$ contains a path R between y_2 and y_3; so $(Q_1 \cup uXx_1) \cup (Q_2 \cup vXx_1) \cup Q_3 \cup Q_4 \cup R \cup K$ is a TK_5 in G with branch vertices x, x_1, x_2, y_2, y_3. Similarly, if $y_2, y_3 \in S_C$ then $G[C + \{y_2, y_3\}] - \{u, v\}$ has a path R between y_2 and y_3; again $(Q_1 \cup uXx_1) \cup (Q_2 \cup vXx_1) \cup Q_3 \cup Q_4 \cup R \cup K$ is a TK_5 in G with branch vertices x, x_1, x_2, y_2, y_3.

Hence we may assume that $\{y_2, y_3\} \not\subseteq N(B)$ and $\{y_2, y_3\} \not\subseteq S_C$. Therefore, $y_1, y_3 \in N(B)$ and $y_3 \notin S_C$. Thus $G[B + \{y_1, y_3\}]$ contains a path R_{13} between y_1 and y_3, and $G[C + \{y_1, y_2\}] - \{u, v\} - V(C \cap B)$ contains a path R_{12} between y_1 and y_2. If $G[D + \{y_2, y_3\}] - d$ contains disjoint paths R_1, R_2 from u, y_2 to v, y_3, respectively, then $R_{12} \cup R_{13} \cup R_2 \cup (x_1XuR_1 \cup vXx_2) \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, y_3. So we may assume that R_1, R_2 do not exist. Then by Corollary 2.3, $(G[D + \{y_2, y_3\}] - d, u, y_2, v, y_3)$ is planar. Since $y_2, y_3 \in N(D - \{d, u, v\})$, we may assume by Lemma 3.1 that $|V(D)-\{d,u,v\}| \geq 2$. So G contains TK_5 by Corollary 2.9. ∎

Let $\mathcal{B}$ denote the set of B-bridges of $H-X$. For each $D \in \mathcal{B}$, $V(B) \cap V(D)$ consists of exactly one vertex, denoted by r_D. For any $x, y \in V(X)$, we denote $x \leq y$ if $x \in V(X[x_1, y])$. If $x \leq y$ and $x \neq y$, then we write $x < y$. By Lemma 3.4, we may assume that, for each $D \in \mathcal{B}$, $D - r_D$ has at least two neighbors on X. Let l_D and h_D denote the the neighbors of $D - r_D$ on X such that $l_D < h_D$ and $l_D X h_D$ is maximal. For each vertex u of $H - X$, we define $u^* = r_D$ if $u \in V(D)$ for some $D \in \mathcal{B}$, and $u^* = u$ if $x \in V(B)$. We say that a member D of $\mathcal{B}$ is a *nice bridge* if there exist $u, v \in N_H(l_D X h_D - \{l_D, h_D\})$ such that $u, v \notin V(D - r_D) \cup V(X)$ and $u^* \neq v^*$.

Lemma 3.6 *There is no nice B-bridge in H, or G contains TK_5.*

Proof. Suppose D is a nice bridge in H. There exist $u, v \in N_H(l_D X h_D - \{l_D, h_D\})$ such that $u, v \notin V(D - r_D) \cup V(X))$ and $u^* \neq v^*$. We now use Lemma 2.1 to find a path P in $G[D + \{l_D, h_D\}] - r_D$ from l_D to h_D.

Let $B_1 \ldots B_k$ denote the chain of blocks in $G[D+\{l_D, h_D\}]-r_D$ from l_D to h_D, with $l_D \in B_1$ and $h_D \in B_k$. Let C' be obtained from $G[D \cup_D X h_D]$ by identifying $G[D \cup l_D Z h_D] - \bigcup_{i=1}^k B_i$ to a single vertex u. Then by Lemma 3.4, we may assume that $C' + u_1 u_2$ is 3-connected. So by Lemma 2.1, $C' + u_1 u_2$ contains an induced cycle T such that $u_1 u_2 \in E(T)$, $u \notin V(T)$ and $C' - T$ is connected. Let $P := T - u_1 u_2$. Then $G[D \cup l_D X h_D] - P$ has at most two components each containing r_D or $l_D X h_D - \{l_D, h_D\}$.

Let $Q' := x_1 X l_D \cup P \cup h_D X x_2$. Then Q' is an induced x_1-x_2 path in H and $H - X'$ is connected. However, $H - X'$ has a block properly containing $B(X)$, contradicting the choice of X. ∎

We say that two B-bridges C and D in $\mathcal{B}$ *overlap* if $E(l_C X h_C) \cap E(l_D X h_D) \neq \emptyset$. Define an auxiliary graph $\mathcal{G}$ with $V(\mathcal{G}) = \mathcal{B}$ such that $C, D \in \mathcal{B}$ are adjacent in $\mathcal{G}$ if, and only if, C and D overlap. The following lemma is similar to results in [2, 3]. The difference is that we need Lemma 3.5 here instead of 4-connectedness in [2, 3].

Lemma 3.7 *Let $D_1 D_2 D_3$ be a path in $\mathcal{G}$. Then $|\{r_{D_i} : i = 1, 2, 3\}| \leq 2$ or G contains TK_5. Moreover, if $D_1 D_2 D_3$ is an induced path in $\mathcal{G}$ then $r_{D_1} = r_{D_3}$ or G contains TK_5.*

Proof. First, suppose $D_1 D_2 D_3$ is an induced path in $\mathcal{G}$. Then D_1 and D_3 do not overlap. Thus we may assume $l_{D_1} < h_{D_1} \leq l_{D_3} < h_{D_3}$. Moreover, $l_{D_2} < h_{D_1}$ and $l_{D_3} < h_{D_2}$. Let $u \in V(D_1) - \{r_{D_1}\}$ such that $u h_{D_1} \in E(G)$ and let $v \in V(D_3) - \{r_{D_3}\}$ such that $v l_{D_3} \in E(G)$. Clearly, $u, v \in N_H(l_{D_2} X h_{D_2} - \{l_{D_2}, h_{D_2}\})$, $u, v \notin (V(D_2) - \{r_{D_2}\}) \cup V(X)$, and $u^* = r_{D_1}$ and $v^* = r_{D_3}$. So by Lemma 3.6, $r_{D_1} = r_{D_3}$ or G contains TK_5.

Now assume that D_1 and D_3 overlap. By symmetry, we may assume that $l_{D_1} X h_{D_1}$ is not properly contained in $l_{D_i} X h_{D_i}$ for $i = 2, 3$. Then for each $i \in \{2, 3\}$, either $l_{D_i} X h_{D_i} = l_{D_1} X h_{D_1}$, or $l_{D_i} \in l_{D_1} X h_{D_1} - \{l_{D_1}, h_{D_1}\}$, or $h_{D_i} \in l_{D_1} X h_{D_1} - \{l_{D_1}, h_{D_1}\}$. Therefore, by Lemma 3.5 and by relabeling D_1, D_2, D_3 (if necessary), we may assume that there exist $x \in V(l_{D_1} X h_{D_1} - \{l_{D_1}, h_{D_1}\}) \cap N(D_2 - r_{D_2})$ and $y \in V(l_{D_1} X h_{D_1} - \{l_{D_1}, h_{D_1}\}) \cap N(D_3 - r_{D_3})$. Let u be a neighbor of x in $D_2 - r_{D_2}$, and v be a neighbor of y in $D_3 - r_{D_3}$. Then $u^* = r_{D_2}$ and $v^* = r_{D_3}$. By Lemma 3.6, we may assume $u^* = v^*$; so $|\{r_{D_i} : i = 1, 2, 3\}| \leq 2$. ∎

Lemma 3.8 *Let $\mathcal{G}_i$, $i = 1, \ldots, k$, denote the components of the graph $\mathcal{G}$. Then $|\{r_D : D \in V(\mathcal{G}_i)\}| \leq 2$ for all $i = 1, \ldots, k$, or G contains TK_5.*

Proof. For suppose $|\{r_D : D \in V(\mathcal{G}_i)\}| \geq 3$ for some $1 \leq i \leq k$. Choose $D_1, D_2, D_3 \in V(\mathcal{G}_i)$ such that $r_{D_1}, r_{D_2}, r_{D_3}$ are pairwise distinct and, subject to this, the connected subgraph of $\mathcal{G}_i$ containing $\{r_{D_1}, r_{D_2}, r_{D_3}\}$, denote by $\mathcal{T}$, has minimum number of edges.

Thus, $\mathcal{T}$ is a tree whose leaves must be contained in $\{D_1, D_2, D_3\}$. So we may assume that D_1 and D_2 are two leaves of $\mathcal{T}$. Then by the minimality of $\mathcal{T}$, $r_{D_j} \neq r_D$ for $j = 1, 2$ and for all $D \in V(\mathcal{G}_i) - \{D_j\}$. Moreover, $|\mathcal{T}| \geq 4$; otherwise, G contains TK_5 by Lemma 3.7. Thus, D_3 is not a leaf of $\mathcal{T}$; otherwise, $\mathcal{T} - D_3$ contradicts the minimality of $\mathcal{T}$. Therefore, $\mathcal{T}$ is actually a path between D_1 and D_2. Hence, since $|\mathcal{T}| \geq 4$ and $|\mathcal{T}|$ is minimum, $\mathcal{T}$ has a subpath of length 2 with ends D_1 and D such that the path is induced in $\mathcal{G}$ and $r_{D_1} \neq r_D$; so G contains TK_5 by Lemma 3.7. ∎

We are now ready to show that $H - B$ is a 3-planar chain.

Lemma 3.9 *$H - B$ is a 3-planar chain from x_1 to x_2, or G contains TK_5.*

Proof. Let $\mathcal{G}_i$, $i = 1, \ldots, k$, denote the components of the graph $\mathcal{G}$. For each i, $\bigcup_{D \in V(\mathcal{G}_i)} l_D X h_D$ is a subpath of X; and let $u_i \leq v_i$ denote the ends of this path. By Lemma 3.4, we may assume $u_i < v_i$ for all i. Let B_i denote the subgraph of $H - B$ that is the union of $u_i X v_i$ and $D - r_D$ fro all $D \in V(\mathcal{G}_i)$. Then $B_i \cap X_i$, $i = 1, \ldots, k$, are pairwise edge-disjoint, and no cut vertex of B_i separates u_i from v_i. By Lemma 3.8, $|N(B_i - \{u_i, v_i\}) \cap V(B)| \leq 2$.

Suppose $|V(B_i)| \geq 3$. Then B_i is 2-connected. Since X is induced and $H - X$ is connected, $|N(B_i - \{u_i, v_i\}) \cap V(B)| \geq 1$. If $|N(B_i - \{u_i, v_i\}) \cap V(B)| = 1$ then by Lemma 3.5, $B_i - \{u_i, v_i\}$ is connected. Now assume $N(B_i - \{u_i, v_i\}) \cap V(B) = \{w_1, w_2\}$.

We may assume that $(G[B_i + \{w_1, w_2\}] - w_1 w_2, u_i, w_1, v_i, w_2)$ is 3-planar. For, otherwise, it follows from Lemma 2.2 that $B_i' := G[B_i + \{w_1, w_2\}]$ contains disjoint paths P, Q from u_i, w_1 to v_i, w_2, respectively. Let X' be obtained from X by replacing $u_i X v_i$ by P. Then $B \cup Q$ is contained in a 2-connected block of $H - X'$. So by the choice of X, $H - X'$ is not connected and hence, by Lemma 3.2, $y_1, y_2, y_3 \in N(B)$. Let C denote a chain of blocks in $B_i' - Q$ from u_i to v_i. Since B_i is 2-connected, $B_i' - C$ is connected. Let C' be obtained from $B_i' + u_i v_i$ by contracting $B_i' - C$ to a single vertex u. Note that C' is 2-connected and $C' - u$ is 2-connected. Suppose C' is 3-connected. Then by Lemma 2.1, C' contains an induced path P' from u_i to v_i such that $u \notin P'$ and $C' - P'$ is connected. Let X'' be obtained from X by replacing $u_i X v_i$ by P'. Then $H - X''$ is connected, and $B \cup Q$ is contained in a 2-connected block of $H - X''$, contradicting the maxmimality of B. Thus, let $\{v, w\}$ be a 2-cut of C'. Since $C' - u$ is 2-connected, $u \notin \{v, w\}$. So $\{v, w\}$ is a cut in $B_i + u_i v_i$. Let A denote a $\{v, w\}$-bridge of $B_i + u_i v_i$ (so that $u_i v_i \notin A$). Since B_i is 2-connected, B_i contains disjoint paths P_v, P_w from $\{u_i, v_i\}$ to v, w, respectively. By choosing notation we may assume $v_i \in P_v$ and $u_i \in P_w$. Since G is 5-connected, $y_1, y_2, y_3 \in N(A - \{v, w\})$. So by Menger's theorem, $G[A + \{y_1, y_2\}]$ contains four independent paths P_1, P_2, P_3, P_4 from some vertex $x \in A - \{v, w\}$ from x to y_1, y_2, v, w, respectively. Let Q be a path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2. Then $P_1 \cup P_2 \cup (P_3 \cup P_v \cup v_i X x_2) \cup (P_4 \cup P_u \cup u_i X x_1) \cup Q$ is a TK_5 in G with branch vertices x, x_1, x_2, y_1, y_2.

We may assume that $B_i - \{u_i, v_i\}$ is connected. For suppose not, and let C_1, C_2 dneote two components of $B_i - \{u_i, v_i\}$. Since B_i is 2-connected, $\{u_i, v_i\} \subseteq N(C_j)$ for $j = 1, 2$. So by the above claim we may assume that $w_1 \notin N(C_2)$ and $w_2 \notin N(C_1)$. Now by Lemma 3.5, G contains TK_5.

Therefore, $H - B$ is a 3-planar chain. ∎

We adopt the following notation throughout the rest of this paper. Let D be a block in $H - B$, and let $u_D, v_D \in V(D \cap X)$ with $u_D X v_D$ maximal such that x_1, u_D, v_D, x_2 occur on X in order. If $|N(D - \{u_D, v_D\}) \cap V(B)| = 2$, let $N(D - \{u_D, v_D\}) \cap B(X) = \{b_D, c_D\}$, and we say that D is a block (of $H - B$) of *type I.* If $|N(D - \{u_D, v_D\}) \cap V(B)| = 1$, let $N(D - \{u_D, v_D\}) \cap B(X) = \{b_D\}$ and $c_D = b_D$, and call D a block (of $H - B$) of *type II.* Also, let D' be obtained from $G[D + \{b_D, c_D\}]$ by deleting edges from $\{b_D, c_D\}$ to $\{u_D, v_D\}$. Note that $D' - \{b_D, c_D\} = D$ which is 2-connected when $|D| \geq 3$.

4 Blocks of type I

The aim of this section is to show that if there is a block of type I in $H - B$, then G contains TK_5. So let D be a block of $H - B$ of type I, and recall the notation for D', b_D, c_D, u_D, v_D. Also recall that D' contains no edge from $\{b_D, c_D\}$ to $\{u_D, v_D\}$, $b_D, c_D \in B$, and x_1, u_D, v_D, x_2 occur on X in order.

We will be interested in the graph obtained from $G[D' + \{y_1, y_2, y_3\}]$ by identifying y_1, y_2, y_3 as y. The idea is to apply Corollaries 2.11 and 2.12 to this graph; so we need it to be $(5, \{b_D, c_D, u_D, v_D, y\})$-connected. Thus, we need to know when D' is not $(4, \{b_D, c_D, u_D, v_D\})$-connected.

Lemma 4.1 *Suppose S is a minimal cut in D' such that $|S| \leq 3$ and $D' - S$ has a component C disjoint from $\{b_D, c_D, u_D, v_D\}$. Then G contains TK_5, or $|S| = 3$ and one of the following holds:*

(i) $D - C$ contains a path P from u_D to v_D such that $S \not\subseteq V(P)$, or

(ii) $S \cap \{b_D, c_D, u_D, v_D\} = \{v_D\}$, and $S - \{v_D\}$ is a 2-cut in D' separating $C + v_D$ from $\{b_D, c_D, u_D\}$, or

(iii) $S \cap \{b_D, c_D, u_D, v_D\} = \{u_D\}$, and $S - \{u_D\}$ is a 2-cut in D' separating $C + u_D$ from $\{b_D, c_D, v_D\}$.

Proof. Suppose $D - C$ contains no path from u_D to v_D. Then let C_1, C_2 denote the components of $D - C$ containing u_D, v_D, respectively. Since $|S| \leq 3$, $|S \cap V(C_1)| \leq 1$ or $S \cap V(C_2)| \leq 1$. Suppose $|S \cap V(C_2)| \leq 1$. Because D is 2-connected, we must have $S \cap V(C_2) = \{v_D\}$, $|S| = 3$, and $b_D, c_D \notin S$. Note that b_D, c_D have no neighbors in C and, in D', neither b_D nor c_D is adjacent to v_D. So $S - \{v_D\}$ is a 2-cut in D' separating $C + v_D$ from $\{b_D, c_D, u_D\}$, and (ii) holds. Similarly, if $|S \cap V(C_1)| \leq 1$ then (iii) holds.

Thus we may assume that $D - C$ contains a path P from u_D to v_D. If $S \not\subseteq V(P)$, then (i) holds. So we may assume that $S \subseteq V(P)$ for any path P in $D - C$ from u_D to v_D.

Let $s_1, s_2 \in S$ with $s_1 P s_2$ maximal, and assume that u_D, s_1, s_2, v_D occur on P in order. Since (D', b_D, u_D, c_D, v_D) is 3-planar, D' is the union of two subgraphs D_1 and D_2 such that $D_1 \cap D_2 = P$, $b_D \in D_1$ and $c_D \in D_2$. Note that $s_2 = v_D$, or $\{s_2, c_D\}$ is a 2-cut in D_2 separating v_D from u_D; otherwise we can modify P inside D_2 to avoid s_2. Similarly, $s_2 = v_D$, or $\{b_D, s_2\}$

is a 2-cut in D_1 separating v_D from u_D. Since D is 2-connected, we must have $s_2 = v_D$. By the same argument, we also have $s_1 = u_D$. Since S is minimal and C is connected, $C \subseteq D_1$ or $C \subseteq D_2$. However, as (D', b_D, u_D, c_D, v_D) is 3-planar, $\{u_D, v_D\}$ must be a cut in D' separating b_D from c_D. Thus G contains TK_5 by Lemma 3.5. ∎

The next result will allow us to assume that D' is $(4, \{b_D, c_D, u_D, v_D\})$-connected.

Lemma 4.2 *Suppose S is a minimal cut in D' and C is a component of $D' - S$ such that $|S| \leq 3$ and $V(C) \cap \{b_D, c_D, u_D, v_D\} = \emptyset$. Then G contains TK_5.*

Proof. Note that the minimality of S implies $S \subseteq N(C)$. We choose S and C so that

(1) C is maximal.

Since D is 2-connected, $|S - \{b_D, c_D\}| \geq 2$ and there exist $s, t \in S - \{b_D, c_D\}$ such that

(2) $D - (S - \{s, t\})$ contains disjoint paths P', P'' from s, t to u_D, v_D, respectively.

By Lemma 4.1, we may assume that $|S| = 3$, and (i) or (ii) or (iii) of Lemma 4.1 holds. Let $S - \{s, t\} = \{r\}$. Since G is 5-conected, $|N(C) \cap \{y_1, y_2, y_3\}| \geq 2$. We may assume that

(3) $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(C') \cap \{y_1, y_2, y_3\}|$, where C' is any 2-connected endblock of C. Moreover, $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$.

First, suppose there is a path P in $D - C$ from u_D to v_D such that $S \not\subseteq V(P)$, and let X' be obtained from X by replacing $u_D X v_D$ with P. Then $C' \subseteq H - X'$; so by Lemma 3.2 and the choice of X, we have $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(C') \cap \{y_1, y_2, y_3\}|$ for any 2-connected block C' of C. If C is 2-connected, then $C' = C$ and hence $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(C') \cap \{y_1, y_2, y_3\}| \geq 2$; so (3) holds. Thus we may assume that C is not 2-connetced. Let $C_1, \ldots, C_k$ denote the endblocks of C, where $k \geq 2$. Suppose $|N(C_i) \cap S| \leq 2$ for some i. Then, since G is 5-connected, $|N(C_i) \cap \{y_1, y_2, y_3\}| \geq 2$. Hence by Lemma 3.1, C_i is 2-connected. So C_i is contained in a 2-connected block of $H - X'$, and (3) follows from the choice of X and Lemma 3.2. So we may assume that $|S| = 3$ and $S \subseteq N(C_i)$ for $i = 1, \ldots, k$. This implies that $G[C + (S - V(P))]$ is 2-connected, and hence is contained in a 2-connected block of $H - X'$. By the choice of X and by Lemma 3.2, we have (3).

Now, suppose that there is no path in $D - C$ from u_D to v_D such that $S \not\subseteq V(P)$. Then by symmetry, we may further assume that S, C satisfy (ii) of Lemma 4.1. Then $v_D = t$. Note that $b_D, c_D \notin S$, since D is 2-connected. Since G is 5-connected, $|N(C) \cap \{y_1, y_2, y_3\}| \geq 2$. So by Lemma 3.1, $|V(C)| \geq 3$.

We claim that $v_D = x_2$ and there is no path in H from x_2 to B internally disjoint from $B \cup X \cup C$. For, otherwise, $H - C$ contains a path X' between x_1 and x_2 (which could use a path in $D - C$ from b_D to u_D). So by Lemma 3.2 and the choice of X, $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(C') \cap \{y_1, y_2, y_3\}|$ for any 2-connected block C' of C. Clearly, $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$ if $|N(C') \cap \{y_1, y_2, y_3\}| \geq 2$ for some choice of C'. So assume $|N(C') \cap \{y_1, y_2, y_3\}| \leq 1$ for any choice of C'. Then $C' \neq C$ and $S \subseteq N(C')$ (since G is 5-connected); so $G[C + S] - v_D$ is 2-connected and contained in $H - X'$. It follows from Lemma 3.2 and the choice of X that $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$.

Note that $S - \{v_D\}$ is a 2-cut in D separating $v_D = x_2$ from $\{b_D, c_D, u_D\}$. Let J dneote the $(S - \{v_D\})$-bridge of D containing $v_D = x_2$. Suppose J is not 2-connected, and let z be a cut vertex of J. Since D is 2-connected, z must separate some $r \in S - \{v_D\}$ from $S - \{r\}$. By Lemma 3.4, the v-bridge of J containing r is induced by the edge rv. Let J' be obtained from J by deleting each vertex in $S - \{v_D\}$ that has degree 1 in J; then J' is 2-connected. Let $T = \{v_1, v_2\} \subseteq V(J')$ be the cut of D separating T from $\{b_D, c_D, u_D\}$. Since G is 5-connected and $|C| \geq 3$, we may assume $y_2, y_3 \in N(J' - \{v_1, v_2, x_2\})$. So by Lemma 3.1, $|V(J')| \geq 5$.

Note that $\{v_1, v_2, y_1, y_2, y_3\}$ is a cut in G, and we can write $G = G_1 \cup G_2$ such that $V(G_1 \cap G_2) = \{v_1, v_2, y_1, y_2, y_3\}$, $J' \subseteq G_1$, and $B \subseteq G_2$. Since $G_2 - \{v_1, v_2, y_1\}$ is connected, it contains three independen paths from some vertex $u \in V(G_2) - V(G_1)$ to x_1, y_2, y_3, respectively. Thus by Lemma 2.4, G_2 has five independent paths P_1, P_2, P_3, P_4, P_5 from u to $S' := \{v_1, v_2, x_1, y_1, y_2, y_3\}$ such that $P_i \cap P_j = \{u\}$ for $1 \leq i \neq j \leq 5$, $|V(P_i) \cap S'| = 1$, $x_1 \in P_1$, $y_2 \in P_2$, and $y_3 \in P_3$. We may assume that P_4 ends in $\{v_1, v_2\}$.

We may assume that $y_1 \in N(J' - \{v_1, v_2, x_2\})$. For, suppose not. Then $\{v_1, v_2, x_2, y_3, y_3\}$ is a 5-cut in G. Without loss of generality, assume $v_1 \in P_4$. If $G[J' + \{y_2, y_3\}] - v_2$ contains disjoint paths Q_1, Q_2 from v_1, y_2 to x_2, y_3, respectively, then $P_1 \cup (P_4 \cup Q_1) \cup P_2 \cup P_3 \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. So we may assume such Q_1, Q_2 do not exist. Then by Corollary 2.3, $(G[J' + \{y_2, y_3\}] - v_2, v_1, y_2, x_2, y_3)$ is planar. So G contains TK_5 by Corollary 2.9.

We claim that for any v_i, there exists $\{p, q\} \subseteq \{1, 2, 3\}$ such that $G[J' + \{y_p, y_q\}]$ contains disjoint paths from v_i, y_p to x_2, y_q, respectively. To prove this let J'' be obtained from $G[J' + \{y_1, y_2, y_3\}]$ by identifying y_1 and y_2 as y. If J'' contains disjoint paths from v_1, y to x_2, y_3, respecttively, then this claim holds for some $p \in \{1, 2\}$ and $q = 3$. Otherwise, by Lemma 2.2, (J'', v_1, y, x_2, y_3) is planar. Then since J' is 2-connected, we see that the claim holds for $p = 1$ and $q = 2$.

Now without loss of generality we may assume that $G[J' + \{y_1, y_2\}]$ contains disjoint paths R_1, R_2 from v_1, y_2 to x_2, y_3, respectively. (The notation can be choosen this way so that we can use the paths $P_1, \ldots, P_5$ above.) If $v_1 \in P_k$ for some $k \in \{4, 5\}$, then $P_1 \cup (P_k \cup R_1) \cup P_2 \cup P_3 \cup R_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. So we may assume $v_1 \notin P_4 \cup P_5$. Hence we may further assume that $v_2 \in P_4$ and $y_1 \in P_5$. Now by the above claim there exists $\{p, q\} \subseteq \{1, 2, 3\}$ such that $G[J' + \{y_p, y_q\}]$ contains disjoint paths R_1', R_2' from v_2, y_p to x_2, y_q, respectively. Then $P_1 \cup (P_4 \cup R_1') \cup R_2' \cup K$ and $P_2 \cup P_3$ (if $\{p, q\} = \{2, 3\}$), or $P_2 \cup P_5$ (if $\{p, q\} = \{1, 2\}$), or $P_3 \cup P_5$ (if $\{p, q\} = \{1, 3\}$) is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. This completes the proof of (3).

(4) We may assume $\{y_1, y_2, y_3\} \not\subseteq N(C)$.

Suppose $\{y_1, y_2, y_3\} \subseteq N(C)$. Let $S' := S \cup \{y_1, y_2, y_3\}$.

We may assume $\{y_1, y_2, y_3\} \not\subseteq N(B)$. For, suppose $\{y_1, y_2, y_3\} \subseteq N(B)$. Since $G[C + \{y_1, s, t\}]$ is connected, it contains three independent paths from some vertex $u \in C$ to y_1, s, t, respectively. So Lemma 2.4 implies the existence of five independent paths P_1, P_2, P_3, P_4, P_5 in $G[C + S']$ from u to S', such that $V(P_i \cap P_j) = \{u\}$ for $1 \leq i \neq j \leq 5$, $|V(P_i) \cap S'| = 1$ for $1 \leq i \leq 5$, $y_1 \in P_1$, $s \in P_3$, and $t \in P_4$. We may assume by symmetry (between y_2 and y_3) that P_2 ends at y_2, and let Q denote a path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2. Then $(P_3 \cup P' \cup u_D X x_1) \cup (P_4 \cup P'' \cup v_D X x_2) \cup P_2 \cup P_1 \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

If (i) of Lemma 4.1 holds, then let X' be the path obtained from X by replacing $u_D X v_D$ with P. We may assume that the paths P' and P'' are subpaths of P. Then $G[C+r] \subseteq H - X'$. If $G[C+r]$ is 2-connected then by Lemma 3.2 and the choice of X, $\{y_1, y_2, y_3\} \subseteq N(B)$, a contradiction. So $G[C+r]$ is not 2-connected. Let J be an endblock of $G[C+r]$ and v be the cutvertex of $G[C+r]$ contained in J such that $r \notin J - v$. If $\{y_1, y_2, y_3\} \subseteq N(J-v)$ then by Lemma 3.2 and the choice of X, we have $\{y_1, y_2, y_3\} \subseteq N(B)$, a contradiction. Hence we may assume $y_1, y_2 \in N(J-v)$ and $y_3 \notin N(J-v)$; so $s, t \in N(J-v)$. By Menger's theorem, $G[J + \{s, t, y_1, y_2\}]$ contains five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some $u \in V(J-v)$ to y_1, y_2, s, t, v, respectively. Since $y_3 \in N(C)$ we see that P_5 can be extened through $G[C - (J-v) + y_3]$ to a path Q_5' ending at y_3. If $y_1, y_2 \in N(B)$ then let Q be a path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2; now $Q_1 \cup Q_2 \cup (Q_3 \cup P' \cup u_D X x_1) \cup (Q_4 \cup P'' \cup v_D X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that by (3) that $y_i, y_3 \in N(B)$ for some $i \in \{1,2\}$. Let Q' be a path in $G[B + \{y_i, y_3\}]$ between y_i and y_3. Then $Q_i \cup Q_5' \cup (Q_3 \cup P' \cup u_D X x_1) \cup (Q_4 \cup P'' \cup v_D X x_2) \cup Q' \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_i, y_3.

Therefore, we may assume by symmetry that (ii) of Lemma 4.1 holds. So $t = v_D$. Without loss of generality and by (3), assume $y_1, y_2 \in N(B)$. Note that $G[C + \{t, y_1, y_2\}]$ contains independent paths from some $u \in V(C)$ to y_1, y_2, t, respectively. So by Lemma 2.4, $G[C + \{r, s, t, y_1, y_2, y_3\}]$ contains five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from u to S' such that $V(Q_i \cap Q_j) = \{u\}$ for $1 \leq i \neq j \leq 5$, $|V(Q_i) \cap S'| = 1$ for $1 \leq i \leq 5$, $y_1 \in Q_1$, $y_2 \in Q_2$, and $t \in Q_3$. We may assume that Q_4 ends at $v \in \{r, s\}$. Since D is 2-connected, $D - C$ contains a path R from v to u_D. Let Q be a path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2. Then $Q_1 \cup Q_2 \cup (Q_3 \cup v_D X x_2) \cup (Q_4 \cup R \cup u_D X x_1) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

By (4), let $y_1, y_2 \in N(C)$ and $y_3 \notin N(C)$. Since G is 5-connected, $C' := G[C + (S \cup \{y_1, y_2\})]$ is $(5, S \cup \{y_1, y_2\})$-connected. By Menger's theorem, C' contains five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex $z \in C$ to y_1, y_2, s, t, r, respectively.

If $y_1, y_2 \in N(B)$, then $G[B + \{y_1, y_2\}]$ contains a path A from y_1 to y_2. So by (2), $P_1 \cup P_2 \cup (P_3 \cup P' \cup u_D X x_1) \cup (P_4 \cup P'' \cup v_D X x_2) \cup A \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, z.

Hence we may assume that $y_1 \notin N(B)$. Hence by (3), $y_2, y_3 \in N(B)$. Let Q denote a path in $G[B + \{y_2, y_3\}]$ between y_2 and y_3.

(5) We may assyme $y_3 \notin N(D - \{u_D, v_D\})$.

Suppose $y_3 \in N(D - \{u_D, v_D\})$. First, assume that $G[D - C + y_3]$ contains disjoint paths Q_1, Q_2, Q_3 from S to u_D, v_D, y_3, respectively. Since we will not use P', P'' in this subscase, we have symmetry among r, s and t. So we may assume that $s \in Q_1$ and $t \in Q_2$. Then $P_2 \cup (P_5 \cup Q_3) \cup (P_3 \cup Q_1 \cup u_D X x_1) \cup (P_4 \cup Q_2 \cup v_D X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_2, y_3, z.

So we may assume that $G[D - C + y_3]$ has a minimal cut T, $|T| \leq 2$, separating S from $\{u_D, v_D, y_3\}$. So T is a cut in D separating $C + S$ from $\{u_D, v_D\}$. Since D is 2-connected, $y_3 \notin T$ and $|T| = 2$. Let D_1 denote the T-bridge of D containing C (so $D_1 - T$ is connected), and let D_2 denote the minimal union of T-bridges of D containing $\{u_D, v_D\}$ (so D_2 consists of at most two T-bridges of D).

If neither b_D nor c_D has a neighbor in $D_1 - T$, then T is a cut of D' separating D_1 from $\{b_D, c_D, u_D, v_D\}$; so $T \cup \{y_1, y_2\}$ is a cut in G, a contradiction. Hence, we may assume that b_D has a neighbor in $D_1 - T$.

If c_D has no neighbor in $D_1 - T$ then $T \cup \{b_D\}$ is a minimal cut of D' separating D_1 from $\{b_D, c_D, u_D, v_D\}$; so $T \cup \{b_D\}, D_1$ contradict the choices of S, C in (1). Hence we may assume that c_D also has a neighbor in $D_1 - T$.

Then $G[D_1 - T + \{b_D, c_D\}]$ contains a path from b_D to c_D. Since (D', b_D, u_D, c_D, v_D) is 3-planar, it contains no disjoint paths from b_D to c_D and from u_D to v_D. Hence, u_D and v_D belong to different components of D_2, and this contradicts the 2-connectedness of D and completes the proof of (5).

Observing the symmetry between b_D and c_D, we may assume that y_2 has a neighbor $y_2' \in B - b_D$. Let y_3' be a neighbor of y_3 in B.

(6) We may assume that $D' - c_D$ has disjoint paths R_1, R_2, R_3 from u_D, v_D, b_D to s, t, r, respectively.

Note that we will not be using P' and P'', so we have symmetry among vertices in S. So if (6) fails then there is a minimal cut T in $D' - c_D$, with $|T| \leq 2$, separating $C \cup S$ from $\{b_D, u_D, v_D\}$. Then T or $T \cup \{c_D\}$ contradicts the choice of S in (1).

(7) We may assume $N(y_3) \subseteq u_D X x_1 \cup v_D X x_2 \cup \{y_3'\}$.

Since y_3 has no neighbor in $D - \{u_D, v_D\}$, $G - \{y_1, y_2\}$ has a path R from y_3 to a vertex $y_3'' \in (B - y_3') \cup (u_D X x_1 - x_1) \cup (v_D X x_2 - x_2)$ and internally disjoint from $D' \cup B \cup X$. If $y_3'' \in B - y_3'$, then $G[B \cup R + \{y_2, y_3\}]$ has independent paths Q_1, Q_2 from y_3 to b_D and y_2, respectively; so $P_2 \cup (P_5 \cup R_3 \cup Q_1) \cup (P_3 \cup R_1 \cup u_D X x_1) \cup (P_4 \cup R_2 \cup v_D X x_2) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_2, y_3, z. Thus we may assume that $y_3'' \notin B - y_3'$ for any choice of R. So $y_3'' \in X$, $R = y_3 y_3''$ (as $H - X$ is connected), and $N(y_3) \subseteq u_D X x_1 \cup v_D X x_2 \cup \{y_3'\}$.

(8) We may further assume that $H - B$ has a 2-connected block F such that $y_3 \in N(F)$, $y_3' \in \{b_F, c_F\}$, and $x_1, u_F, v_F, u_D, v_D, x_2$ occur on X in order.

By (7) and by symmetry, we may assume that y_3 has a neighbor $y_3'' \in u_D X x_1 - x_1$. If $y_3 \in N(u_D)$ then we find independent paths L_1, L_2 in $G[D + y_2]$ from u_D to y_2, v_D, respectively; now $u_D X x_1 \cup (L_2 \cup v_D X x_2) \cup L_1 \cup u_D y_3 \cup Q \cup K$ is a TK_5 in G with branch vertices u_D, x_1, x_2, y_2, y_3. Thus we may assume that y_3 has a neighbor $y_3'' \in V(u_D X x_1 - \{u_D, x_1\})$.

Since X is induced, $H - D$ has a path R from y_3'' to B internally disjoint from $B \cup X$.

We claim that R must end at y_3' and we may choose R to be a path of length at least 2. First, we may assume that $C' - y_1$ has disjoint paths L_1, L_2 from s, r to t, y_2, respectively; for otherwise, $(C' - y_1, r, s, y_2, t)$ is not planar by Corollary 2.3, and hence G contains TK_5 by Corollary 2.9. If $G[B \cup R' + \{y_2, y_3\}]$ has disjoint paths M_1, M_2 from y_3'', y_3 to y_2, b_D, respectively, then $M_1 \cup y_3'' y_3 \cup y_3'' X x_1 \cup (y_3'' X u_D \cup R_1 \cup L_1 \cup R_2 \cup v_D X x_2) \cup (M_2 \cup R_3 \cup L_2) \cup K$ is a TK_5 in G with branch vertices $x_1, x_2, y_2, y_3, y_3''$. If $G[B \cup R' + \{y_2, y_3\}]$ has disjoint paths N_1, N_2 from y_3'', y_3 to b_D, y_2, respectively, then $(N_1 \cup R_3 \cup L_2) \cup y_3'' y_3 \cup y_3'' X x_1 \cup (y_3'' X u_D \cup R_1 \cup L_1 \cup R_2 \cup v_D X x_2) \cup N_1 \cup K$ is a TK_5 in G with branch vertices $x_1, x_2, y_2, y_3, y_3''$. So we may assume that M_1, M_2 do not exist, and N_1, N_2 do not exist. Therefore, R must end at y_3'. Moreover, we may choose R to be a path of length at least 2; as otherwise there are two edges from y_3'' to B, and M_1, M_2 or N_1, N_2 would exist.

Note that $R - y_3'$ is contained in a 2-connected block F of $H - B$, and let b_F, c_F, u_F, v_F be defined as before; so $y_3' \in \{b_F, c_F\}$. Then $x_1, u_F, v_F, u_D, v_D, x_2$ occur on X in order.

By (7) and (8), let w denote a neighbor of $y_3 \in N(F)$ in $u_F X v_F - \{u_D, x_1\}$. We may assume that

(9) $w \notin \{u_F, v_F\}$.

Suppose $w \in \{u_F, v_F\}$ for any choice of w. Then $y_3 \notin N(F - \{u_F, v_F\})$. Hence we may assume that $y_1, y_2 \in N(F - \{u_F, v_F\})$, which follows from 5-connectedness of G when $b_F = c_F$, or from the planarity of (F', b_F, u_F, c_F, v_F) when $b_F \neq c_F$ (as otherwise G contains TK_5 by Corollary 2.9).

Let $S' := \{b_F, c_F, u_F, v_F, y_1, y_2\}$. Since $G[F + y_1]$ is connected, it contains three independent paths from some vertex $u \in F - \{u_F, v_F\}$ to u_F, v_F, y_1, respectively. Since $G[F' + \{y_1, y_2\}]$ is $(5, S')$-connected, it follows from Lemma 2.4 that $G[F' + \{y_1, y_2\}]$ contains five independent paths W_1, W_2, W_3, W_4, W_5 from u to S' such that $V(W_i \cap W_j) = \{u\}$ for $1 \leq i \neq j \leq 5$, $|V(W_i) \cap S'| = 1$ for $1 \leq i \leq 5$, $u_F \in W_1$, $v_F \in W_2$, and $y_1 \in W_3$. Without loss of generality, we may assume that W_4 ends in $\{b_F, c_F\}$. Thus W_4 can be extended through $G[B + y_2]$ to a path W_4' ending at y_2.

If $C' - r$ contains disjoint paths L_1, L_2 from y_1, s to y_2, t, respectively, then $W_3 \cup W_4' \cup (W_1 \cup u_F X x_1) \cup (W_2 \cup v_F X u_D \cup R_1 \cup L_2 \cup R_2 \cup v_D X x_2) \cup L_1 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. Thus we may assume that L_1, L_2 do not exist in $C' - r$. By Corollary 2.3, $(C' - r, y_1, s, y_2, t)$ is planar; so G contains TK_5 by Corollary 2.9.

By (9), we may assume that $w \in F - \{u_F, v_F\}$. Let $S' := \{b_F, c_F, u_F, v_F\} \cup (N(F - \{u_F, v_F\}) \cap \{y_1, y_2\})$. It is clear that $G[F' + S']$ is $(4, S')$-connected. Also note that F has independent paths from w to u_F, v_F, as it is 2-connected. So by Lemma 2.4, $G[F' + S']$ contains four independent paths W_1, W_2, W_3, W_4 from w to S' such that $V(W_i \cap W_j) = \{u\}$ for $1 \leq i \neq j \leq 4$, $|V(W_i) \cap S'| = 1$ for $1 \leq i \leq 4$, $u_F \in W_1$ and $v_F \in W_2$. Without loss of generality, we may assume that $b_F = y_3'$ and $c_F \notin W_3$.

If W_3 ends at y_2, then $wy_3 \cup W_3 \cup (W_1 \cup u_F X x_1) \cup (W_2 \cup v_F X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_2, y_3. (Recall that Q is given before (5).)

Now assume that W_3 ends at y_1. If $C' - y_2$ has disjoint paths L_1, L_2 from r, s to y_1, t, respectively, then let Q' denote a path in $G[B + y_3]$ between b_D and y_3; so $wy_3 \cup W_3 \cup (W_1 \cup u_F X x_1) \cup (W_2 \cup v_F X u_D \cup R_1 \cup L_2 \cup R_2 \cup v_D X x_2) \cup (Q' \cup R_3 \cup L_1) \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_1, y_3. So we may assume that L_1, L_2 do not exist. Then by Corollary 2.3, $(C' - y_2, r, s, y_1, t)$ is planar; so G contains TK_5 by Corollary 2.9.

We may thus assume that W_3 ends at $b_F = y_3'$. Recall that $y_2' \neq b_D$. In $G[B + y_2]$ we find independent paths Q_1, Q_2 from b_F to b_D, y_2, respectively. Then $y_3 y_3' \cup W_3 \cup y_3 w \cup (x_1 y_2 \cup Q_2) \cup x_1 y_3 \cup (x_1 X u_F \cup W_1) \cup (x_1 y_1 \cup P_1) \cup (P_5 \cup R_3 \cup Q_1) \cup (P_3 \cup R_1 \cup u_D X v_F \cup W_2) \cup (P_4 \cup R_2 \cup v_D X x_2 \cup x_2 y_3)$ is a TK_5 in G with branch vertices w, x_1, y_3, y_3', z. ∎

Let D^* be obtained from $G[D' + \{y_1, y_2, y_3\}]$ by identifying y_1, y_2, y_3 to a single vertex y, and let $A^* := \{y, b_D, c_D, u_D, v_D\}$. Recall that D' does not contain edges from $\{b_D, c_D\}$ to $\{u_D, v_D\}$, and note that

$$(D^* - y, b_D, u_D, c_D, v_D) \text{ is planar.}$$

So we may assume

$$|N(D - \{u_D, v_D\}) \cap \{y_1, y_2, y_3\}| \geq 2;$$

as otherwise, G contains TK_5 by Corollary 2.9. By Lemma 3.1, $|D| \geq 4$; so $|D^*| \geq 7$. By Lemma 4.2, we may assume that

$$D^* \text{ is } (5, A^*)\text{-connected.}$$

Let C denote the facial walk of $D^* - y$ containing $A^* - \{y\}$ and assume that it is the outer walk of $D^* - y$. Then C is a cycle, or b_D (or c_D) has degree 1 in C and $C - b_D$ (or $C - c_D$) is a cycle, or b_D, c_D both have degree 1 in C and $C - \{b_D, c_D\}$ is a cycle.

We now show that there exist paths in D^* as shown in Corollaries 2.11 and 2.12.

Lemma 4.3 *G contains TK_5, or there exist a vertex $w \in D^* - A^*$ and a cycle C_w in $D^* - y$ such that C_w consists of all vertices of $D^* - y$ cofacial with w, and one of the following holds:*

(1) w is a neighbor of y and $D^* - y$ has three independent paths P_1, P_2, P_3 from w to $\{b_D, c_D, u_D, v_D\}$ such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 3$, and $|V(P_i \cap C_w)| = |V(P_i) \cap A^*| = 1$ for $i = 1, 2, 3$.

(2) y has no neighbor in $D^* - C$, $C \cap C_w = \emptyset$, and $D^* - y$ has four independent paths P_1, P_2, P_3, P_4 from w to A^* such that $V(P_i \cap P_j) = \{w\}$ for $1 \leq i < j \leq 4$, $|V(P_i \cap C_w)| = |V(P_i) \cap A^*| = 1$ for $1 \leq i \leq 4$, and either (a) $y \notin \bigcup_{i=1}^4 P_i$, or (b) $y \in \bigcup_{i=1}^4 P_i$ and we can write $A^* - \{y\} = \{a_1, a_2, a_3, a_4\}$ such that $a \in P_1$, $a_i \in P_i$ for $i = 2, 3, 4$, $a_1, a_2, a_3, P_1 \cap C, a_4$ occur on C in cyclic order.

Proof. If D^* has a 5-separation (F_1, F_2) such that $\{y, b_D, c_D, u_D, v_D\} \subseteq F_1$ and $|F_2| \geq 7$, we choose (F_1, F_2) so that F_2 is minimal and let $A := V(F_1) \cap V(F_2)$; otherwise let $F_2 = D^*$ and $A := \{y, b_D, c_D, u_D, v_D\}$. By the minimality of F_2, A is independent in F_2 and $F_2 - y$ is 2-connected. We may assume $y \in A$; for, otherwise, since (F_2, A) is planar, G contains TK_5 by Lemma 2.6.

By Menger's theorem, there are four disjoint paths in $F_1 - y$ from $A - \{y\}$ to $A^* - \{y\}$, which allows us to extend the paths we will find in F_2 to the desired paths in D^*. Let C' denote the the outer cycle of $F_2 - y$, which contains $A - y$. We may assume $D^* - y$ contains no K_4^- as otherwise G contains K_4^-, and hence G contains TK_5 by Theorem 1.1.

If y has a neighbor inside C', say w, then (1) follows from Corollary 2.11 (after appropriate extension of the paths to A^*). Hence we may assume that C' contains all neighbors of y in F_2. If F_2 is not the exceptional graph in Corollary 2.12, then (2) follows from Corollary 2.12 (after appropriate extensions of the paths to A^*).

So we may assume that F_2 is the exceptional graph. Let $A = \{b', c', u', v'\}$ and $tuvwt$ be the cycle in $F_2 - A$ such that $C' = b'tv'uc'vu'wb'$, and let Q_1, Q_2, Q_3, Q_4 be disjoint paths in $F_1 - y$ from b', c', u', v' to b_D, c_D, u_D, v_D, respectively.

Since G is 5-connected and by Lemma 3.1, each of $\{t, u, v, w\}$ has exactly one neighbor in $\{y_1, y_2, y_3\}$. Since G contains no K_4^-, we may assume by symmetry that $y_3 \in N(u) \cap N(w)$ and that either $y_2 \in N(t) \cap N(v)$ or $y_1 \in N(v)$ and $y_2 \in N(t)$.

Suppose $y_2 \in N(t) \cap N(v)$. Then by Lemma 3.1, $y_1 \notin N(\{t, u, v, w\})$. Note that $G' := G - \{t, u, v, w, y_2, y_3\}$ contains two paths R_1, R_2 from b' to $\{c', u', v'\}$ such that $R_1 \cap R_2 = \{b'\}$;

for otherwise, G' has a cut T, $|T| \le 1$, separating b' from $\{c', u', v'\}$, and so $\{b', y_2, y_3\} \cup T$ would be a cut in G, contradicting 5-connectedness of G. Clearly, R_1, R_2 can be extended, using $u'v$ or $c'v$ and $v'u$ or $c'u$, to give independent paths R'_1, R'_2 in $G - \{t, u, v, w, y_2, y_3\}$ from b' to u, v, respectively. Now $b't \cup b'w \cup R'_1 \cup R'_2 \cup tuvwt \cup ty_2v \cup uy_3w$ is a TK_5 in G with branch vertices b', t, u, v, w.

Thus we may assume that $y_1 \in N(v)$ and $y_2 \in N(t)$. Note the triangle $b'twb'$ is contained in a block of $H - (x_1Xu_D \cup Q_3 \cup u'vuv' \cup Q_4 \cup v_DXx_2)$ and has two neighbors in $\{y_1, y_2, y_3\}$. So by Lemma 3.2 and by the choice of X, $|N(B) \cap \{y_1, y_2, y_3\}| \ge 2$. If $y_1, y_2 \in N(B)$ then let Q be a path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2; now $(twu' \cup Q_3 \cup u_DXx_1) \cup (tv' \cup Q_4 \cup v_DXx_2) \cup (tuvy_1) \cup ty_2 \cup Q \cup K$ is a TK_5 in G with branch vertices t, x_1, x_2, y_1, y_2. So by symmetry we may assume that $y_2, y_3 \in N(B)$. Let R denote a path in $G[B + \{y_2, y_3\}]$ between y_2 and y_3. Then $(tuvu' \cup Q_3 \cup u_DXx_1) \cup (tv' \cup Q_4 \cup v_DXx_2) \cup twy_3 \cup ty_2 \cup R \cup K$ is a TK_5 in G with branch vertices t, x_1, x_2, y_2, y_3. ∎

Lemma 4.4 *Suppose D^* contains w, C_w, P_1, P_2, P_3 which satisfy (1) of Lemma 4.3. Then G contains TK_5.*

Proof. Without loss of generality, we may assume that $y_1w \in E(G)$. Let $L = C_w \cup P_1 \cup P_2 \cup P_3 \cup y_1w$. We may assume that

(1) We may assume that $\{b_D, c_D\} \subseteq L$, and $v_D \in L$ (by symmetry).

If $\{u_D, v_D\} \subseteq L$, then (1) holds by letting $v_D \in L$ using symmetry between u_D and v_D. So assume $\{u_D, v_D\} \subseteq L$. By symmetry, we may assume $b_D \in L$.

We may assume that $x_1 = u_D$ and $x_2 - v_D$. Otherwise, we may assume by symmetry that $x_1 \ne u_D$. Then H has a path Q from x_1 to b_D and internally disjoint from $X \cup D'$. Now $L \cup Q \cup x_1y_1 \cup x_1Xu_D \cup (x_1y_2x_2 \cup x_2Xv_D)$ is a TK_5 in G.

If $|V(x_2C_wx_1)| = 2$ then $x_1x_2 \in E(G)$; so $G[x_1, x_2, y_1, y_2] \cong K_4^-$, and G contains TK_5 by Theorem 1.1. So we may assume that $|V(x_2C_wx_1)| \ge 3$.

Suppose w has no neighbor in $x_2C_wx_1 - \{x_1, x_2\}$. Since D^* is $(5, A^*)$-connected, $\{x_1, x_2, c_D\}$ cannot be a cut in D separating $\{b_D, c_D, x_1, x_2\}$ from some vertex. Therefore, $x_2C_wx_1 = x_2c_Dx_1$. As D is of type I, $c_Dw \in E(G)$. Now $G[\{c_D, w, x_1, x_2\}] \cong K_4^-$, and G contains TK_5 by Theorem 1.1.

Therefore, we may assume that w has a neighbor $w' \in x_2C_wx_1 - \{x_1, x_2\}$. If D contains a path Q from w' to c_D and internally disjoint from C_w, then replacing the path in L from w to u_D with $Q + \{w, ww'\}$ we get (1). So we may assume that such Q does not exist. Then since $(D^* - y, b_D, u_D, c_D, v_D)$ is planar, there exist $u \in V(w'C_wx_1 - w')$ and $v \in V(x_2C_ww' - w')$ such that $\{u, v, w\}$ is a cut in D separating $\{b_D, c_D, x_1, x_2\}$ from w', contradicting the fact that D^* is $(5, A^*)$-connected.

(2) $x_1 \notin C_w$.

For if $x_1 \in C_w$ then $L \cup x_1y_1 \cup (x_1y_2x_2 \cup x_2Xv_D)$ and a path in B between b_D and c_D form a TK_5 in G with branch vertices w, x_1 and $P_i \cap C_w$, $i = 1, 2, 3$.

(3) We may assume that $D - u_D$ and $D - v_D$ are 2-connected, and $D' - \{u_D, v_D\}$ is a chain of blocks from b_D to c_D.

First, suppose $D - u_D$ is not 2-connected. Then let C be an endblock of $D - u_D$ and v be the cut vertex of $D - u_D$ contained in C such that $v_D \notin C - v$. Since D is 2-connected, $u_D \in N(C - v)$ and $u_D \in N(D - u_D - C)$. In particular, $D - (C - v)$ contains a path from u_D to v_D. Thus, since (D', b_D, u_D, c_D, v_D) is planar, $b_D \notin N(C - v)$ or $c_D \notin N(C - v)$, say the former. Then $\{c_D, u_D, v\}$ is a cut in D' separating C from $\{b_D, c_D, u_D, v_D\}$, contradicting the assumption that D^* is $(5, A^*)$-connected.

Thus we may assume that $D - u_D$ is 2-connected. Similarly, we may also assume that $D - v_D$ is 2-connected.

By the definition of planar chain, $D - \{u_D, v_D\}$ is connected. So $D' - \{u_D, v_D\}$ is connected. Now suppose $D' - \{u_D, v_D\}$ is not a chain of blocks from b_D to c_D. Then let C be an endblock of $D' - \{u_D, v_D\}$ and v be the cut vertex of $D - \{u_D, v_D\}$ such that $D' - \{u_D, v_D\} - (C - v)$ has a path between b_D and c_D. Then $\{u_D, v_D, v\}$ is a cut in D' separating C from $\{b_D, c_D, u_D, v_D\}$, contradicting the assumption that D^* is $(5, A^*)$-connected.

(4) We may assume $u_D = x_1$, and H contains no path from x_1 to B internally disjoint from $B \cup D' \cup X$.

Suppose (4) fails. Note that if $u_D \neq x_1$ then H contains a path from x_1 to B internally disjoint from $B \cup D' \cup X$. So let R be an arbitrary path in H from x_1 to $x \in V(B)$ and internally disjoint from $B \cup D' \cup v_D X x_2$.

Suppose x may be choosen so that there exists some $y_i \in N(B - x)$. Then $G[B \cup R + y_i]$ contains disjoint paths Q_1, Q_2 from $\{b_D, c_D\}$ to x_1, y_i, respectively. Recall $x_1 \notin C_w$ from (2). If $i = 1$ then $(y_1x_1 \cup Q_1) \cup Q_2 \cup (y_1x_2 \cup x_2Xv_D) \cup L$ is a TK_5 in G. So assume $i \neq 1$. Then $Q_1 \cup (x_1y_i \cup Q_2) \cup x_1y_1 \cup (x_1y_{5-i}x_2 \cup x_2Xv_D) \cup L$ is a TK_5 in G.

Therefore, we may assume that x is unique and $y_i \notin N(B - x)$ for all $i = 1, 2, 3$. So by Lemma 3.1, $|N(B) \cap \{y_1, y_2, y_3\}| \leq 1$. If H has a path from x_2Xv_D to B internally disjoint from $B \cup D' \cup X$, then H has a path from x_1 to x_2 disjoint from $D - v_D$; so by Lemma 3.2 and the choice of X, $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(D - v_D) \cap \{y_1, y_2, y_3\}| \geq 2$, a contradiction.

Thus we may assume that H has no path from x_2Xv_D to B internally disjoint from $B \cup D' \cup X$; so $x_2 = v_D$. Since $\{b_D, c_D, u_D, x\}$ cannot be a cut in G, we see that $|B| = 3$ and $x \notin \{b_D, c_D\}$. Since x has at least three neighbors outside B, $G - D'$ contains independent paths Q_1, Q_2 from x to x_1, y_i, respectively, for some $i \in \{1, 2, 3\}$. If $i = 1$ then $(Q_1 \cup x_1y_2x_2) \cup Q_2 \cup (B - b_Dc_D) \cup L$ is a TK_5 in G; and if $i \neq 1$ then $(Q_1 \cup x_1y_2) \cup (Q_2 \cup y_ix_2 \cup x_2Xv_D) \cup (B - b_Dc_D) \cup L$ is a TK_5 in G.

(5) We may assume that $y_1 \notin N(B - \{b_D, c_D\})$ and $|N(y_1) \cap B| \leq 1$.

First, suppose $|N(y_1) \cap B| \geq 2$. Then $G[B + y_1]$ has two independent paths Q_1, Q_2 from y_1 to b_D, c_D, respectively. So $Q_1 \cup Q_2 \cup (y_1x_2 \cup x_2Xv_D) \cup L$ is a TK_5 in G.

Now let $y \in N(y_1) \cap V(B - \{b_D, c_D\})$. Since G is 5-connected, $x_2Xv_D + \{y_2, y_3\}$ has a neighbor in $B - \{b_D, c_D\}$. If $G[B \cup x_2Xv_D + \{y_2, y_3\}]$ has three independent paths Q_1, Q_2, Q_3 from y to $b_D, c_D, x_2Xv_D + \{y_2, y_3\}$, respectively, then we may assume Q_3 ends at v_D; now $Q_1 \cup Q_2 \cup Q_3 \cup yy_1 \cup L$ is a TK_5 in G. So we may assume that such Q_1, Q_2, Q_3 do not exist. Then there is a 2-cut S in $G[B \cup x_2Xv_D + \{y_2, y_3\}]$ separating y from $b_D, c_D, x_2Xv_D + \{y_2, y_3\}$. Since B is 2-connected $S \subseteq B$. But then by (4), $S \cup \{y_1\}$ is a 3-cut in G, a contradiction.

Let $S := \{b_D, c_D, y_2, y_3\} \cup V(x_2 X v_D)$. Then by (4) and (5), $G' := G - y_1 - (D - v_D)$ is $(5, S)$-connected, and $G' - \{y_2, y_3\}$ contains a path from B to $v \in V(x_2 X v_D)$ and internally disjoint from X. We choose v so that vXv_D is minimal. Note that $G' - \{y_2, y_3\} - (x_2 X v_D - v)$ has independent paths from some $u \in V(B) - \{b_D, c_D\}$ to b_D, c_D, v, respectively. So by Lemma 2.4, $G - \{x_1, y_1\} - D''$ contains five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from u to b_D, c_D, v, z_1, z_2, respectively, where $z_1, z_2 \in S - \{v\}$ such that $|V(Q_i) \cap S| = 1$ for $1 \le i \le 5$. If $v \ne x_2$ then Q_4 can be extended through $G[(x_2 X v - v) + \{y_1, y_2, y_3\}]$ to a path Q_4' ending at y_1; so $Q_1 \cup Q_2 \cup (Q_3 \cup vXv_D) \cup Q_4' \cup L$ is a TK_5 in G. So assume $v = x_2$. Then by the minimality of vXv_D, we see that $z_1 \in \{y_2, y_3\}$, sat $z_1 = y_2$. Now by (2), $(Q_1 \cup Q_2 \cup (Q_3 \cup vXv_D) \cup (Q_4 \cup y_2 x_1 y_1) \cup L$ is a TK_5 in G. ∎

Lemma 4.5 *Suppose D^* contains $w, C_w, P_1, P_2, P_3, P_4$ satisfying (2) of Lemma 4.3. Then G contains TK_5.*

Proof. Let $L = C_w \cup P_1 \cup P_2 \cup P_3 \cup P_4$. If $y \notin L$ then L, $u_D X x_1 \cup x_1 y_1 x_2 \cup x_2 X v_D$ and a path in B between b_D and c_D form a TK_5 in G. So we may assume that $y \in P_1$. Since D^* is $(5, A^*)$-connected, D' is $(4, \{b_D, c_D, u_D, v_D\})$-connected. Recall that $(D^* - y, b_D, u_D, c_D, v_D)$ is planar. Let C denote the outerwalk of $D^* - y$; note that C is a cycle, or $C - b_D$ is a cycle and b_D is of degree 1 in C, or $C - c_D$ is a cycle and c_D is of degree 1 in C, or $C - \{b_D, c_D\}$ is a cycle and both b_D and c_D have degree 1 in C. Without loss of generality we may assume that b_D, u_D, c_D, v_D occur on C in couterclockwise order.

Recall that $C_w \cap C = \emptyset$. We have two cases: $u_D, v_D \in L$, or $b_D, c_D \in L$.

Case 1. $u_D, v_D \in L$

By symmetry, we may assume that $u_D \in P_2$, $b_D \in P_3$, and $v_D \in P_4$. Without loss of generality we may view P_1 as a path in G with $y_1 \in P_1$. Further, we may assume by symmetry that $c_D, u_D, b_D, P_1 \cap C, v_D$ occur on C in clockwise order.

We may assume that $x_2 = v_D$, H has no path from x_2 to B internally disjoint from $B \cup D \cup X$, and $N(\{y_2, y_3\}) \subseteq D \cup X$. For, otherwise, $G - y_1$ has a path Q from v_D to b_D disjoint from $(D - v_D) \cup u_D X x_1$, and $L \cup Q \cup (y_1 x_1 \cup x_1 X u_D)$ is a TK_5 in G.

Then $u_D \ne x_1$; as otherwise $\{b_D, c_D, x_1, y_1\}$ would be a 4-cut in G. Hence H contains a path X_1 from x_1 to some $x_1' \in V(B)$ and internally disjoint from $X \cup B \cup D'$.

We may also assume $N(y_1) \subseteq B \cup D \cup \{x_1, x_2\}$. Otherwise, $G - \{y_2, y_3\}$ contain a path P from y_1 to $u_D X x_1 - x_1$ and internally disjoint from $B \cup X \cup D'$. Now $P \cup X_1 \cup B \cup u_D X x_1$ contains disjoint paths from x_1, y_1 to b_D, u_D, respectively, which, together with $L \cup x_1 y_2 x_2$, forms a TK_5 in G.

Suppose y_2, y_3 have neighbors u, v, respectively, in $u_D X x_1 - x_1$. Without loss of generality let x_1, u, v, u_D occur on X in order. Since $H - X$ is connected and $|N(u) \cap \{y_1, y_2, y_3\}| \le 1$ (by Lemma 3.1), u has a neighbor in B or there is a path in H from u to B internally disjoint from $X \cup B$. Thus H contains a path Q from u to b_D internally disjoint from $X \cup D'$. Now $L \cup (x_2 y_2 u \cup Q) \cup (y_1 x_1 y_3 v \cup v X u_D)$ is a TK_5 in G.

So we may assume that $N(y_3) \subseteq D' \cup \{x_1, x_2\}$.

We may assume that y_2 has a neighbor in $u_D X x_1 - \{x_1, u_D\}$, say u, and choose u so that uXx_1 is minimal. For, otherwise, $\{b_D, c_D, u_D, x_1, y_1\}$ is a cut in G separating $B \cup u_D X x_1$ from D'. Let G_1 denote the $\{b_D, c_D, u_D, x_1, y_1\}$-bridge of G containing $B \cup u_D X x_1$. If $G_1 - c_D$ contains disjoint paths Q_1, Q_2 from b_D, u_D to x_1, y_1, respectively, then $L \cup (Q_1 \cup x_1 y_2 x_2) \cup Q_2$

is a TK_5 in G. Hence we may assume that such paths do not exist. Then by Corollary 2.3, $(G_1 - c_D, b_D, u_D, x_1, y_1)$ is planar. It follows from Corollary 2.9 that G contains TK_5.

We may assume that y_1 has a neighbor in $B - \{b_D, c_D\}$. For, if y_1 has no neighbor in $B - \{b_D, c_D\}$, then $\{b_D, c_D, u_D, x_1, y_2\}$ is a cut in G separating $B \cup u_D X x_1$ from D'. Let G_1 denote the $\{b_D, c_D, u_D, x_1, y_2\}$-bridge of G containing $B \cup u_D X x_1$. If $G_1 - c_D$ contains disjoint paths Q_1, Q_2 from b_D, u_D to y_2, x_1, respectively, then $L \cup (Q_1 \cup y_2 x_2) \cup (Q_2 \cup x_1 y_1)$ is a TK_5 in G. Hence we may assume that such paths do not exist. Then by Corollary 2.3, $(G_1 - c_D, b_D, u_D, y_2, x_1)$ is planar. So by Corollary 2.9, G contains TK_5.

We may further assume that D is the only block of $H - B$ that is 2-connected. For, suppose F is another block of $H - B$ that is 2-connected. Since $N(y_1) \subseteq B \cup D \cup \{x_1, x_2\}$ and $N(y_3) \subseteq D' \cup \{x_1, x_2\}$, $b_F \neq c_F$ and $\{b_F, c_F, u_F, v_F, y_2\}$ is a cut of G separating F from $B \cup D$. Now $G[F' + y_2]$ is $(5, \{b_F, c_F, u_F, v_F, y_2\})$-connected, and (F', b_F, u_F, c_F, v_F) is planar. Hence G contains TK_5 by Corollary 2.9.

In particular, this and Lemma 3.4 allow us to assume that all $(B \cup X)$-bridges of H not contained in D' are induced by edges between B and $u_D X x_1$.

Subcase 1.1. $N(y_2) - \{u, x_1, x_2\} \not\subseteq v_D C c_D$.

In $G[B + \{u, y_1\}]$ we find two independent paths Q_1, Q_2 from u to y_1, c_D, respectively.

Suppose y_2 has a neighbor in $D' - v_D C c_D$. Note that, because of P_1, y_1 has a neighbor on $b_D C v_D - \{b_D, v_D\}$. So by planarity and since D' is $(4, \{b_D, c_D, u_D, v_D\})$-connected, $G[D + \{y_1, y_2\}] - v_D C_D c_D$ contains a path Q from y_1 to y_2. Now $Q_1 \cup (Q_2 \cup v_D C c_D) \cup u X x_1 \cup u y_2 \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

Now assume that y_2 has a neighbor v in $u_D X x_1 - x_1$ and $v \neq u$. Then $v \in u_D X u - u$ by the minimality of $u X x_1$. Again, by planarity and since D is 2-connected and D' is $(4, \{b_D, c_D, u_D, v_D\})$-connected, $G[D + \{y_1, y_2\}] - b_D - v_D C_D c_D$ contains a path Q' from y_1 to u_D. Now $Q_1 \cup (Q_2 \cup v_D C c_D) \cup u X x_1 \cup u y_2 \cup (Q' \cup u_D X v \cup v y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

Subcase 1.2. $N(y_2) - \{u, x_1, x_2\} \subseteq v_D C c_D$.

Let v_1 be the neighbor of y_1 in P_1 and let v_2 be the neighbor of y_2 in $v_D C c_D$ with $v_2 C c_D$ maximal (so $|V(v_D C v_2)| \geq 3$).

Since D' is $(4, \{b_D, c_D, u_D, v_D\})$-connectced, D' has no 2-cut $\{s_1, s_2\}$ separating v_D from $\{b_D, c_D, u_D\}$, with $s_1 \in b_D C v_1$ and $s_2 \in v_2 C c_D$. Thus by planarity D' contains three disjoint paths Q_1, Q_2, Q_3 from v_1, v_2, v_D to b_D, c_D, u_D, respectively. If $G[B + \{y_1, u\}]$ has disjoint paths R_1, R_2 from y_1, u to c_D, b_D, respectively, then $u X x_1 \cup (u X u_D \cup Q_3) \cup u y_2 \cup (R_2 \cup Q_1 \cup v_1 y_1) \cup (R_1 \cup Q_2 \cup v_2 y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume R_1, R_2 do not exist. Then by Lemma 2.2, $(G[B + \{y_1, u\}], y_1, u, c_D, b_D)$ is 3-planar. Hence $G[B + \{y_1, u\}]$ contains disjoint paths L_1, L_2 from y_1, b_D to u, c_D, respectively. Then $u X x_1 \cup (u X u_D \cup Q_3) \cup u y_2 \cup L_1 \cup (y_1 v_1 \cup Q_1 \cup L_2 \cup Q_2 \cup v_2 y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

Case 2. $\{b_D, c_D\} \subseteq L$.

By symmetry, we may assume that $c_D \in P_2$, $v_D \in P_3$, and $c_D \in P_4$. Again, we view P_1 as a path in G, with $y_1 \in P_1$. Further, we may assume by symmetry that $u_D, b_D, v_D, P_1 \cap C, b_D$ occur on C in counterclockwise order.

Since $C_w \cap C = \emptyset$, we can modify L to L' by extending P_4 to u_D (possibly $b_D \in L'$), and modify L to L'' by extending P_2 to u_D (possibly $c_D \in L''$).

We may assume that H contains no path from x_2Xv_D to $B - \{b_D, c_D\}$ and internally disjoint from $B \cup D \cup X$. For, otherwise, H contains a path Q from v_D to b_D disjoint from $(D - v_D) \cup x_1Xu_D + c_D$. Now $Q \cup (y_1x_1 \cup x_1Xu_D) \cup L''$ is a TK_5 in G.

Therefore, $S := \{b_D, c_D, u_D, y_1, y_2, y_3\}$ is a cut in G separating $B \cup u_DXx_1$ from $D \cup v_DXx_2$. Let K denote the minimal union of S-bridges of G containing $B \cup u_DXx_1$, and let K' be obtained from K by identifying y_2 and y_3 as y and identifying u_D and c_D as u.

We may assume that (K', y_1, y, u, b_D) is 3-planar. For, otherwise, it follows from Lemma 2.2 that K' contains disjoint paths from y_1, y to u, b_D, respectively. Hence, K contains disjoint paths R_1, R_2 from y_1, y_i (for some $i \in \{2,3\}$) to $z \in \{u_D, c_D\}, b_D$, respectively, with $V(R_2) \cap \{u_D, c_D\} = \{z\}$. If $z = u_D$ then $R_1 \cup (R_2 \cup y_ix_2 \cup x_2Xv_D) \cup L''$ is a TK_5 in G; and if $z = c_D$ then $R_1 \cup (R_2 \cup y_ix_2 \cup x_2Xv_D) \cup L$ is a TK_5 in G.

Let K'' be obtained from K by identifying y_2 and y_3 as y. Suppose $K'' - b_D$ contains disjoint paths from y_1, y to c_D, u_D, respectively. Then $K - b_D$ contains disjoint paths R_1, R_2 from y_1, y_i (for some $i \in \{2,3\}$) to c_D, u_D, respectively. Then $R_1 \cup (R_2 \cup y_ix_2 \cup x_2Xv_D) \cup L'$ is a TK_5 in G.

Thus we may assume that $K'' - b_D$ does not contain disjoint paths from y_1, y to c_D, u_D, respectively. So by Lemma 2.2, $(K'' - b_D, y_1, y, c_D, u_D)$ is 3-planar. Note that $B - c_D$ is connected and disjoint from $u_DXx_1 \cup x_1y_1$. So the 3-planarity of (K', y_1, y, u, b_D) implies that $K'' - c_D$ has a cut vertex c separating $\{y_1, y\}$ from $\{b_D, u_D\}$. Since B is 2-connected, $\{b_D, c_D, c, u_D\}$ is a 4-cut in G, a contradiction. ∎

We can now summarize the results in this section as the following

Lemma 4.6 *If some block of $H - B$ is of type I then G contains TK_5.*

5 Blocks of type II

In this section we show, with the help of Lemma 4.6, that if $H - B$ has a block of type II then G contains TK_5. Let D be a block of $H - B$ of type II, and recall the notation D', b_D, u_D, v_D. Let $D'' := D - \{u_D, v_D\}$ which is connected. Since G is 5-connected and D is of type II, $|N(D'') \cap \{y_1, y_2, y_3\}| \geq 2$. An important step is to show that $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$.

Lemma 5.1 *If $H - B$ has a block of type II then G contains TK_5 or $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$.*

Proof. First, we may assume $K_4^- \not\subseteq G$, as otherwsie G contains TK_5 by Theorem 1.1. Since G is 5-connected, $|N(D'') \cap \{y_1, y_2, y_3\}| \geq 2$.

(1) We may assume that D'' or $G[D'' + b_D]$ is 2-connected.

Since G is 5-connected, $|N(D'') \cap \{y_1, y_2, y_3\}| \geq 2$. So $|D''| \geq 2$ by Lemma 3.1. In fact, $|D''| \geq 3$ as D is 2-connetced and $K_4^- \not\subseteq G$. Let $C_1, \ldots, C_k$ denote the endblocks of D''.

We may assume $k \geq 2$, as otherwise D'' is 2-connected and (1) holds. Let $v_i \in V(C_i)$ such that v_i is a cut vertex of D''.

Suppose there is some endblock of D'', say C_k, such that $u_D, v_D \in N(C_k - v_k)$. Let X' be obtained from X by replacing u_DXv_D with a path in $G[C_k + \{u_D, v_D\}] - v_k$ between u_D and v_D. If $|N(C_i) \cap \{y_1, y_2, y_3\}| \geq 2$ for some $1 \leq i \leq k-1$, then by Lemma 3.1, C_i

is 2-connected; so by the choices of X, we have $|N(B) \cap \{y_1, y_2, y_3\}| \geq 2$. Thus we may assume that for $1 \leq i \leq k-1$, $|N(C_i) \cap \{y_1, y_2, y_3\}| \leq 1$. Then, since G is 5-connected, $\{b_D, u_D, v_D\} \subseteq N(C_i - v_i)$ for $1 \leq i \leq k-1$. This shows that $H - B - C_k$ has a path X'' from x_1 to x_2 (by replacing $u_D X v_D$ with a path in $G[(C_1 - v_1) + \{u_D, v_D\}]$ from u_D to v_D). Lemma 3.2 and the choice of X imply that $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(C_k) \cap \{y_1, y_2, y_3\}|$. Hence, we may assume $|N(C_k) \cap \{y_1, y_2, y_3\}| \leq 1$, which in turn forces $b_D \in N(C_k - v_k)$ as G is 5-connected. Thus, $G[D'' + b_D]$ is 2-connected.

Hence we may assume that $\{u_D, v_D\} \not\subseteq N(C_i - v_i)$ for $1 \leq i \leq k$. If $b_D \in N(C_i - v_i)$ for $1 \leq i \leq k$ then $G[D'' + b_D]$ is 2-connected. So we may assume that for some i, $b_D \notin N(C_i - v_i)$. Then $y_1, y_2, y_3 \in N(C_i - v_i)$ as G is 5-connected. Note that X may be revised so that $X \cap C_i = \emptyset$. Hence by the choice of X and Lemma ref1comp, $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(C_i - v_i) \cap \{y_1, y_2, y_3\}| = 3$.

(2) $D - u_D$ and $D - v_D$ are 2-connected.

Now assume $D - u_D$ is not 2-connected. Since D is 2-connected, $D - u_D$ is connected. Let C be an endblock of $D - u_D$ and let v be the cut vertex of $D - u_D$ such that $v_D \notin C - v$. Since G is 5-connected, $|N(C - v) \cap \{y_1, y_2, y_3\}| \geq 2$. So C is 2-connected by Lemma 3.1.

Since D'' is connected, $v_D \neq v$; so $D - C$ contains a path P from u_D to v_D. By replacing $u_D X v_D$ with P we obtain from X a path X' in H between x_1 and x_2 such that C is contained in a 2-connected block of $H - X'$. Hence by Lemma 3.2 and the choice of X, $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(C) \cap \{y_1, y_2, y_3\}| \geq 2$.

(3) We may assume $u_D \neq x_1$, $v_D = x_2$, and H contains no path from x_2 to B internally disjoint from $B \cup X \cup D'$.

If $u_D = x_1$ and $v_D = x_2$ then, since G is 5-connected, $|N(B - b_D) \cap \{y_1, y_2, y_3\}| \geq 2$. So we may assume by symmetry that $x_1 \neq u_D$. Then H has a path from x_1 to B internally disjoint from $B \cup D' \cup X$.

Suppose H also has a path from x_2 to B internally disjoint from $B \cup D' \cup X$. Then H contains a path X' between x_1 and x_2 and disjoint from $D - v_D$. So by (2) and Lemma 3.2 and by the choice of X, $|N(B) \cap \{y_1, y_2, y_3\}| \geq |N(D - v_D) \cap \{y_1, y_2, y_3\}| \geq 2$.

So we may assume $x_2 = v_D$ and H contains no path from x_2 to B internally disjoint from $B \cup X \cup D'$.

Since D'' is connected, we have

(4) for any $y_i, y_j \in N(D'')$, $G[D'' + \{x_2, y_i, y_j\}]$ contains three independent paths from some vertex $u \in D''$ to x_2, y_i, y_j, respectively.

By (3), there are at most two 2-connected blocks in $H - B$. So we have two cases.

Case 1. D is the unique 2-connected block in $H - B$.

Subcase 1.1. $N(y_i) \subseteq D' + \{x_1, x_2\}$ for some $i \in \{1, 2, 3\}$, say $i = 1$.

Then $S := \{b_D, u_D, x_1, y_2, y_3\}$ is a cut in G. Let $G_1 := G - (D'' + \{x_2, y_1\}$.

Suppose $y_2, y_3 \in N(D'')$. Then by (4), $G[D'' + \{x_2, y_2, y_3\}]$ has independent paths from some $u \in V(D'')$ to x_2, y_2, y_3, respectively. So by Lemma 2.4 there exist four independent paths P_1, P_2, P_3, P_4 in $G[D' + \{y_2, y_3\}]$ from u to $x_2, y_2, y_3, s \in \{b_D, u_D\}$, respectively, such that $|V(P_i) \cap \{u_D, v_D, x_2, y_2, y_3\}| \le 1$ for $1 \le i \le 4$. Let $t \in \{b_D, u_D\} - \{s\}$. If $G_1 - t$ has disjoint paths Q_1, Q_2 from x_1, y_2 to s, y_3, respectively, then $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup Q_1) \cup Q_2 \cup K$ is a TK_5 in G with branch vertice u, x_1, x_2, y_2, y_3. So we may assume that such paths do not exist. Then by Corollary 2.3, $(G_1 - t, x_1, y_2, s, y_3)$ is planar; and so G contains TK_5 by Corollary 2.9.

So we may assume that $y_3 \notin N(D'')$. Then $\{b_D, u_D, x_2, y_1, y_2\}$ is a cut in G separating D'' from $B \cup u_D X x_1$.

We may assume that $G[D' + y_2]$ contains disjoint paths Q_1, Q_2 from u_D, b_D to x_2, y_2, respectively; for, otherwise, it follows from Corollary 2.3 that $(G[D' + y_2], u_D, b_D, x_2, y_2)$ is planar; and so G contains TK_5 by Corollary 2.9. Similarly, we may assume that $G[D' + y_2]$ contains disjoint paths Q_1', Q_2' from u_D, b_D to y_2, x_2, respectively.

Suppose $|N(y_3) \cap V(B)| \ge 2$. We may assume $y_2 \notin N(B)$, or else the assertion of the lemma holds. Hence y_2 has a neighbor $u \in u_D X x_1 - \{u_D, x_1\}$ (otherwise $\{x_1, b_D, u_D, y_3\}$ would be a 4-cut in G). Now $G[B + \{u, y_3\}]$ contains independent paths R_1, R_2 from y_3 to u, b_D, respectively, and $uy_2 \cup R_1 \cup uXx_1 \cup (uXu_D \cup Q_1) \cup (R_2 \cup Q_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3.

Thus we may assume that there exist distinct $v, v' \in N(y_3) \cap V(u_D X x_1 - x_1)$, and assume that x_1, v, v', u_D occur on X in order. We may assume that $y_2 \notin N(B - b_D)$; for otherwise $G[B + \{y_2, v\}]$ has independent paths R_1, R_2 from v to y_2, b_D, respectively, and $vy_3 \cup R_1 \cup vXx_1 \cup (R_2 \cup Q_2') \cup (y_3 v' \cup v' X u_D \cup Q_1') \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_2, y_3. So y_2 has a neighbor $u \in u_D X x_1 - \{u_D, x_1\}$.

Suppose $u \in x_1 X v - v$. Let R be a path in $G[B + u]$ from u to b_D. Then $uy_2 \cup (uXv \cup vy_3) \cup uXx_1 \cup (R \cup Q_2') \cup (y_3 v' \cup v' X u_D \cup Q_1') \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3.

Now assume $u \in vXv' - \{v, v'\}$. Then in $G[B + v]$ we find a path R from v to b_D. So $vy_3 \cup (vXu \cup uy_2) \cup vXx_1 \cup (R \cup Q_2') \cup (y_3 v' \cup v' X u_D \cup Q_1') \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_2, y_3.

Therefore, we may assume $u \in v' X u_D - \{u_D, v'\}$. If $G[B + \{u, v, x_1\}]$ has disjoint paths R_1, R_2 from x_1, v to u, b_D, respectively, then $uy_2 \cup (uXv' \cup v'y_3) \cup R_1 \cup (uXu_D \cup Q_1) \cup (y_3 v \cup R_2 \cup Q_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. So we may assume R_1, R_2 do not exist. Then by Lemma 2.3, $(G[B + \{u, v, x_1\}], v, x_1, b_D, u)$ is 3-planar. Thus, $G[B + \{u, v, x_1\}]$ contains disjoint paths L_1, L_2 from x_1, v to b_D, u, respectively. Hence $X' := Q_2' \cup L_1$ is a path in H between x_1 and x_2, and $uXv \cup L_2$ is a cycle in $H - X'$ and contains neighbors of both y_1 and y_2. It now follows from Lemma 3.2 and the choice of X that $|N(B) \cap \{y_1, y_2, y_3\}| \ge 2$.

Subcase 1.2. $N(y_i) \not\subseteq D' + \{x_1, x_2\}$ for all $i = 1, 2, 3$.

We may assume $|N(B) \cap \{y_1, y_2, y_3\}| \le 1$, as otherwise the assertion of the lemma holds. So by symmetry let $y_1, y_2 \notin N(B)$; hence $y_1, y_2 \in N(x_1 X u_D - \{x_1, u_D\})$. Further, if $y_3 \in N(x_1 X u_D - \{x_1, u_D\})$ then we may assume that the neighbor of $\{y_1, y_2, y_3\}$ on $x_1 X u_D$ closest to u_D is a neighbor of y_3, denoted by v_3. Let $v_i \in N(y_i) \cap V(x_1 X u_D - \{x_1, u_D\})$, $i = 1, 2$. We may assume that x_1, v_1, v_2, u_D occur on X in order. Note that each v_i has at least two neighbors in B. Let X_1 denote a path in $G[B + x_1]$ from x_1 to b_D.

We may assume $y_3 \in N(D'')$. For, suppose $y_3 \notin N(D'')$. Then $\{b_D, u_D, x_2, y_1, y_2\}$ is a 5-cut in G separating D' from $B \cup u_D X x_1$. In $G[D' + \{y_1, y_2\}]$ we apply Menger's theorem to find five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex $u \in D''$ to y_1, y_2, x_2, b_D, u_D, respectively. Now $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup X_1) \cup (y_1v_1 \cup v_1Xv_2 \cup v_2y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

Next we show that we may also assume $y_1, y_2 \in N(D'')$. For suppose, by symmetry, that $y_1 \notin N(D'')$. Then $y_2, y_3 \in N(D'')$ as G is 5-connected, and $\{b_D, u_D, x_2, y_2, y_3\}$ is a cut in G separating D' from $B \cup u_DXx_1$. By Menger's theorem, $G[D' + \{y_2, y_3\}]$ has five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex $u \in D''$ to y_2, y_3, x_2, b_D, u_D. If v_3 is defined then $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup X_1) \cup (y_2v_2 \cup v_2Xv_3 \cup v_3y_3) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. So assume that v_3 is not defined. Thus $y_3 \in N(B)$ (otherwise $\{b_D, u_D, x_2, y_2\}$ would be a 4-cut in G), and $G[B + \{v_2, y_3\}]$ contains a path R from v_2 to y_3. If $G[D' + \{y_2, y_3\}] - b_D$ has disjoint paths Q_1, Q_2 from u_D, y_2 to x_2, y_3, respectively, then $v_2y_2 \cup R \cup v_2Xx_1 \cup (v_2Xu_D \cup Q_1) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices v_2, x_1, x_2, y_2, y_3. So assume that Q_1, Q_2 do not exist. Then by Corollary 2.3, $(G[D' + \{y_2, y_3\}] - b_D, u_D, y_2, x_2, y_3)$ is planar; so G contains TK_5 by Corollary 2.9.

Hence, by (4), $G[D'' + \{x_2, y_1, y_2\}]$ has three inpdependent paths from some vertex $u \in D''$ to y_1, x_2, y_2, respectively. Let $S := \{b_D, u_D, x_2, y_1, y_2, y_3\}$. By Lemma 2.4, $G[D'' + S]$ has five independent paths P_1, P_2, P_3, P_4, P_5 from u to S such that $|V(P_i \cap P_j) = \{u\}$ for $1 \le i \ne j \le 5$, $|V(P_i) \cap S| = 1$ for $1 \le i \le 5$, $y_1 \in P_1$, $y_2 \in P_2$, and $x_2 \in P_3$. We may assume that P_4 ends in $\{b_D, u_D\}$. We may further assume that P_4 ends at u_D; or else, $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup X_1) \cup (y_1v_1 \cup v_1Xv_2 \cup v_2y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

We may also assume that v_3 is not defined. For, othewise, $v_3 \in u_DXv' - \{u_D, v'\}$ by the definition of v_3. Let X_1' be a path in $G[B + \{v_3, x_1\}]$ from x_1 to v_3. Then $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup u_DXv_3 \cup X_1') \cup (y_1v_1 \cup v_1Xv_2 \cup v_2y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

So $y_3 \in N(B - b_D)$ since $N(y_3) \not\subseteq D' + \{x_1, x_2\}$. Let D^* be obtained from $G[D' + \{y_1, y_2, y_3\}]$ by identifying u_D and b_D as w.

Suppose $D^* - y_3$ contains disjoint paths Q_1, Q_2 from y_1, w to y_2, x_2, respectively. We view Q_2 as a path in G; so $u_D \in Q_2$ or $b_D \in Q_2$. If $b_D \in Q_2$ then let Q be a path in $G[B + v_1]$ from v_1 to b_D; now $v_1y_1 \cup (v_1Xv_2 \cup v_2y_2) \cup v_1Xx_1 \cup (Q \cup Q_2) \cup Q_1 \cup K$ is a TK_5 in G with branch vertices v_1, x_1, x_2, y_1, y_2. So we may assume $u_D \in Q_2$. Let R be a path in $G[B + \{v_2, x_1\}]$ from v_2 to x_1. Then $v_2y_2 \cup (v_2Xv_1 \cup v_1y_1) \cup R \cup (v_2Xu_D \cup Q_2) \cup Q_1 \cup K$ is a TK_5 in G with branch vertices v_2, x_1, x_2, y_1, y_2.

Therefore, we may assume that such Q_1, Q_2 do not exist in $D^* - y_3$. So by Lemma 2.2, $(D^* - y_3, y_1, w, y_2, x_2)$ is 3-planar. Since D is 2-connected, $D^* - \{y_1, y_2, y_3\}$ is 2-connected. Thus, $D^* - y_1$ contains disjoint paths R_1, R_2 from y_2, x_2 to y_3, w, respectively, or $D^* - y_2$ contains disjoint paths R_1, R_2 from y_1, x_2 to y_3, w, respectively. We may assume the latter. We view R_2 as a path in G; so $b_D \in R_2$ or $u_D \in R_2$. Note that $G[B + \{v_1, y_3\}]$ contains independent paths L_1, L_2 from v_1 to y_3, b_D, respectively. If $b_D \in R_2$, then $v_1y_1 \cup L_1 \cup v_1Xx_1 \cup (L_2 \cup R_2) \cup R_1 \cup K$ is a TK_5 in G with branch vertices v_1, x_1, x_2, y_1, y_3. So we may assume $u_D \in R_2$. Then $v_1y_1 \cup L_1 \cup v_1Xx_1 \cup (v_1Xu_D \cup R_2) \cup R_1 \cup K$ is a TK_5 in G with branch vertices v_1, x_1, x_2, y_1, y_3.

Case 2. $H - B$ has a 2-connected block D_1 such that $D_1 \ne D$.

Then by (3), $u_{D_1} = x_1$, and $H - B$ has exactly two 2-connected blocks, D_1 and $D_2 := D$. Let $b_i := b_{D_i}$ for $i = 1, 2$, and $v_1 := v_{D_1}$ and $u_2 := u_{D_2}$.

Subcase 2.1. $y_1, y_2, y_3 \in N(D_i'')$ for $i = 1, 2$.

We may assume $|N(B) \cap \{y_1, y_2, y_3\}| \leq 1$, or else we have the assertion of this lemma. So, since G is 5-connected, $b_1 \neq b_2$ and there is an edge between $v_1 X u_2$ and $B - \{b_1, b_2\}$.

We claim that there exist $\{i, j\} \subseteq \{1, 2, 3\}$ such that $G[D_1' + \{y_i, y_j\}]$ contains disjoint paths Q_1, Q_2 from x_1, y_i to v_1, y_j, respectively. This is clear if there exist y_i and y_j both with neighbors on $v_1 X x_1$, for X is induced, D_1 is 2-connected, and $D_1' - v_1 X x_1$ is connected. Thus we may assume (by pigeonhole principle) that there exist y_i and y_j both with neighbors in $D_1 - v_1 X x_1$. So, since $H - X$ is connected, $G[D_1' + \{y_i, y_j\}] - v_1 X x_1$ has a path between y_i and y_j.

Without loss of generality, we may assume that $\{i, j\} = \{1, 2\}$. By (4), $G[D_2'' + \{x_2, y_1, y_2\}]$ has independent paths from some vertex $u \in D_2''$ to y_1, y_2, x_2, respectively. So $G[D_2' + \{y_1, y_2, y_3\}]$ contains five independent paths P_1, P_2, P_3, P_4, P_5 from u to $S := \{b_2, u_2, x_2, y_1, y_2, y_3\}$ such that $V(P_i \cap P_j) = \{u\}$ for $1 \leq i \neq j \leq 5$, $|V(P_i) \cap S| = 1$ for $1 \leq i \leq 5$, $y_1 \in P_1$, $y_2 \in P_2$, and $x_2 \in P_3$. We may assume that P_4 ends in $\{b_2, u_2\}$.

If P_4 ends at u_2 then $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup u_2 X v_1 \cup Q_1) \cup Q_2 \cup K$ is a TK_5 with branch vertices u, x_1, x_2, y_1, y_2. So assume that P_4 ends at b_2. Since there is an edge between $v_1 X u_2$ and $B - \{b_1, b_2\}$ and because $b_1 \neq b_2$, we see that $G[B \cup v_1 X u_2] - b_1$ contains a path Q from b_2 to v_1. Hence $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup Q \cup Q_1) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

So by symmetry, we may assume that $y_1, y_2 \in N(D_1'')$, $y_3 \notin N(D_1'')$, and $y_1 \in N(D_2'')$.

Subcase 2.2. $y_3 \notin N(D_1'')$ and $y_2 \in N(D_2'')$.

Then by (4), $G[D_2'' + \{x_2, y_1, y_2\}]$ has three independent paths from some $u \in D''$ to y_1, y_2, x_2, respectively. So by Lemma 2.4, $G[D_2' + \{y_1, y_2, y_3\}]$ contains five independent paths P_1, P_2, P_3, P_4, P_5 from u to $S := \{b_2, u_2, x_2, y_1, y_2, y_3\}$ such that $V(P_i \cap P_j) = \{u\}$ for $1 \leq i \neq j \leq 5$, $|V(P_i) \cap S| = 1$ for $1 \leq i \leq 5$, $y_1 \in P_1$, $y_2 \in P_2$, and $x_2 \in P_3$. We may assume that P_4 ends in $\{b_2, u_2\}$.

First, assume that P_4 ends at u_2. If $G[D_1' + \{y_1, y_2\}] - b_1$ has disjoint paths Q_1, Q_2 from v_1, y_2 to x_1, y_1, respectively, then $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup u_2 X v_1 \cup Q_1) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So assume that Q_1, Q_2 do not exist. Then by Corollary 2.3, $(G[D_1' + \{y_1, y_2\}] - b_1, v_1, y_2, x_1, y_1)$ is planar. So G contains TK_5 by Corollary 2.9.

Now assume P_4 ends at b_2. Let Q be a path in B from b_2 to b_1. If $G[D_1' + \{y_1, y_2\}] - v_1$ has disjoint paths Q_1, Q_2 from b_1, y_2 to x_1, y_1, respectively, then $P_1 \cup P_2 \cup P_3 \cup (P_4 \cup Q \cup Q_1) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So assume that Q_1, Q_2 do not exist. Then by Corollary 2.3, $(G[D_1' + \{y_1, y_2\}] - v_1, b_1, y_2, x_1, y_1)$ is planar. So G contains TK_5 by Corollary 2.9.

Subcase 2.3. $y_3 \notin N(D_1'')$, $y_2 \notin N(D_2'')$, and $y_2 \in N(B \cup u_2 X v_1)$.

In $G[D_1 + \{y_1, y_2\}]$ we use Menger's theorem to find five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some $u \in V(D_1'')$ to y_1, y_2, x_1, b_1, v_1, respectively. Since $y_2 \in N(B \cup u_2 X v_1)$, $G[B \cup u_2 X v_1 + y_2]$ has disjoint paths R_1, R_2 from $s \in \{b_1, v_1\}, y_2$ to $\{b_2, u_2\}$.

We may assume that $G[D_2' + y_1]$ contains disjoint paths L_1, L_2 from b_2, u_2 to x_2, y_1, respectively; as otherwise by Corollary 2.3, $(G[D_2' + y_1], b_2, u_2, x_2, y_1)$ is planar, and so G contains TK_5 by Corollary 2.9. Similarly, we may assume that $G[D_2' + y_1]$ contains disjoint paths L_1', L_2' from b_2, u_2 to y_1, x_2, respectively.

Let $s \in Q_i$ where $i \in \{4, 5\}$. If $b_2 \in R_1$, then $Q_1 \cup Q_2 \cup Q_3 \cup (Q_i \cup R_1 \cup L_1) \cup (R_2 \cup L_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So assume $u_2 \in R_1$. Then $Q_1 \cup Q_2 \cup Q_3 \cup$

$(Q_i \cup R_1 \cup L_2') \cup (R_2 \cup L_1') \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

Subcase 2.4. $y_3 \notin N(D_1'')$, $y_2 \notin N(D_2'')$ and $y_2 \notin N(B \cup u_2Xv_1)$.

Let $v \in N(x_1) \cap V(D_1'')$ and $G' := G[D_1' + \{y_1, y_2\}]$. By Menger's theorem, $G' - x_1$ has four independent paths Q_1, Q_2, Q_3, Q_4 from v to y_1, y_2, b_1, v_1, respectively. We amy assume that Q_i, $1 \le i \le 4$, are induced in G', and let $L = \bigcup_{i=1}^5 Q_i$, where $Q_5 = vx_1$.

Note that $|N(y_2) \cap V(D_1'')| \ge 3$. So G' has an L-bridge, say J, containing an edge y_2u such that $u \notin Q_2 + x_1$. We now show that L, J may be choosen so that J has an attachment in $(Q_1 \cup Q_3 \cup Q_4) - v$. For, otherwise, all attachments of J are contained in $Q_2 + x_1$. Since G is 5-connected, J has an attachment on Q_2, say z; and we choose z so that zQ_2v is minimal. Again since G is 5-connected, there is a path in $G' - x_1$ from $y_2Q_2z - \{y_2, z\}$ to $(Q_1 \cup Q_3 \cup Q_4) - v$. Now letting Q_2' be obtained from Q_2 by replacing y_2Q_2z with a path in J from y_2 to z internally disjoint from $Q_2 + x_1$, we see that for Q_1, Q_2', Q_3, Q_4, the corresponding J, L satisfy the desired properties.

Therefore, J contains a path Y from y_2 to $y \in V(Q_1 \cup Q_3 \cup Q_4 - v)$ internally disjoint from L. Let R be a path in B between b_1 and b_2. As in Subcase 2.3, we may assume that $G[D_2' + y_1]$ contains disjoint paths L_1, L_2 from b_2, u_2 to x_2, y_1, respectively, as well as disjoint paths L_1', L_2' from b_2, u_2 to y_1, x_2, respectively.

If $y \in Q_1 - v$ then $vx_1 \cup Q_2 \cup (Q_3 \cup R \cup L_1) \cup (Q_4 \cup v_1Xu_2 \cup L_2) \cup (Y \cup yQ_1y_1) \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_1, y_2. If $y \in Q_3 - v$ then $vx_1 \cup Q_1 \cup Q_2 \cup (Q_4 \cup v_1Xu_2 \cup L_2') \cup (Y \cup yQ_3b_1 \cup R \cup L_1') \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_1, y_2. So $y \in Q_4 - v$. Then $vx_1 \cup Q_1 \cup Q_2 \cup (Q_3 \cup R \cup L_1) \cup (Y \cup yQ_4v_1 \cup v_1Xu_2 \cup L_2) \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_1, y_2. ∎

Lemma 5.2 *If $H - B$ has a 2-connected block then G contains TK_5.*

Proof. By Lemma 4.6, we may assume that no 2-connected block of H is of type I. For any 2-connected block D of $H - B$, recall the notation D'', D', b_D, u_D, v_D. Since G is 5-connected, $|N(D'') \cap \{y_1, y_2, y_3\}| \ge 2$.

Case 1. $|N(D'') \cap \{y_1, y_2, y_3\}| = 2$ for any 2-connected block D of $H - B$.

Let D be a 2-connected block of $H - B$, Without loss of generality, let $y_1, y_2 \in N(D'')$ and $y_3 \notin N(D'')$. By Menger's theorem, we find independent paths P_1, P_2, P_3, P_4, P_5 in $G[D' + \{y_1, y_2\}]$ from some vertex $u \in D''$ to y_1, y_2, u_D, v_D, b_D, respectively.

If $y_1, y_2 \in N(B)$ then in $G[B + \{y_1, y_2\}]$ we find a path Q from y_1 to y_2, and $P_1 \cup P_2 \cup (P_3 \cup u_DXx_1) \cup (P_4 \cup v_DXx_2) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that $y_1 \notin N(B)$; hence by Lemma 5.1 we may assume $y_2, y_3 \in N(B)$.

Subcase 1.1. $N(y_1) \not\subseteq D + \{x_1, x_2\}$.

Then $G - \{y_2, y_3\}$ contains a path P from y_1 to some vertex $u \in (B \cup X) - (D' + \{x_1, x_2\})$ internally disjoint from $B \cup D' \cup X$. If $u \in B$ then $G[B \cup P + y_2]$ has a path Q between y_1 and y_2, and $P_1 \cup P_2 \cup (P_3 \cup u_DXx_1) \cup (P_4 \cup v_DXx_2) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

So we may assume that $u \notin B$ for any choice of P. Hence, since $H - X$ is connected, all neighbors of y_1 outside $D + \{x_1, x_2\}$ are on X; in particular, $u \in (u_DXx_1 - \{u_D, x_1\}) \cup (v_DXx_2 - \{v_D, x_2\})$ and $V(P) = \{y_1, u\}$. By symmetry we may assume that $u \in u_DXx_1 - \{u_D, x_1\}$. Since X is induced and $H - X$ is connected and by Lemma 3.1, H contains a path from u

to B and internally disjoint from $B \cup X \cup D$, which can be extended through $G[B + y_2]$ to a path R from u to y_2. If $G[D' + \{y_1, y_2\}] - b_D$ has disjoint paths R_1, R_2 from y_1, u_D to y_2, v_D, resepctively, then $uy_1 \cup R \cup uXx_1 \cup (uXu_D \cup R_2 \cup v_DXx_2) \cup R_1 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. Thus we may assume that such R_1, R_2 do not exist. So by Corollary 2.3, $(G[D' + \{y_1, y_2\}] - b_D, y_1, u_D, y_2, v_D)$ is planar. Now G contains TK_5 by Corollary 2.9.

Subcase 1.2. $N(y_1) \subseteq D + \{x_1, x_2\}$, and $N(y_2) \subseteq D' + \{x_1, x_2\}$.

Then $N(y_2) \cap V(B) = \{b_D\}$, and $\{b_D, u_D, v_D, x_1, x_2\}$ is a cut in G separating $B + y_3$ from $D' + \{y_1, y_2\}$. So $x_1 \neq u_D$ and $x_2 \neq v_D$, as G is 5-connected. Therefore, $H - D$ contains a path X' from x_1 to x_2. Note that D is 2-connected; so it is contained in a 2-connected block of $H - X'$. Also note that y_1 and y_2 each have at least two neighbors in D. So it follows from Lemma 3.2 and the choice of X that y_2, y_3 should each have at least two neighbors in B, contradicting the assumption that $N(y_2) \subseteq D' + \{x_1, x_2\}$.

Subcase 1.3. $N(y_1) \subseteq D + \{x_1, x_2\}$, and $y_2 \in N(F'')$ for some 2-connected block F of $H - B$.

Let $v \in N(y_2) \cap V(F'')$. Without loss of generality, assume that $x_1, u_F, v_F, u_D, v_D, x_2$ occur on X in order. Since $N(y_1) \subseteq D + \{x_1, x_2\}$, $y_1 \notin N(F'')$; and since G is 5-connected, $y_3 \in N(F'')$. Let Q be a path in $G[B + y_3]$ from y_3 to b_D. If $G[F' + \{y_2, y_3\}] - b_F$ contains disjoint paths Q_1, Q_2 from u_F, y_2 to v_F, y_3, respectively, then $P_2 \cup (P_5 \cup Q) \cup (P_3 \cup u_DXv_F \cup Q_1 \cup u_FXx_1) \cup (P_4 \cup v_DXx_2) \cup Q_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3. So we may assume that Q_1, Q_2 do not exist. Then by Corollary 2.3, $G[F' + \{y_2, y_3\}] - b_F, u_F, y_2, v_F, y_3)$ is planar. Hence G contains TK_5 by Corollary 2.9.

Subcase 1.4. $N(y_1) \subseteq D + \{x_1, x_2\}$, $N(y_2) \not\subseteq D' + \{x_1, x_2\}$, and $y_2 \notin N(F'')$ for any 2-connected block F of $H - B$ other than D.

Therefore, since G is 5-connected, D is the unique 2-connected block of $H - B$. So let $v \in N(y_2)$ such that $v \in (B - b_D) \cup (X - (u_DXv_D + \{x_1, x_2\})$. By symmetry, we may assume that $v \in (B - b_D) \cup (x_1Xu_D - \{x_1, u_D\})$.

We may further assume that $v \in B - b_D$. For, otherwise, $N(y_2) \cap V(B) = \{b_D\}$. Hence by Lemma 3.1, $y_3 \in N(B - b_D)$. Thus, $G[B + \{v, y_3\}]$ contains independent paths R_1, R_2 from b_D to y_3, v, respectively. Now $y_2b_D \cup R_2 \cup vy_2 \cup (x_1y_2 \cup x_1Xv \cup x_1y_3 \cup R_1) \cup (P_2 \cup P_5 \cup P_1 \cup y_1x_1 \cup P_3 \cup u_DXv)$ is a TK_5 in G with branch vertices b_D, u, v, x_1, y_2.

We may assume that $G[D' + y_2]$ contains disjoint paths Q_1, Q_2 from b_D, v_D to y_2, u_D, respectively; for otherwise by Corollary 2.3, $(G[D' + y_2], b_D, v_D, y_2, u_D)$ is planar, and so G contains TK_5 by Corollary 2.9. Similarly, we may assume that $G[D' + y_2]$ contains disjoint paths Q_1', Q_2' from b_D, v_D to u_D, y_2, respectively, as well as disjoint paths Q_1'', Q_2'' from b_D, u_D to v_D, y_2, respectively.

Suppose y_3 has at least two neighbors in B. Then $G[B + \{v, y_3\}]$ contains independent paths R_1, R_2 from y_3 to v, b_D, respectively. Then $P_2 \cup (P_5 \cup R_2) \cup (P_3 \cup u_DXx_1) \cup (P_4 \cup v_DXx_2) \cup (R_1 \cup vy_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_2, y_3.

Thus we may assume that y_3 has only one neighbor in B. Therefore y_3 must have at least two neighbors in $(u_DXx_1 - x_1) \cup (v_DXx_2 - x_2)$.

First, assume that y_3 has two neighbors $w_1, w_2 \in v_DXx_2 - x_2$, with $w_1 \in x_2Xw_2$. Since $v \in B - b_D$, $G[B + \{w_1, y_2\}]$ has independent paths R_1, R_2 from w_1 to b_D, y_2, respectively. So

$w_1Xx_2 \cup (R_1 \cup Q_1' \cup u_DXx_1) \cup R_2 \cup w_1y_3 \cup (y_3w_2 \cup w_2Xv_D \cup Q_2') \cup K$ is a TK_5 in G with branch vertices w_1, x_1, x_2, y_2, y_3.

Next assume that y_3 has exactly one neighbor $w_1 \in v_DXx_2 - x_2$. Then y_3 also has a neighbor $w_2 \in u_DXx_1 - x_1$. Clearly, $x_1, x_2 \in N(B)$; so $G[B + \{x_1, x_2\}]$ contains a path X' between x_1 and x_2. We claim that $|N(y_2) \cap V(B)| \geq 2$; otherwise, we have a contradiction to the choice of X and Lemma 3.2 because D is in a 2-connected block of $H - X'$, $y_1, y_2 \in N(D'')$, and $|N(y_1) \cap V(D)| \geq 3$. Thus y_2 has a neighbor $w \in B$ such that $x_1 \in N(B - w)$. Suppose $w_1 = v_D$. In $G[D + y_2]$ we find independent paths R_1, R_2 from w_1 to u_D, y_2, respectively, and let R be a path in $G[B+\{y_2, y_3\}]$ from y_2 to y_3. Now $w_1y_3 \cup R_2 \cup w_1Xx_2 \cup (R_1 \cup u_DXx_1) \cup R \cup K$ is a TK_5 in G with branch vertices w_1, x_1, x_2, y_2, y_3. So we may assume that $w_1 \neq v_D$. In $G[D' + \{y_1, y_2\}] - \{b_D, u_D\}$ we find a path Q from v_D to y_2 through y_1, which exists because D is 2-connected and $N(y_1) \subseteq D' + \{x_1, x_2\}$. In $G[B \cup u_DXx_1 + \{w, w_1\}]$ we find independent paths R_1, R_2 from w_1 to x_1, w, respectively. Then $R_1 \cup w_1Xx_2 \cup (R_2 \cup wy_2) \cup w_1Xv_D \cup Q \cup K$ is a TK_5 in G with branch vertices w_1, x_1, x_2, y_1, y_2.

Thus we may assume that y_3 has at least two neighbors in $u_DXx_1 - x_1$. In particular, let $w \in N(y_3) \cap V(u_DXx_1 - \{u_D, x_1\})$. If $G[B + \{w, y_2, y_3\}]$ contains disjoint paths R_1, R_2 from w, b_D to y_2, y_3, respectively, then $wy_3 \cup R_1 \cup wXx_1 \cup (wXu_D \cup Q_2 \cup v_DXx_2) \cup (Q_1 \cup R_2) \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_2, y_3. So assume that R_1, R_2 do not exist. Then $G[B + \{w, y_2, y_3\}]$ contains disjoint paths R_1', R_2' from w, y_2 to b_D, y_3, respectively. So $wy_3 \cup (R_1' \cup Q_1) \cup wXx_1 \cup (wXu_D \cup Q_2 \cup v_DXx_2) \cup R_2' \cup K$ is a TK_5 in G with branch vertices w, x_1, x_2, y_2, y_3.

Case 2. There exists a 2-connected block D of $H - B$ such that $\{y_1, y_2, y_3\} \subseteq N(D'')$.

By Lemma 5.1, we may assume that $y_1, y_2 \in N(B)$.

We may further assume that $G[H+y_3]$ contains no path from y_3 to B internally disjoint from $B \cup X \cup D'$. For, let P be such a path in H. Then, for any $\{i, j\} \subseteq \{1, 2, 3\}$, $G[B \cup P + \{y_i, y_j\}]$ contains a path Q_{ij} between y_i and y_j. Note that D contains independent paths from some $u \in V(D'')$ to u_D, v_D, respectively. So by Lemma 2.4, $G[D' + \{y_1, y_2, y_3\}]$ has five independent paths P_1, P_2, P_3, P_4, P_5 from u to $S := \{b_D, u_D, v_D, y_1, y_2, y_3\}$ such that $V(P_i \cap P_j) = \{u\}$ for $1 \leq i \neq j \leq 5$, $|V(P_i) \cap S| = 1$ for $1 \leq i \leq 5$, $u_D \in P_1$, and $v_D \in P_2$. Without loss of generality, we may assume that P_3 ends at y_i and P_4 ends at y_j. Now $(P_1 \cup u_DXx_1) \cup (P_2 \cup v_DXx_2) \cup P_3 \cup P_4 \cup Q_{ij} \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_i, y_j.

In particular, $N(y_3) \subseteq D \cup X$.

Subcase 2.1. $D - u_D$ is not 2-connected or $D - v_D$ is not 2-connected.

By symmetry we may assume that $D - u_D$ is not 2-connected. Let C denote an endblock of $D - u_D$, and let $v \in V(C)$ be the cut vertex of $D - u_D$ contained in C such that $v_D \notin C - v$. By Lemma 3.5 we may assume that $v_D \neq v$. Since G is 5-connected, $|N(C - v) \cap \{y_1, y_2, y_3\}| \geq 2$; hence by Lemma 3.1 C is 2-connected.

Since D is 2-connected, $D - C$ has a path P from u_D to v_D. So C is contained in a 2-connected block of $H - (x_1Xu_D \cup P \cup v_DXx_2)$. Hence, $|N(C - v) \cap \{y_1, y_2, y_3\}| = 2$, for, otherwise, it follows from Lemma 3.2 and the choice of X that $\{y_1, y_2, y_3\} \subseteq N(B)$, a contradiction. Hence $b_D \in N(C - v)$.

Suppose $y_1, y_2 \in N(C - v)$. Then since G is 5-connected, there are five inpdepentdent paths Q_1, Q_2, Q_3, Q_4, Q_5 in $G[C + \{b_D, u_D, y_1, y_2\}]$ from some vertex $u \in C - v$ to u_D, v, y_1, y_2, b_D, respectively. Let Q denote a path in $G[B + \{y_1, y_2\}]$ from y_1 to y_2, and let R denote a path in

$D - u_D - (C - v)$ from v to v_D. Then $(Q_1 \cup u_D X x_1) \cup (Q_2 \cup R \cup v_D X x_2) \cup Q_3 \cup Q_4 \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

Thus, by symmetry, we may assume that $y_2, y_3 \in N(C - v)$. So $y_1 \notin N(C)$. Let $C' := (D - u_D) - (C - v)$.

We may assume that $G[C' + \{u_D, y_1\}]$ has three independent paths from some vertex $u \in C' - \{v, v_D\}$ to u_D, v_D, y_1, respectively. For, suppose not. Then v is a cut vertex of C' separating v_D from $N(y_1) \cap V(C')$. Let C_v denote the v-bridge of C' containing v_D, and let C_y be a v-bridge of C' such that $y_1 \in N(C_y - v)$. Let X' be the path obtained from X by replacing $u_D X v_D$ with a path in $G[C_v + u_D] - v$ from u_D to v_D. Then $X' \cap (B \cup C \cup C_y) = \emptyset$. Suppose $y_3 \in N(C_y - v)$. Then $G[C_y + \{y_1, y_3\}] - v$ has a path Q_1 berween y_1 and y_3. Let Q_2 be path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2, and Q_3 be a path in $G[C + \{y_2, y_3\}] - v$ between y_2 and y_3. Now $Q_1 \cup Q_2 \cup Q_3 \cup X' \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, y_3. Thus we may assume assume that $y_3 \notin N(C_y - v)$. Hence, since G is 5-connected, $b_D, y_1, y_2 \in N(C_y - v)$. So by Menger's theorem, $G[C_y + \{b_D, u_D, y_1, y_2\}]$ contains five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from some vertex $u \in C_y - v$ to u_D, v, y_1, y_2, b_D, respectively. Note that the path Q_2 can be extended through C_v to a path Q_2' ending at v_D. Let Q be a path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2. Then $(Q_1 \cup u_D X x_1) \cup (Q_2' \cup v_D X x_2) \cup Q_3 \cup Q_4 \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

So by Lemma 2.4, $G[C' + \{b_D, u_D, y_1, y_2, y_3\}]$ has five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from u to $S := \{b_D, u_D, v_D, v, y_1, y_2, y_3\}$ such that $V(Q_i \cap Q_j) = \{u\}$ for $1 \le i \ne j \le 5$, $|V(Q_i) \cap S| = 1$ for $1 \le i \le 5$, $u_D \in Q_1$, $v_D \in Q_2$, and $y_1 \in Q_3$. We may assume P_4 ends in $\{v, y_2, y_3\}$.

If $y_2 \in Q_4$ then let Q be a path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2; now $Q_3 \cup Q_4 \cup (Q_1 \cup u_D X x_1) \cup (Q_2 \cup v_D X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. If $v_1 \in Q_4$ then we extend Q_4 through $G[C + y_2]$ to a path Q_4' ending at y_2; now $Q_3 \cup Q_4' \cup (Q_1 \cup u_D X x_1) \cup (Q_2 \cup v_D X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So assume that $y_3 \in Q_4$ ends at y_3. Let Q be a path in $G[B \cup C + \{y_1, y_3\}] - v$ between y_1 and y_3; then $Q_3 \cup Q_4 \cup (Q_1 \cup u_D X x_1) \cup (Q_2 \cup v_D X x_2) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_3.

Subcase 2.2. $D - u_D$ and $D - v_D$ are 2-connected.

First, assume $u_D = x_1$ and $v_D = x_2$. Then since $N(y_3) \subseteq D \cup X$, $\{b_D, x_1, x_2, y_1, y_2\}$ is a cut in G separating B from D. In $G[B + \{x_1, x_2, y_1, y_2\}]$ we use Menger's theorem to find five independent paths P_1, P_2, P_3, P_4, P_5 from some vertex u to x_1, x_2, y_1, y_2, b_D, respectively. In $G[D'' + \{y_1, y_2\}]$ we find a path Q between y_1 and y_2. Now $P_1 \cup P_2 \cup P_3 \cup P_4 \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2.

Thus we may assume that $u_D \ne x_1$. We may further assume that $v_D = x_2$, and H contains no path from v_D to B internally disjoint from $B \cup D' \cup X$. For, otherwise, since $u_D \ne x_1$, H contains a path X' from x_1 to x_2 internally disjoint from $D \cup X$. Thus $D - v_D$ is contained in a 2-connected block of $H - X'$. Since $y_1, y_2, y_3 \in N(D'')$, it follows from Lemma 3.2 and the choice of X that $y_1, y_2, y_3 \in N(B)$, a contradiction.

Suppose $N(y_3) \subseteq N(D)$. Then $\{b_D, u_D, x_1, y_1, y_2\}$ is a cut in G separating $B \cup u_D X x_1$ from D'. Let G_1 denote the $\{b_D, u_D, x_1, y_1, y_2\}$-bridge of G containing $B \cup u_D X x_1$. Since $D - u_D$ is 2-connected, $G[D'' + \{v_D, y_1, y_2\}]$ has independent paths from some $u \in D''$ to y_1, y_2, v_D, respectively. So in $G[D' + \{y_1, y_2, y_3\}]$ we use Lemma 2.4 to find five independent paths Q_1, Q_2, Q_3, Q_4, Q_5 from u to $S := \{b_D, u_D, v_D, y_1, y_2, y_3\}$ such that $V(Q_i \cap Q_j) = \{u\}$

for $1 \le i \ne j \le 5$, $|V(Q_i) \cap S| = 1$ for $1 \le i \le 5$, $y_1 \in Q_1$, $y_2 \in Q_2$, and $v_D \in Q_3$. We may assume Q_4 ends in $\{b_D, u_D\}$. If $u_D \in Q_4$ then let Q be a path in $G[B + \{y_1, y_2\}]$ between y_1 and y_2; now $Q_1 \cup Q_2 \cup Q_3 \cup (Q_4 \cup u_D X x_1) \cup Q \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume $b_D \in Q_4$. If $G_1 - u_D$ contains disjoint paths R_1, R_2 from x_1, y_2 to b_D, y_1, respectively, then $Q_1 \cup Q_2 \cup Q_3 \cup (Q_4 \cup R_1) \cup R_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that such R_1, R_2 do not exist; then by Corollary 2.3, $(G_1 - u_D, x_1, y_2, b_D, y_1)$ is planar. Hence G contains TK_5 by Corollary 2.9.

Thus, we may assume that there exists $u \in N(y_3) \cap V(u_D X x_1 - \{u_D, x_1\})$.

We claim that for any permutation ijk of $\{1, 2, 3\}$ there are (not necessarily distinct) vertices v_1, v, v_2 on X in order from x_1 to u_D or there exist a 2-connected block $F \ne D$ of $H - B$ and $v \in F''$ with $v_1 = u_F$ and $v_2 = v_F$, and there are independent paths P_1, P_2, P_3, P_4 in $H + \{y_i, y_j\}$ from v to v_1, v_2, y_i, y_j, respectively, and internally disjoint from $v_1 X x_1 \cup v_2 X x_2 \cup D \cup K$. This is easy to verify when $u \notin F$ for any 2-connected block F of $H - B$; as in this case u has at least two neighbors in B and, since $y_1, y_2 \in N(B)$, we get the desired paths by setting $v = v_1 = v_2 = u$. So we may assume that $u \in F$ for some 2-connected block F in $H - B$. By Lemma 3.1, we see that F contains a path from u to b_F internally disjoint from X; so, because $y_1, y_2 \in N(B)$, the claim holds whenever $3 \in \{i, j\}$ by taking $v_1 = v_2 = v = u$. Now suppose $\{i, j\} = \{1, 2\}$. Let $v_1 = u_F$ and $v_2 = v_F$. First, assume $y_i \in N(F'')$ and $y_j \notin N(F'')$. Then by Menger's theorem we find five independent paths $P_1, P_2, P_3, P_4', P_5'$ in $G[F + \{y_i, y_3\}]$ from some vertex $v \in F''$ to v_1, v_2, y_i, b_F, y_3, respectively. By extending P_4' through $G[B + y_j]$ to a path P_4 ending at y_j, we find the desired paths. So we may assume that $y_i, y_j \in N(F'')$. Note that $G[F + y_i]$ contains independent paths from some vertex v to v_1, v_2, y_i, respectively (as F is 2-connected). So by Lemma 2.4, $G[F' + \{y_1, y_2, y_3\}]$ contains five independent paths $P_1, P_2, P_3, P_4', P_5'$ from v to $S := \{b_F, v_1, v_2, y_i, y_j, y_3\}$, such that $|V(P_i \cap P_j) = \{v\}$ for $1 \le i \ne j \le 5$, $|V(P_i) \cap S| = 1$ for $1 \le i \le 5$, $v_1 \in P_1$, $v_2 \in P_2$, and $y_i \in P_3$. We may assume P_4' ends in $\{b_F, y_j\}$. If P_4' ends at y_j then let $P_4 := P_4'$; if P_4' ends at b_F then we extend P_4' through $G[B + y_j]$ to a path P_4 ending at y_j. Now P_1, P_2, P_3, P_4 give the desired paths.

Let D^* be obtained from $G[D + \{y_1, y_2, y_3\}]$ by identifying y_1 and y_2, and use y to denote the new vertex.

Suppose D^* contains disjoint paths Q_1, Q_2 from u_D, y to v_D, y_3, respectively. Then in G, Q_2 is a path from y_i to y_3 for some $i \in \{1, 2\}$. Using the paths P_1, P_2, P_3, P_4 for $\{i, j\} = \{i, 3\}$, we see that $(P_1 \cup v_1 X x_1) \cup (P_2 \cup v_2 X u_D \cup Q_1) \cup P_3 \cup P_4 \cup Q_2 \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_i, y_3.

Thus we may assume that such Q_1, Q_2 do not exist. So by Lemma 2.3, (D^*, u_D, y, v_D, y_3) is 3-planar. Since D is 2-connected, we see that $G[D + \{y_1, y_2\}]$ has disjoint paths R_1, R_2 from u_D, y_2 to v_D, y_1, respectively. Therefore, using the paths P_1, P_2, P_3, P_4 for $\{i, j\} = \{1, 2\}$, we see that $(P_1 \cup v_1 X x_1) \cup (P_2 \cup v_2 X u_D \cup R_1) \cup P_3 \cup P_4 \cup R_2 \cup K$ is a TK_5 in G with branch vertices v, x_1, x_2, y_1, y_2. ∎

6 $H - B = X$

By Lemmas 4.6 and 5.2, it suffices to deal with the case when $H - B = X$ is simply an induced path. First, we show that at least two of $\{y_1, y_2, y_3\}$ each have at least two neighbors in B.

Lemma 6.1 *Suppose $H - B = X$. Then $|\{y_i : |N(y_i) \cap V(B)| \geq 2, i = 1, 2, 3\}| \geq 2$.*

Proof. Suppose on the contrary that $|\{y_i : |N(y_i) \cap V(B)| \geq 2, i = 1, 2, 3\}| \leq 1$. Then since G is 5-connected and X is induced in G, there exist distinct vertices $v_1, v_2 \in X - \{x_1, x_2\}$ such that each v_i is a neighbor of some $\{y_1, y_2, y_3\}$ with at least two neighbors in $X - \{x_1, x_2\}$. We choose v_1 and v_2 so that $v_1 X v_2$ is maximal.

Without loss of generality, we may assume that x_1, v_1, v_2, x_2 occur on X in this order, $|N(y_i) \cap V(B)| \leq 1$ for $i = 1, 2$, and $v_1 \in N(y_1)$ and $v_2 \in N(\{y_1, y_2\})$. Note that, since G is 5-connected and by Lemma 3.1, each v_i has at least two neighbors in B.

First, assume that $v_2 \in N(y_1)$. Without loss of generality, let $w_2, u_2 \in N(y_2) \cap V(X - \{x_1, x_2\})$ such that v_1, w_2, u_2, v_2 occur on X in order. In $G[B + \{v_1, x_2\}]$ there is a path P from v_1 to x_2. Thus $v_1 X x_1 \cup P \cup v_1 y_1 \cup (v_1 X w_2 \cup w_2 y_2) \cup (y_2 u_2 \cup u_2 X v_2 \cup v_2 y_1) \cup K$ is a TK_5 in G with branch vertices v_1, x_1, x_2, y_1, y_2.

Hence we may assume that $v_2 \in N(y_2)$. For $i = 1, 2$, let $w_i \in N(y_i) \cap V(v_1 X v_2 - \{v_1, v_2\})$. Note that the only possible cut vertex in $G[B + \{v_1, v_2, x_1\}]$ exists when x_1 has a unique neighbor in B. Thus $G[B + \{v_1, v_2, x_1\}]$ has independent paths P, Q from v_2 to x_1, v_1, respectively. Then $P \cup v_2 X x_2 \cup v_2 y_2 \cup (Q \cup v_1 y_1) \cup (y_1 w_1 \cup w_1 X w_2 \cup w_2 y_2) \cup K$ is a TK_5 in G with branch vertices v_2, x_1, x_2, y_1, y_2. ∎

We now reduce the problem to that case when $N(y_i) \subseteq V(B) \cup \{x_1, x_2\}$ for $i = 1, 2, 3$. We will make use of Lemma 2.5.

Lemma 6.2 *G contains TK_5, or $N(y_i) \subseteq V(B) \cup \{x_1, x_2\}$ for $i = 1, 2, 3$.*

Proof. By Lemma 6.1, we may assume that $|N(y_i) \cap V(B)| \geq 2$ for $i = 1, 2$.

Suppose there exists some $i \in \{1, 2, 3\}$ such that $y_i \in N(B)$ and $y_i \in N(X - \{x_1, x_2\})$. Let $u \in N(y_i) \cap V(X - \{x_1, x_2\})$. Then there exists $j \in \{1, 2\} - \{i\}$ such that $G[B + \{u, y_i, y_j\}]$ contains two independent paths P_1 and P_2 from y_j to u, y_i respectively. Now $u y_i \cup P_1 \cup X \cup P_2 \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_i, y_j.

Thus, we may assume that $N(y_i) \subseteq V(B) \cup \{x_1, x_2\}$ for $i = 1, 2$, and $N(y_3) \subseteq V(X)$ or $N(y_3) \subseteq V(B) \cup \{x_1, x_2\}$. We may further assume that $N(y_3) \subseteq V(X)$, or else the assertion of the lemma holds. Let $u_1, u_2 \in N(y_3) \cap V(X - \{x_1, x_2\})$ such that $u_1 \in x_1 X u_2 - \{x_1, u_2\}$. Since G is 5-connected, x_1 has a neighbor in B, say x. Note that $G[B + \{u_1, u_2, y_1, y_2\}]$ is 2-connected. Let G^* denote the graph obtained from $G[B + \{u_1, u_2, y_1, y_2\}]$ by identifying y_1 and y_2, and let y denote the new vertex. Then G^* is also 2-connected.

Suppose there exist disjoint paths P_1 and P_2 in G^* from u_1, u_2 to y, x, respectively. Without loss of generality, we may assume that P_1 is a path in G ending at y_1. Then $(P_1 \cup y_1 x_2) \cup (P_2 \cup x x_1) \cup X \cup u_1 y_3 \cup u_2 y_3 \cup (K - y_1)$ is a TK_5 with branch vertices u_1, u_2, x_1, x_2, y_3.

Thus we may assume that such paths do not exist. Then by Lemma 2.3, (G^*, u_1, u_2, y, x) is 3-planar. Note that $R := G[B + \{u_2, y_1, y_2\}]$ is 2-connected.

We now show that R has a cycle T containing $\{u_2, y_1, y_2\}$. For, otherwise, by Lemma 2.5, R has 2-cuts S_i, $i = 1, 2, 3$, such that if D_i (for $i = 1, 2$) denotes the components of $R - S_i$ containing y_i and D_3 denotes the component of $R - S_3$ containing u_2 then D_1, D_2, D_3 are pairwise disjoint. If some y_i is a cut vertex of $R[D_i \cup S_i]$ separating the vertices in S_i then, since y_i has at least three neighbors in D_i, $R - y_i$ is not 2-connected, a contradiction. Thus, for each $i \in \{1, 2\}$, $R[D_i \cup S_i] - y_i$ contains a path Q_i between the vertices in S_i. So Q_1 and

Q_2 can be used to form a cycle in $R - \{u_2, y_1, y_2\}$ which separates u_2 from $\{y_1, y_2\}$. But this contradicts the fact that (G^*, u_1, u_2, y, x) is 3-planar.

Then $T \cup X \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, u_2. ∎

We now show that G contains TK_5. By Lemma 6.2, we may assume that $N(y_i) \subseteq V(B) \cup \{x_1, x_2\}$ for $i = 1, 2, 3$; so $R := G[B + \{y_1, y_2, y_3\}]$ is 2-connected and each y_i has degree at least 3 in R.

If R has a cycle C containing $\{y_1, y_2, y_3\}$, then $C \cup X \cup K$ is a TK_5 in G with branch vertices x_1, x_2, y_1, y_2, y_3. So we may assume that such a cycle does not exist in R. Then by Lemma 2.5, we have three cases to consider.

Case 1. There exists a 2-cut S in R and there exist three distinct components D_1, D_2, D_3 of $R - S$ such that $y_i \in V(D_i)$ for each $i \in \{1, 2, 3\}$.

Let $S = \{a, b\}$. Since each y_i has degree at least 3 in R, $|D_i - y_i| \geq 1$ for $1 \leq i \leq 3$. Since G is 5-connected, $N(D_i - y_i) \cap V(X - \{x_1, x_2\}) \neq \emptyset$. Moveover, since B is 2-connected, $G[D_i + S] - y_i$ is a chain of blocks from a to b; so let $Q_i \subset G[D_i \cup S]$ be a path from a to b containing y_i.

We may assume $ab \notin E(G)$. For, suppose $ab \in E(G)$. Since X is induced, x_1 has at least two neighbors in some D_i, say $i = 3$. Then $G[D_3 + S + x_1]$ has independent paths L_1, L_2 from x_1 to a, b, respectively. Now $Q_1 \cup Q_2 \cup ab \cup y_1x_2y_2 \cup L_1 \cup L_2 \cup x_1y_1 \cup x_1y_2$ is a TK_5 in G with branch vertices a, b, x_1, y_1, y_2.

Let A_i be a path in G from a to some $a_i \in N(D_i - y_i) \cap V(X - \{x_1, x_2\})$ which is internally disjoint from $(B - D_i) \cup X$. We may assume $|\{a_1, a_2, a_3\}| \geq 2$. For otherwise, $a_1 = a_2 = a_3$. Then by symmetry, we may assume that $G[B + a_1]$ has independent paths P_i from a_1 to $q_i \in V(y_1Q_ib)$ and internally disjoint from Q_i. Now $a_1Xx_1 \cup a_1Xx_2 \cup (P_1 \cup q_1Q_1y_1) \cup (P_2 \cup q_2Q_2y_2) \cup (y_1Q_1a \cup aQ_2y_2) \cup K$ is a TK_5 in G with branch vertices a_1, x_1, x_2, y_1, y_2.

We may further assume that $R - S$ has only three components and $N(a) \cap V(X) = \emptyset$. Otherwise, there exists a path A from a to some $a' \in V(X)$ which is internally disjoint from $D_1 \cup D_2 \cup D_3 \cup X$. Without loss of generality, we may assume that $a' \in x_1Xa_3 - a_3$. Then $aQ_1y_1 \cup aQ_2y_2 \cup (y_1Q_1b \cup bQ_2y_2) \cup (A_3 \cup a_3Xx_2) \cup (A \cup a'Xx_1) \cup K$ is a TK_5 in G with branch vertices a, x_1, x_2, y_1, y_2.

Therefore, a has degree at least 5 in R. By Lemma 3.1, $|N(a) \cap \{y_1, y_2, y_3\}| \leq 1$. Hence, since $ab \notin E(G)$, there exists some $i \in \{1, 2, 3\}$ such that $|(N(a) \cap V(D_i)) - y_i| \geq 2$, say $i = 1$.

We claim that $G[D_1 \cup X + a] - y_1$ has independent paths P_1, P_2 from a to some $c_1, c_2 \in V(X)$ internally disjoint from X. For, suppose P_1, P_2 do not exist. Then $G[D_1 \cup X + a] - y_1$ has a cut vertex c separating a from X. Hence, $\{a, b, c, y_1\}$ is a cut in G as $|(N(a) \cap V(D_1)) - y_1| \geq 2$, a contradiction.

Now $(P_1 \cup c_1Xx_1) \cup (P_2 \cup c_2Xx_2) \cup aQ_2y_2 \cup aQ_3y_3 \cup (y_2Q_2b \cup bQ_3y_3) \cup K$ is a TK_5 in G with branch vertices a, x_1, x_2, y_2, y_3.

Case 2. There exist a vertex b of R, 2-cuts S_1, S_2, S_3 in R and components D_i of $R - S_i$ containing y_i, for all $i \in \{1, 2, 3\}$, such that $S_1 \cap S_2 \cap S_3 = \{b\}$ and $S_i - \{b\} = \{a_i\}$ where a_1, a_2, a_3 are distinct.

For convenience, let $R' := R - (D_1 \cup D_2 \cup D_3)$. We choose S_1, S_2, S_3 such that $D_1 \cup D_2 \cup D_3$ is maximal. Then $R' - b$ is connected.

As in Case 1, let $Q_i \subseteq G[D_i \cup S_i]$ be a path from a_i to b which passes through y_i, and let A_i be a path from a_i to $c_i \in N(D_i - y_i) \cap V(X - \{x_1, x_2\})$ and internally disjoint from

$(B - D_i) \cup X$. We may choose c_i so that $|\{c_1, c_2, c_3\}| \geq 2$; the proof is the same as in Case 1 (for showing $|\{a_1, a_2, a_3\}| \geq 2$) since $R' - b$ is connected.

Suppose there exists a vertex $u \in R' - \{a_1, a_2, a_3, b\}$ such that $R' - b$ has two independent paths from u to two distinct vertices of $\{a_1, a_2, a_3\}$, say a_1 and a_2. Let $S = \{a_1, a_2, a_3, b\} \cup (N(R') \cap V(X))$. Note that $G[R' + S] - b$ is $(4, S - \{b\})$-connected and $R' - a_3$ contains independent paths from u to a_1, a_2, respectively. So by Lemma 2.4, there exist four independent paths P_1, P_2, P_3, P_4 in $G[R' + S] - b$ from u to $S - \{b\}$ such that $|V(P_i \cap P_j) = \{u\}$ for $1 \leq i \neq j \leq 4$, $|V(P_i) \cap (S - \{b\})| = 1$ for $1 \leq i \leq 4$, $a_1 \in P_1$, and $a_2 \in P_2$. We may assume that P_3 ends at some vertex $v \in V(X)$ and P_4 ends at some vertex $w \in V(X) \cup \{a_3\}$. If $w \in X$ then by symmetry we may assume $v \in x_1Xw$; now $(P_1 \cup a_1Q_1y_1) \cup (P_2 \cup a_2Q_2y_2) \cup (P_3 \cup vXx_1) \cup (P_4 \cup wXx_2) \cup (y_1Q_1b \cup bQ_2y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that $w = a_3$. If $v \neq c_3$ then by symmetry we may assume $v \in x_1Xc_3$; now $(P_1 \cup a_1Q_1y_1) \cup (P_2 \cup a_2Q_2y_2) \cup (P_3 \cup vXx_1) \cup (P_4 \cup A_3 \cup c_3Xx_2) \cup (y_1Q_1b \cup bQ_2y_2) \cup K$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_2. So we may assume that $v = c_3$. Then $v \neq c_1$ or $v \neq c_2$. By symmetry, we may assume that $v \neq c_2$, and $v \in x_1Xc_2$. Then $(P_1 \cup a_1Q_1y_1) \cup (P_4 \cup a_3Q_3y_3) \cup (P_3 \cup vXx_1) \cup (P_2 \cup A_2 \cup c_2Xx_2) \cup (y_1Q_1b \cup bQ_3y_3)$ is a TK_5 in G with branch vertices u, x_1, x_2, y_1, y_3.

So we may assume that for any vertex $u \in R' - \{a_1, a_2, a_3, b\}$, there exists a 2-cut $S_u = \{b, b_u\}$ in R' separating u from $\{a_1, a_2, a_3\}$. We choose u and S_u so that the S_u-bridge of R' containing u is maximal. Then $b_u \in \{a_1, a_2, a_3\}$, say $b_u = a_3$, and $R' - \{a_1, a_2\}$ is the unique b_u-bridge of $R' - b$ containing u. Since $R - \{y_1, y_2, y_3\}$ is 2-connected, $R[\{a_1, a_2, a_3\}]$ must be connected.

We may assume that $R[\{a_1, a_2, a_3\}]$ is a triangle. Otherwise, for some permutation ijk of $\{1, 2, 3\}$, we have $a_ia_j \notin E(G)$ and $a_ia_k, a_ja_k \in E(G)$. Then $\{b, a_k\}$ is a 2-cut such that y_1, y_2, y_3 belong to three different components of $G - \{b, a_k\}$ whose union properly contains $D_1 \cup D_2 \cup D_3$, contradicting the choice of S_1, S_2, S_3 to maximize $D_1 \cup D_2 \cup D_3$.

Suppose for some $i \in \{1, 2\}$, $N(a_i) \not\subseteq \{a_1, a_2, a_3, b\} \cup V(D_i)$. Then H has an edge a_iv_i with $v_i \in X$. Since $\{a_i, b, y_i, v_i\}$ is not a cut in G, we see that A_i may be choosen so that $c_i \neq v_i$. Without loss of generality, we may assume that $v_i \in x_1Xc_i - c_i$. Let $\{i, j\} = \{1, 2\}$. Now $(A_i \cup c_iXx_2) \cup (a_iv_i \cup v_iXx_1) \cup (a_ia_j \cup a_jQ_jy_j) \cup (a_ia_3 \cup a_3Q_3y_3) \cup (y_jQ_jb \cup bQ_3y_3) \cup K$ is a TK_5 in G with branch vertices a_i, x_1, x_2, y_j, y_3.

Thus we may assume that for all $i \in \{1, 2\}$, $N(a_i) \subset \{a_1, a_2, a_3, b\} \cup V(D_i)$. We may further assume that there exists some $i \in \{1, 2\}$ such that $a_ib \notin E(G)$, say $i = 1$; otherwise, $G[\{a_1, a_2, a_3, b\}]$ is a K_4^-, and so G contains TK_5 by Theorem 1.1.

Then $|N(a_1) \cap V(D_1 - y_1)| = |N(a_1) - \{a_2, a_3, y_1\}| \geq 2$. So 5-connectedness of G implies that there exist two independent paths P_1, P_2 in $G[(D_1 + a_1) \cup X] - y_1$ from a_1 to $c_1, c_2 \in V(X)$ respectively, and internally disjoint from X. Without loss of generality, assume $c_1 \in x_1Xc_2$.

Now $(P_1 \cup c_1Xx_1) \cup (P_2 \cup c_2Xx_2) \cup (a_1a_3 \cup a_3Q_3y_3) \cup (a_1a_2 \cup a_2Q_2y_2) \cup (y_3Q_3b \cup bQ_2y_2) \cup K$ is a TK_5 in G with branch vertices a_3, x_1, x_2, y_2, y_3.

Case 3. There exist pairwise disjoint 2-cuts S_1, S_2, S_3 in R and components D_i of $R - S_i$ containing y_i, for all $i \in \{1, 2, 3\}$, such that D_1, D_2, D_3 are pairwise disjoint and $R - D_1 \cup D_2 \cup D_3$ has exactly two components, each containing exactly one vertex from S_i, for all $i \in \{1, 2, 3\}$.

Let $S_i = \{a_i, t_i\}$ for all $i \in \{1, 2, 3\}$ such that $\{a_1, a_2, a_3\}$ is contained in a component A of $R - (D_1 \cup D_2 \cup D_3)$ and $\{t_1, t_2, t_3\}$ is contained in a component T of $R - (D_1 \cup D_2 \cup D_3)$.

Note that any TK_5 we found in Case 2 only uses b to connect y_1 and y_2, which can be done in this case by using T. So by treating T, A as $b, R' - b$, respectively, in Case 2, the arguments in Case 2 work for Case 3 as weel and produce a TK_5 in G. ∎

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