COMBINATORICA
Bolyai Society – Springer-Verlag

Combinatorica 1–26

COUNTING CRITICAL SUBGRAPHS IN k-CRITICAL GRAPHS

JIE MA*, TIANCHI YANG

Received June 29, 2019
Revised May 28, 2020

Gallai asked in 1984 if any k-critical graph on n vertices contains at least n distinct $(k-1)$-critical subgraphs. The answer is trivial for $k \le 3$. Improving a result of Stiebitz [10], Abbott and Zhou [1] proved in 1995 that for all $k \ge 4$, any k-critical graph contains $\Omega(n^{1/(k-1)})$ distinct $(k-1)$-critical subgraphs. Since then no progress had been made until very recently, Hare [4] resolved the case $k=4$ by showing that any 4-critical graph on n vertices contains at least $(8n-29)/3$ odd cycles.

In this paper, we mainly focus on 4-critical graphs and develop some novel tools for counting cycles of specified parity. Our main result shows that any 4-critical graph on n vertices contains $\Omega(n^2)$ odd cycles, which is tight up to a constant factor by infinitely many graphs. As a crucial step, we prove the same bound for 3-connected non-bipartite graphs, which may be of independent interest. Using the tools, we also give a short solution to Gallai's problem when $k=4$. Moreover, we improve the longstanding lower bound of Abbott and Zhou to $\Omega(n^{1/(k-2)})$ for the general case $k \ge 5$. We will also discuss some related problems on k-critical graphs in the final section.

1. Introduction

In this paper, all graphs are simple (no loops or parallel edges), unless otherwise specified. The *chromatic number* $\chi(G)$ of a graph G is the minimum number of colors to be assigned to its vertices so that no adjacent vertices receive the same color. A graph G is called *k-critical* if it has chromatic

Mathematics Subject Classification (2010): .... Provide, please

* Partially supported by National Natural Science Foundation of China grant 11622110, the project "Analysis and Geometry on Bundles" of Ministry of Science and Technology of the People's Republic of China, and Anhui Initiative in Quantum Information Technologies grant AHY150200.

number k but every proper subgraph has chromatic number less than k. Note that all 3-critical graphs are odd cycles.

In 1984, Gallai asked the following problem (see Problem 5.9 of [5] or the discussion in [10]).

Problem 1.1 (Gallai). *If G is a k-critical graph on n vertices, is it true that G contains n distinct $(k-1)$-critical subgraphs?*

This problem is trivial for $k \leq 3$. From now on, we will assume $k \geq 4$. For convenience, for each $s \geq 3$ we denote by $f_s(G)$ the number of distinct s-critical subgraphs in a graph G. For $s = 3$, we will simply write $f(G)$ instead. Let G be an n-vertex k-critical graph. Stiebitz [10] first proved that $f_{k-1}(G) \geq \log_2 n$. This was improved by Abbott and Zhou [1] to

$$f_{k-1}(G) \geq ((k-1)!n)^{\frac{1}{k-1}}$$

in 1995 and there has been no further improvement for general k. Very recently, Hare [4] answered Gallai's problem in the case $k = 4$ by showing that every 4-critical graph on n vertices contains at least $\frac{8}{3}n - \frac{29}{3}$ odd cycles.

Our first result improves the general bound of Abbott and Zhou [1] for every $k \geq 4$.

Theorem 1.2. *For $k \geq 4$, every k-critical graph G on n vertices satisfies $\binom{f_{k-1}(G)}{k-2} \geq e(G)$. Thus*

$$f_{k-1}(G) \geq ((k-1)!n/2)^{\frac{1}{k-2}}.$$

Proof. For each $e \in E(G)$, $G-e$ has a proper $(k-1)$-coloring, say with color classes $A_1, \ldots, A_{k-1}$, where A_1 contains the ends of e. For each $2 \leq i \leq k-1$, we see that $G - A_i$ has chromatic number $k-1$ and thus contains a $(k-1)$-critical subgraph G_i^e. It is also clear that $e \in E(G_i^e)$. Let $L(e) = \{G_2^e, \ldots, G_{k-1}^e\}$. Note that each graph in $L(e)$ is $(k-1)$-critical and contains e. We claim that for any $f \in E(G-e)$ there is at least one subgraph in $L(e)$ not containing f. To see this, we may assume $f = uv$ with $u \in A_i$ and $v \in A_j$ for some $1 \leq i < j \leq k-1$, implying that $f \notin E(G_j^e)$. This claim shows that $L(e)$ are distinct for all edges e in G and so $\binom{f_{k-1}(G)}{k-2} \geq e(G)$. Since any k-critical graph has minimum degree at least $k-1$, we have $e(G) \geq (k-1)n/2$ and this further implies $f_{k-1}(G) \geq ((k-1)!n/2)^{\frac{1}{k-2}}$. ∎

For the rest of the paper, we focus on the case of 4-critical graphs. Our main result is a tight bound on the number of odd cycles in 4-critical graphs. This in fact is proved in a stronger form, that reveals a relationship between the numbers of odd cycles and edges. To state it, we introduce a parameter that will play an important role in the proofs: for any graph G, let

$$t(G) = |E(G)| - |V(G)| + 1.$$

Note that if G is 2-connected, then any ear-decomposition of G (we postpone its definition to Section 2) has exactly $t(G)$ ears; also, for a 4-critical graph G, since every vertex has degree at least 3, we have $t(G) \geq |E(G)|/3 \geq |V(G)|/2$.

Theorem 1.3. *If G is a 4-critical graph on n vertices and m edges, then $f(G) \geq 0.02t^2(G)$. Thus*

$$f(G) \geq \Omega(m^2) \geq \Omega(n^2).$$

We remark that this is tight up to a constant factor. To see this, by an n-vertex *d-wheel* $W(n,d)$ we denote the graph obtained from a cycle C_{n-d} and a clique K_d by joining each vertex of C_{n-d} to each vertex of K_d. It is *odd* if $n-d$ is odd and *even* otherwise. For simplicity, we just call a 1-wheel a wheel. Now we observe that the odd wheel $W = W(n,1)$ is 4-critical and has $\binom{n-1}{2}+1$ odd cycles; it also has $O(|E(W)|^2)$ and $O(t^2(W))$ odd cycles.

The proof of Theorem 1.3 consists of two cases. Let G be a 4-critical graph. The first case is that G contains some 2-cut $\{x, y\}$. By a structural result (Lemma 2.1), G can be decomposed into two subgraphs F_1, F_2 which are 'close' to being 4-critical. On the one hand, we can find relatively many odd cycles in each of F_1 and F_2 by using induction; on the other hand, using results we prove for counting paths between two given vertices (i.e., Lemmas 2.4 and 5.2), we can find many paths of specified parity between x and y in F_1 and in F_2, which together give a good number of odd cycles distinct from those above. This would give $\Omega(t^2(G))$ odd cycles when G contains some 2-cut.

The other case in the proof of Theorem 1.3 is that G does not contain 2-cuts, that is, G is 3-connected (and 4-critical). As an intermediate step and a result of independent interest, we prove the following similar bound for graphs that are 3-connected and non-bipartite.

Theorem 1.4. *If G is a 3-connected non-bipartite graph, then $f(G) \geq 0.02t^2(G)$.*

We note that Theorem 1.4 is (slightly) stronger than what we need to complete the proof of Theorem 1.3. For the proof of Theorem 1.4, the main idea is to find an induced non-separating (defined in Section 2) odd cycle C and to find many paths with end-vertices in $V(C)$ that are internally disjoint from C. A crucial observation is that each such path can be extended to an odd cycle by adding exactly one of the two subpaths in C between its end-vertices. Along the way to obtaining these results, we develop some tools for counting cycles with specified parity and passing through some fixed vertex (such as Lemma 5.1). The key ingredient in proving these results (including Lemmas 5.1 and 5.2) is a novel application of ear-decompositions, together with the use of non-separating cycles. To facilitate this approach, we also consider and establish analogous results in signed graphs, which may be of independent interest.

We also would like to mention here that using the idea of induced non-separating odd cycles, one can give a short proof to the case $k=4$ of Problem 1.1 (see Theorem 4.2 in Section 4).

The rest of the paper is organized as follows. In Section 2, we define notation and collect basic lemmas for later use. We then prove some lemmas for 3-connected non-bipartite signed graphs in Section 3. In Section 4, we illustrate how to use the idea of non-separating odd cycles in a short proof of the case $k=4$ of Problem 1.1. In Section 5, we prove two technical lemmas as tools for counting cycles of each parity. In Section 6, we complete the proof of Theorem 1.4 by detouring to signed graphs. In Section 7, we prove Theorem 1.3. The final section contains some concluding remarks and related problems. We do not attempt to optimize the constant factors in our results, preferring rather to provide a simpler presentation.

2. Preliminaries

The following structural lemma on k-critical graphs was first proved by Dirac [2,3], and a detailed proof can also be found in [9] (see its Lemma 3.2).

Lemma 2.1 ([2,3]). *Let $k\geq 4$ be an integer, G be a k-critical graph and $\{u,v\}$ be a 2-cut of G. Then $uv\notin E(G)$ and there are unique proper induced subgraphs F_1,F_2 of G such that*

(a) $G=F_1\cup F_2$ and $V(F_1)\cap V(F_2)=\{u,v\}$,
(b) u and v have no common neighbor in F_2, and
(c) both F_1+uv and $F_2/\{u,v\}$ are k-critical.[1]

Answering a long-standing conjecture of Ore from 1967 on the number of edges in 4-critical graphs, Kostochka and Yancey [7] proved the following tight result. Let $e(G)$ be the number of edges in a graph G.

[1] The graph $F_2/\{u,v\}$ is obtained from F_2 by contracting u and v into a new vertex.

Theorem 2.2 **([7]).** *If G is a 4-critical graph, then $e(G)\geq\frac{5}{3}|V(G)|-\frac{2}{3}$.*

Given a subgraph F in a graph G, by $G-F$ we denote the subgraph obtained from G by deleting all vertices in F. We say a cycle C is *non-separating* in G if $G-C$ is connected. In 1980 Krusenstjerna-Hafstrøm and Toft proved the following theorem (Theorems 4 and 5 in [8]).

Theorem 2.3 **([8]).** *Let G be a graph which is either 4-critical or 3-connected and let F be a connected subgraph of G such that $G-F$ contains an odd cycle. Then G contains a non-separating induced odd cycle C such that $V(C)\cap V(F)=\emptyset$.*

A path with end-vertices x and y is called an *(x,y)-path*. Let G be a given graph (not necessarily connected). A vertex $v\in V(G)$ is called a *cut-vertex* of G if $G-v$ has more components than G. A *block* B of G is a maximal connected subgraph of G such that there exists no cut-vertex of B. So a block is either an isolated vertex, an edge or a 2-connected graph. For a subgraph F in G, an *F-ear* in G is a path in G whose two end-vertices lie in F but whose internal vertices do not. An *ear-decomposition* of G is a nested sequence $(G_0,G_1,\ldots,G_s)$ of subgraphs of G such that G_0 is a cycle, $G_{i+1}=G_i\cup P_{i+1}$ where P_{i+1} is a G_i-ear in G for $0\leq i<s$, and $G_s=G$. We also identify the ear-decomposition by the union $P_0\cup P_1\cup\ldots\cup P_s$, where $P_0=G_0$.

Lemma 2.4. *For any two distinct vertices x,y in a block B, there are at least $t(B)+1$ distinct (x,y)-paths in B.*

Proof. If B is an edge xy, then this holds trivially. So we may assume that B is 2-connected. Let $t:=t(B)$ and C be any cycle containing x and y. Using the standard ear decomposition of a 2-connected graph, there exist $t-1$ paths $P_1,P_2,\ldots,P_{t-1}$ in B such that $B_i:=C\cup(\cup_{j=1}^{i}P_j)$ is 2-connected for each $0\leq i\leq t-1$, where $B_0=C$ and $B_{t-1}=B$. For each $1\leq i\leq t-1$, let a_i and b_i be the end-vertices of P_i. As B_{i-1} is 2-connected, there exist two disjoint paths from $\{a_i,b_i\}$ to $\{x,y\}$ in B_{i-1}. This gives an (x,y)-path in B_i containing the path P_i. Together with the two (x,y)-paths in C, we get at least $t+1$ distinct (x,y)-paths in B. ∎

Let $\mathcal{B}$ be the set of blocks in a graph G and $\mathcal{C}$ be the set of cut-vertices of G. The *block structure* of G is the bipartite graph with bipartition $(\mathcal{B},\mathcal{C})$, where $c\in\mathcal{C}$ is adjacent to $B_i\in\mathcal{B}$ if and only if $c\in V(B_i)$. Note that the block structure of any connected graph is a tree. An *end-block* in G is a block containing at most one cut-vertex of G.

Proposition 2.5. *Let G be a connected graph. Then $t(G) = \sum_{B \in \mathcal{B}} t(B)$.*

Proof. This can be proved easily by induction on the number of blocks using the block structure. ∎

A *signed graph* is a graph G associated with a function $p: E(G) \to \{0,1\}$. For $e \in E(G)$, we refer to $p(e)$ as the *parity* of e. The parity of a path or a cycle C in G is the parity of the sum of the parities of all edges in $E(C)$, and we say C is *even* if its parity is 0 and *odd* otherwise. A signed (multi-)graph is *bipartite* if every cycle is even and *non-bipartite* otherwise. In this paper we view every graph as a signed graph by assigning 1 to every edge. The following property is an easy observation.

Proposition 2.6. *A signed (multi-)graph (G,p) is bipartite if and only if there exists a bipartition $V(G) = A \cup B$ such that each $e \in E(A,B)$ is odd and each $e \in E(G) \setminus E(A,B)$ is even.*

We also need a lemma proved by Kawarabayashi, Reed and Lee (see Lemma 2.1 in [6]).

Lemma 2.7 ([6]). *If s is a vertex in a 3-connected signed graph G such that $G - s$ is not bipartite, then there is a non-separating induced odd cycle C in G with $s \notin V(C)$.*

Throughout the rest of this paper, a set of edges is called *independent* if their sets of endpoints are pairwise disjoint. For any integer $k \geq 1$, we write $[k]$ to denote $\{1, 2, \ldots, k\}$.

3. Lemmas on 3-connected non-bipartite signed graphs

Throughout this section, let G be a 3-connected non-bipartite signed graph. By Lemma 2.7, there exists an induced odd cycle C in G such that $G - C$ is connected. Fix such a cycle C and let $H = G - C$, $t = t(H)$ and $m = |E(C,H)|$. Then it is straightforward to see that $t(G) = t + m$.

A pair of edges $xa, yb \in E(C,H)$ with $x, y \in V(C)$ and $a, b \in V(H)$ is called *good* if $x \neq y$. Given such a pair $\{xa, yb\}$, we call any (a,b)-path contained in H a *good path.* It is easy to see that any good (a,b)-path in H can be uniquely extended to an odd cycle in G by adding xa, yb and one of the two (x,y)-paths in C. Such an odd cycle will be called *basic* in G for the good pair $\{xa, yb\}$. We remark that the odd cycle C is not basic, and each basic cycle corresponds to a unique even cycle.

Lemma 3.1. *If H is 2-connected, then there are at least $(t+1)m$ distinct basic cycles in G.*

Proof. Clearly we have $|C| \geq 3$ and $|V(H)| \geq 3$. Since G is 3-connected, there are 3 independent edges $x_ia_i \in E(C,H)$ with $x_i \in V(C)$ and $a_i \in V(H)$ for all $i \in [3]$. By Lemma 2.4, for distinct $i,j \in [3]$, we get at least $t+1$ distinct (a_i,a_j)-paths in H. This gives at least $3(t+1)$ distinct basic cycles in G using exactly two of $\{x_1a_1, x_2a_2, x_3a_3\}$. For any $yb \in E(C,H)$ other than $\{x_ia_i\}$, there is at least one edge (say x_1a_1) in $\{x_ia_i\}$ independent of yb. Using Lemma 2.4, similarly one can find at least $t+1$ distinct basic cycles using yb and x_1a_1. Together we see at least $3(t+1)+(m-3)(t+1)=(t+1)m$ distinct basic cycles in G. ∎

Let $\mathcal{B}$ be the set of blocks in H and $\mathcal{C}$ be the set of cut-vertices in H. For $a,b \in V(H)$, by $\mathcal{P}_{a,b}$ we denote the shortest path $B_{j_1}c_1B_{j_2}c_2\cdots c_{\ell-1}B_{j_\ell}$ in the block structure $(\mathcal{B},\mathcal{C})$ of H satisfying that $a \in V(B_{j_1})$ and $b \in V(B_{j_\ell})$, where $B_i \in \mathcal{B}$ and $c_j \in \mathcal{C}$.

Lemma 3.2. *Let $a,b \in V(H)$ be two distinct vertices. Then there are at least $\prod_{B\in\mathcal{P}_{a,b}\cap\mathcal{B}}(t(B)+1) \geq \left(\sum_{B\in\mathcal{P}_{a,b}\cap\mathcal{B}} t(B)\right)+1$ distinct (a,b)-paths in H.*

Proof. Let $B_1c_1B_2c_2\cdots c_{\ell-1}B_\ell$ be the path $\mathcal{P}_{a,b}$, where $a \in V(B_1)$ and $b \in V(B_\ell)$. Let $c_0 = a$ and $c_\ell = b$. By Lemma 2.4, there are at least $t(B_i)+1$ distinct (c_{i-1},c_i)-paths in B_i for each $1 \leq i \leq \ell$, implying this lemma. ∎

In the rest of this section, we assume that H is connected but not 2-connected. For each end-block B_i in H, we define the unique cut-vertex of H in B_i to be c_i. We now define a good pair of edges $\{e_i, f_i\}$ in $E(C, B_i - c_i)$, called *staple edges* of the end-block B_i, as follows. If B_i is an edge say a_ic_i, as a_i has at least two neighbors $x_i, y_i \in V(C)$, let $e_i = x_ia_i$ and $f_i = y_ia_i$. Otherwise B_i is 2-connected with $|V(B_i)| \geq 3$. There are 3 disjoint paths from B_i to C in G (as G is 3-connected) at most one of which uses the cut-vertex c_i, so the other two paths must be two independent edges say $e_i = x_ia_i$ and $f_i = y_ib_i$ in $E(C, B_i - c_i)$.

Lemma 3.3. *Let k be the number of end-blocks in H. If $k \geq 2$, then there are at least $(m-k)(t+k)+\lceil \frac{k}{2} \rceil$ basic cycles in G.*

Proof. Let $B_1, B_2, \ldots, B_k$ be all end-blocks in H. Let uv be a non-staple edge in $E(C,H)$ with $v \in V(H)$. For each B_i, at least one of e_i, f_i has an end-vertex in $V(C)-u$; let $e_i = x_ia_i$ be such an edge with $a_i \in V(B_i)-c_i$ and thus $\{uv, x_ia_i\}$ is a good pair. Since the block structure of H is a tree, the union of the k paths $\mathcal{P}_{v,a_i}$ over $i \in [k]$ contains all blocks in $\mathcal{B}$. By Lemma 3.2

and Proposition 2.5, the number of distinct (v, a_i)-paths, summed over all $i \in [k]$, is at least $(\sum_{B\in\mathcal{B}} t(B)) + k = t + k$. This gives $t+k$ basic cycles in G using uv and exactly one staple edge. Since there are $m-2k$ non-staple edges in $E(C,H)$, we have at least $(m-2k)(t+k)$ distinct basic cycles in G using exactly one staple edge.

We then consider basic cycles with two staple edges. For end-blocks B_i, B_j, we can always pair the four staple edges e_i, f_i, e_j, f_j into two good pairs $\mathcal{A}_\ell$ for $\ell \in [2]$ with $|\mathcal{A}_\ell \cap \{e_i, f_i\}| = 1$. Thus each of the $2k$ staple edges (say e_1) appears in k good pairs $\{e_1, g_j\}$ for $j \in [k]$, where g_j is a staple edge of B_j. Similarly as above, each staple edge is contained in at least $t+k$ basic cycles using two staple edges. By double-counting, this gives at least $k(t+k)$ basic cycles using two staple edges.

Now consider the staple edges e_i, f_i of each B_i. As G is 3-connected, there exists $g \in E(C,H)$ independent of e_i, f_i. Thus $\{g, e_i\}$ and $\{g, f_i\}$ are both good pairs. Note that such an edge g may be a staple edge or not, and we have only considered one good pair for g in the above counting. By double-counting (as g can be a staple edge), we can get $\lceil \frac{k}{2} \rceil$ more good pairs, which lead to $\lceil \frac{k}{2} \rceil$ more distinct basic cycles in G. This lemma follows by adding up all basic cycles above. ∎

As we point out earlier that each basic cycle also corresponds to a unique even cycle, Lemmas 3.1 and 3.3 give the same number of distinct even cycles in G.

4. A short proof to Gallai's problem when $k = 4$

In this section, to illustrate the idea of non-separating odd cycles and as a warm-up for the more complicated proofs to come, we give a short proof of the case $k=4$ of Problem 1.1. The following result was first proved by Hare (see Theorem 1.5* in [4]). Here we give a different proof for completeness.

Lemma 4.1. *Every 3-connected non-bipartite graph G contains at least $2t(G)-2$ distinct odd cycles.*

Proof. Following the notation in Section 3, let C be an induced odd cycle in G such that $G-C$ is connected. Let $H = G-C$, $t = t(H)$ and $m = |E(C,H)|$. Then we have $t(G) = t+m$. If H is 2-connected, then $t \geq 1$ and $m \geq 3$. Since $(t+1)m - (2t(G)-2) = (t-1)(m-2) \geq 0$, by Lemma 3.1, G contains at least $(t+1)m \geq 2t(G)-2$ odd cycles. So assume H is not 2-connected. Let ℓ be the number of end-blocks in H.

If $\ell \geq 2$, then $m \geq 2\ell \geq \ell+2$ and thus $(m-\ell)(t+\ell)+2 = ((m-\ell-2)+2)((t+\ell-2)+2)+2 \geq 2(m-\ell-2)+2(t+\ell-2)+6 = 2t(G)-2$. By Lemma 3.3 (plus the cycle C), G contains at least $(m-\ell)(t+\ell)+2 \geq 2t(G)-2$ odd cycles. It remains to consider $\ell=1$, that is, H is an isolated vertex or an edge. If H is a vertex, then every two edges in $E(C,H)$ form a good pair. If H is an edge ab, then any non-good pair in $E(C,H)$ must be $\{ax,bx\}$ for some $x \in V(C)$, which also defines a triangle abx. Hence in either case, it holds that $t=0$, $t(G)=m$ and any pair in $E(C,H)$ contributes a distinct odd cycle in G. Adding the cycle C, there are at least $\binom{m}{2}+1=\frac{1}{2}t(G)(t(G)-1)+1 \geq 2t(G)-2$ odd cycles in G, where the inequality holds as $t(G) \geq |V(G)|/2+1 \geq 2$. This completes the proof. ∎

Now we are ready to prove the following theorem using the idea of non-separating odd cycles.

Theorem 4.2. *If G is a 4-critical graph on n vertices, then $f(G) \geq 2t(G)-2 = 2e(G)-2n$. In particular, $f(G) \geq n$, where the unique 4-critical graph achieving the equality is K_4 when $n=4$.*

Proof. Let G be a 4-critical graph on n vertices. We prove $f(G) \geq 2t(G)-2$ by induction on n. It is clear that if $n=4$, then $G=K_4$ has exactly 4 odd cycles. So we may assume that this holds for all 4-critical graphs with at most $n-1$ vertices.

Clearly, G is 2-connected and non-bipartite. If G is 3-connected, then Lemma 4.1 implies $f(G) \geq 2t(G)-2$. So we may assume that there exists a 2-cut $\{u,v\}$ in G. By Lemma 2.1, $uv \notin E(G)$ and there exist induced subgraphs G_1 and G_2 of G such that $G=G_1 \cup G_2$, $V(G_1) \cap V(G_2)=\{u,v\}$, and $H_1 := G_1+uv$ and $H_2 := G_2/\{u,v\}$ are 4-critical. Also u,v have no common neighbor in G_2, so $e(H_2)=e(G_2)$, from which we can derive that $t(H_1)+t(H_2)=t(G)+1$.

We claim that both G_1 and G_2 contain two (u,v)-paths of different parities. Since H_1 is 4-critical and thus 2-connected, there exist an odd cycle C not containing u and two disjoint paths from $\{u,v\}$ to C in H_1 (also in G_1). Then we can easily get two (u,v)-paths of different parities in G_1. Similarly, H_2 has an odd cycle D avoiding the new vertex contracted from $\{u,v\}$. There are two disjoint paths from $\{u,v\}$ to D in the 2-connected G. Clearly, these paths are also contained in G_2. Thus, we can get two (u,v)-paths of different parities in G_2.

Suppose that the numbers of (u,v)-paths of even length in G_1,G_2 are α,γ, and the numbers of (u,v)-paths of odd length in G_1,G_2 are β,θ, respectively. By induction $f(H_i) \geq 2t(H_i)-2$ for each $i \in \{1,2\}$. Then G_1 has $f(H_1)-\alpha$ odd cycles and G_2 has $f(H_2)-\theta$ odd cycles. In total, G

has at least $(f(H_1)-\alpha)+(f(H_2)-\theta)+\alpha\theta+\beta\gamma$ odd cycles. We know $\alpha,\beta,\gamma,\theta \geq 1$. So $\alpha\theta+\beta\gamma-\alpha-\theta \geq (\alpha-1)(\theta-1)+\beta\gamma-1 \geq 0$. Thus $f(G)\geq f(H_1)+f(H_2)\geq (2t(H_2)-2)+(2t(H_2)-2)=2t(G)-2$.

By Theorem 2.2, we have $f(G)\geq 2t(G)-2=2e(G)-2n\geq \frac{4}{3}(n-1)\geq n$, with equality if and only if $n=4$ and $G=K_4$. This completes the proof of Theorem 4.2. ∎

5. Counting cycles with parity via ear-decompositions

In this section we prove two lemmas for counting cycles of specified parities passing through a given vertex or a given edge in 3-connected non-bipartite (signed) graphs. The key idea is to choose some ear-decomposition with particular properties, based on a given non-separating induced odd cycle. For a path P with $x,y\in V(P)$, we denote by xPy the subpath of P between x and y.

Lemma 5.1. *Let G be a 3-connected non-bipartite signed graph, x be a vertex in G, and D be a non-separating induced odd cycle in G such that $x\notin V(D)$. Let R_i for $i\in[3]$ be three disjoint paths from x to $z_i\in V(D)$ with $xy_i\in E(R_i)$.*

Suppose there exists an edge-coloring g assigning colors to every edge incident to x such that $g(xy_i)$ for $i\in[3]$ are distinct.[2] *Then G contains at least $t(G)$ cycles of each parity passing through x such that the two edges incident to x in every such cycle have different colors assigned by g.*

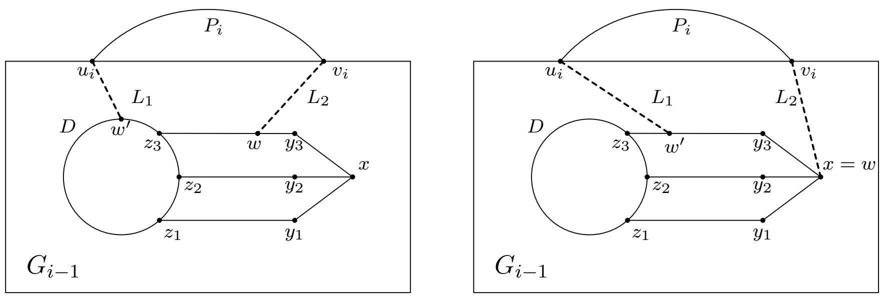

Figure 1. A key step in the proof of Lemma 5.1

[2] The coloring g does not need to be proper, and it does not have any bound on the number of allowed colors.

Proof. Let $t = t(G)$. We claim that there is an ear-decomposition $P_1 \cup P_2 \cup \ldots \cup P_t$ of G such that $P_1 = D, P_2 = R_1 \cup R_2, P_3 = R_3$ and for each $i \geq 3$, at least one of the ends of P_i is not in D and thus D is non-separating in $G_i := \cup_{j=1}^{i} P_j$. To see this, suppose we already get desired ears $\{P_j\}_{1\leq j\leq i-1}$ for some $4 \leq i \leq t$; since D is induced and non-separating in G, one can always find a new ear P_i (a single edge or not) internally disjoint from G_{i-1} with one end not in D.[3] For $i \geq 4$, let the ends of P_i be u_i, v_i with $v_i \notin V(D)$. Since D is non-separating in G_{i-1}, there is a path L in $G_{i-1} - D$ from v_i to some vertex $w \in V(R_1 \cup R_2 \cup R_3) - V(D)$. As G_{i-1} is 2-connected, there are two disjoint paths L_1, L_2 in G_{i-1} from $\{v_i, u_i\}$ to $D \cup R_1 \cup R_2 \cup R_3$. By concatenating with the path L and renaming if necessary[4], we may assume that L_1, L_2 are from $\{v_i, u_i\}$ to $\{w, w'\} \subseteq D \cup R_1 \cup R_2 \cup R_3$, where $w \notin V(D)$. Now we see that for each $i \geq 4$, there exists a path $Q_i := P_i \cup L_1 \cup L_2$ in G_i with ends w, w' containing the ear P_i and internally disjoint from $D \cup R_1 \cup R_2 \cup R_3$, where the ends w, w' are in $D \cup R_1 \cup R_2 \cup R_3$ and at most one of them is in D; see Figure 1 for an illustration.

We observe that it will suffice to extend Q_i to a path Q_i' in G_i with both ends in D passing through x such that its two edges incident to x have different colors assigned by g. Indeed, if true, then since D is odd, by adding one of the two paths between two ends of Q_i' in D to Q_i', we can get a desired cycle of each parity for every $4 \leq i \leq t$. Since $P_i \subseteq Q_i' \subseteq G_i$, this provides $t-3$ distinct such cycles. Also $D \cup R_1 \cup R_2 \cup R_3$ contains three desired cycles of each parity, so the lemma follows.

Finally, we show how to extend Q_i to Q_i' in G_i. This can be verified by considering all possible locations of the ends w, w' of Q_i in $D \cup R_1 \cup R_2 \cup R_3$. Note that at most one of w, w' is in D. In case that $w, w' \in V(D \cup R_1 \cup R_2 \cup R_3) - x$, we omit the straightforward details. So it remains to consider when $x \in \{w, w'\}$ (say $x = w$). Let $xy \in E(Q_i)$ and by symmetry, $w' \notin V(R_1 \cup R_2)$. There exists some $j \in [2]$ such that $g(xy_j) \neq g(xy)$. If $w' \in V(D)$, then Q_i' can be chosen as $Q_i \cup R_j$; otherwise $w' \in V(R_3)$, then Q_i' can be chosen as $z_3 R_3 w' \cup Q_i \cup R_j$. This completes the proof. ∎

Lemma 5.2. *Let x, y be two distinct vertices in a 3-connected graph G such that both $G - x$ and $G - y$ are non-bipartite. Then G contains at least*

[3] To do this, start at an edge e incident to, but outside, G_{i-1} with an endpoint not in D, and then follow a cycle in G through e and some edge in G_{i-1} until it reaches G_{i-1}.

[4] By this, we mean the following process. Let s be the vertex in $V(L) \cap V(L_1 \cup L_2)$ such that the subpath wLs is as short as possible. If $s = v_i \in V(L_j)$ for $j \in [2]$, then we rename L_j to be L. Otherwise, $s \in V(L - v_i) \cap V(L_\ell)$ for $\ell \in [2]$. Let $r \in \{v_i, u_i\}$ be an end of L_ℓ. Now we rename L_ℓ to be $wLs \cup sL_\ell r$. Note that the new L_1, L_2 are still disjoint.

$t(G)-1$ distinct (x,y)-paths of each parity (not including the possible edge xy).

Proof. Let H be obtained from G by adding the edge xy (if it does not exist) and let $t=t(H)$. Then H also satisfies the hypothesis of the lemma with $t(G)\le t\le t(G)+1$.

First we consider that $H-\{x,y\}$ is bipartite. By Theorem 2.3 (or Lemma 2.7), we see that there exists a non-separating induced odd cycle D in H with $x\notin V(D)$. Since $H-\{x,y\}$ is bipartite, such D must contain y. There exist two disjoint paths P_1,P_2 from x to D in $H-y$, internally disjoint from D. Let H' be obtained from H by deleting all edges incident to y except the two edges (say yu,yv) in D. So H' is 2-connected and D is non-separating in H'. Similarly to the proof of Lemma 5.1, we can find an ear-decomposition $F_1\cup\ldots\cup F_m$ in H' such that $F_1=D$, $F_2=P_1\cup P_2$ and for each $i\ge 3$, at least one end of the path F_i is not in D, where $m=t(H')$. So for $i\ge 3$, D is non-separating in $H_i:=\cup_{j=1}^{i}F_j$. By similar analysis as in Lemma 5.1, there exists a path Q_i in H_i containing the ear F_i from x to some vertex in $D-y$, which can be extended to an (x,y)-path of each parity in H_i containing F_i for each $i\ge 3$. Adding two such paths in $F_1\cup F_2$, we get m desired (x,y)-paths in H'. Also by Theorem 2.3, there exists a non-separating induced odd cycle D' in H with $x\in V(D')$ and $y\notin V(D')$. Note that there are at least $t(G)-m-1$ edges yz in $E(H)-E(H')$ where z is allowed to range over all possible vertices in H except u, v, and x. We claim that for each such edge yz, there exists a path in H from y to some vertex in $D'-x$ which uses yz. This is clear if $z\in V(D')$; for $z\notin V(D')$, since H is 3-connected, there exists a path in $H-\{x,y\}$ from z to $D'-x$, from which the claim holds. Using this claim, it is easy to find at least $t(G)-m-1$ many (x,y)-paths in G of each parity, which are also distinct from the above m paths. This finishes the proof when $H-\{x,y\}$ is bipartite.

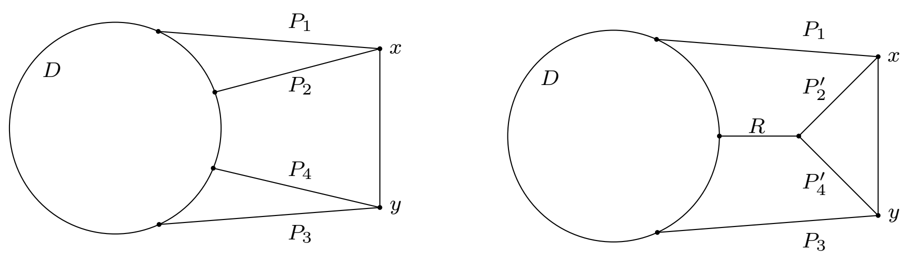

Figure 2. A key step in the proof of Lemma 5.2

Now we may assume that $H-\{x,y\}$ contains an odd cycle. By Theorem 2.3 there exists a non-separating induced odd cycle D in H such that $H-D$ contains xy. We claim that there are four paths P_1,P_2,P_3,P_4 in H from $\{x,y\}$ to D such that (see Figure 2)

(a) x is an end of P_1,P_2 and y is an end of P_3,P_4,
(b) any P_i,P_j are internally disjoint, except possibly when $\{i,j\}=\{2,4\}$, and
(c) if P_2 and P_4 intersect, then $P_2=P_2'\cup R$ and $P_4=P_4'\cup R$ such that P_2',P_4',R are internally disjoint paths and $x,y\notin V(R)$.

To prove this, since H is 3-connected, we begin by choosing three internally disjoint paths P_1,P_2,R in H from x,x,y to $a,b,c\in V(D)$, respectively. There are also two disjoint paths P_3,P_4 in $H-x$ from y to $D\cup P_1\cup P_2-x$, which are internally disjoint from $D\cup P_1\cup P_2$. By concatenating P_3,P_4 with the path R and renaming if necessary[5], we may assume that P_3 is from y to $c\in V(D)$ and by symmetry (between P_1 and P_2), P_4 is from y to $D\cup P_2$. Thus, there do exist paths P_1,P_2,P_3,P_4 satisfying conditions (a-c), as desired.

Next we build an ear-decomposition $F_1\cup\ldots\cup F_t$ of H such that $F_1=D$, $F_2=P_1\cup P_2$, $F_3=P_3\cup P_4$ (in case P_2 and P_4 intersect, let $F_3=P_3\cup P_4'$), $F_4=xy$, and for each $i\geq 5$, at least one end of the path F_i is not in D and x,y cannot be the two ends of F_i. The construction is similar to that in the previous lemma (following the facts that D is induced and non-separating in H and $\{x,y\}$ is not a 2-cut of H), and we omit the details here. Let $H_i:=\cup_{j=1}^{i}F_j$ and A be the vertex set of $F_1\cup\ldots\cup F_4$.

For fixed $i\geq 5$, let the ends of F_i be u,v with $v\notin V(D)$. Since H_{i-1} is 2-connected, D is non-separating in H_{i-1} and $\{x,y\}$ is not a 2-cut in H_{i-1}, there exist two disjoint paths L_1,L_2 in H_{i-1} from $\{u,v\}$ to $\{w_1,w_2\}\subseteq A$ and internally disjoint from A such that $w_1\notin V(D)$ and $\{w_1,w_2\}\neq\{x,y\}$. So $Q_i=F_i\cup L_1\cup L_2$ is a (w_1,w_2)-path in H_i containing the ear F_i. By considering all possible locations of w_1,w_2 in A, it can be verified that there exist two disjoint paths X_i,Y_i in H_i from $\{x,y\}$ to $V(D)$ such that one of X_i and Y_i contains Q_i. Since D is odd, this provides an (x,y)-path of each parity in H_i containing F_i for every $5\leq i\leq t$. So we get $t-4$ desired paths. Also observing that $F_1\cup F_2\cup F_3$ contains at least three (x,y)-paths of each parity (not including the edge xy), we see that G has at least $t-1\geq t(G)-1$ desired (x,y)-paths. This completes the proof. ∎

We remark that in Lemma 5.2 if xy is an edge, then G contains at least $t(G)-1$ distinct cycles of each parity passing through xy.

[5] Let s be the vertex in $V(R)\cap V(P_3\cup P_4)$ such that cRs is as short as possible. Similar to the previous footnote (regarding the location of s), we can rename P_3,P_4 accordingly.

6. Proof of Theorem 1.4

We first prove a lemma as follows. Let $\Delta(G)$ denote the maximum degree of a (signed) graph G.

Lemma 6.1. *Let T be a positive integer and G be a 3-connected non-bipartite signed graph such that $t(G) \geq 0.8T$, $f(G) < 0.02T^2$ and $\Delta(G) \leq 0.2T+1$. Let C be any non-separating induced odd cycle in G and let $H = G-C$. Then $e(C,H) \leq 0.2T$ and $t(H) \geq 0.6T$. Moreover, it is not possible for a vertex of degree $0.2T+1$ to be contained in a non-separating induced odd cycle in G.*

Proof. Let $t = t(H)$ and $m = e(C,H)$. So $t(G) = t+m \geq 0.8T$.

First we show $f(G) \geq mt/2$. This holds trivially when $|V(H)| \in \{1,2\}$ (as we have $t=0$). So $|V(H)| \geq 3$. If H is 2-connected, then by Lemma 3.1 we get $f(G) \geq (t+1)m \geq mt/2$. So we may assume that H has $k \geq 2$ end-blocks. Then Lemma 3.3 shows that $f(G) \geq (m-k)(t+k) \geq mt/2$, where the last inequality holds because $m \geq 2k$ and thus $m-k \geq m/2$. This proves $f(G) \geq mt/2$.

Let $C = x_1x_2\cdots x_\ell x_1$ and $d_j = |N_H(x_j)|$. For any two edges $x_ia_i, x_ja_j \in E(C,H)$ with $x_i \neq x_j$, one can find an (a_i,a_j)-path in H. Since C is odd, together with one of the two (x_i,x_j)-paths in C, this provides an odd cycle[6] in G. Thus $f(G) \geq \sum_{i<j} d_id_j$. If $m > 0.6T$, since $\Delta(G) \leq 0.2T+1$ it is easy to divide $V(C)$ into two sets X,Y such that $\sum_{x_i\in X} d_i \geq 0.2T$ and $\sum_{x_j\in Y} d_j \geq 0.2T$. Then by the above, $f(G) \geq (\sum_{x_i\in X} d_i)(\sum_{x_j\in Y} d_j) \geq 0.02T^2$, a contradiction. So we have $m \leq 0.6T$, implying $t = t(G)-m \geq 0.2T$. Since $0.02T^2 > f(G) \geq mt/2$, it follows that $m \leq \frac{0.04T^2}{0.2T} \leq 0.2T$ and thus $t = t(G)-m \geq 0.6T$. Now suppose there exists some vertex u of degree $0.2T+1$. By the condition, one may assume that u is contained in the above non-separating induced odd cycle C. So $d(u) = e(\{u\},H)+2 \leq e(C,H) = m \leq 0.2T$, a contradiction. This completes the proof. ∎

The next result is the core of the proof of Theorem 1.4.

Theorem 6.2. *Let G be a 3-connected non-bipartite signed graph with maximum degree at most $0.2t(G)$. Then $f(G) \geq 0.02t^2(G)$.*

Proof. Throughout this proof, let $T = t(G)$ and $\mathcal{G}_T$ be the family of all 3-connected non-bipartite signed graphs with maximum degree at most $0.2T$. So $G \in \mathcal{G}_T$. We aim to show $f(G) \geq 0.02T^2$.

[6] Recall that such an odd cycle is called *basic* in Section 3.

In the remainder of the proof, we assume that $f(G) < 0.02T^2$. Under this assumption, our plan is to construct a sequence of signed graphs $G_0, G_1, \ldots, G_q$ with the following properties:

(i) $G_i \in \mathcal{G}_T$ for each $i \geq 0$, where $G_0 = G$, and
(ii) for each $i \geq 1$, $f(G_{i-1}) - f(G_i) \geq \frac{1}{2}T \cdot (T_{i-1} - T_i)$ and $1 \leq T_{i-1} - T_i \leq 0.4T$, where $T_i = t(G_i)$.

To do this, we will apply an iterative algorithm as follows: Suppose that for some integer $s \geq 0$, we have constructed signed graphs $G_0, G_1, \ldots, G_s$ which satisfy (i) and (ii). If G_s satisfies either $T_s < 0.8T$, or $T_s \geq 0.8T$ and $f(G_s) \geq 0.02T_i^2$, then we terminate this algorithm. Otherwise, we will construct a new signed graph G_{s+1} which satisfies (i) and (ii). This algorithm will eventually terminate as by (ii), $t(G_i)$ is strictly decreasing as i increases.

Before defining these G_i's, let us show how this desired sequence contradicts our assumption $f(G) < 0.02T^2$ and thus finishes the proof of Theorem 6.2. If this process terminates at G_q when $T_q \geq 0.8T$ and $f(G_q) \geq 0.02T_q^2$, then by (ii) we have

$$f(G) = f(G_q) + \sum_{i=1}^{q} (f(G_{i-1}) - f(G_i)) \geq 0.02T_q^2 + \frac{1}{2}T \cdot (T - T_q) \geq 0.02T^2.$$

Otherwise it terminates when $T_q < 0.8T$, then by (ii) we can also get $f(G) \geq \frac{1}{2}T \cdot (T - T_q) \geq 0.02T^2$.

Now suppose for some $s \geq 0$, we have defined G_i for every i such that $0 \leq i \leq s$, as required. We may assume

$$T_s \geq 0.8T \quad \text{and} \quad f(G_s) < 0.02T_s^2. \tag{1}$$

In the rest of the proof, as we demonstrate, it suffices to define G_{s+1} satisfying (i) and (ii). In steps to construct G_{s+1}, we will define several intermediate signed (multi-)graphs M_ℓ for $0 \leq \ell \leq 3$.[7]

First we construct M_0 from G_s as following. Since $G_s \in \mathcal{G}_T$, by Lemma 2.7 there exists a non-separating induced odd cycle C in G_s. If $|E(C, G_s - C)| \geq 4$, we simply define $M_0 = G_s$. Now consider $|E(C, G_s - C)| = 3$. As G_s is 3-connected and C is induced, we see that C is a triangle say $xyzx$ and $E(C, G_s - C)$ consists of three independent edges say xa, yb, zc. Now let M_0

[7] For a multi-graph M, its *underlying graph* is a simple graph obtained from M by deleting certain edges so that only one edge of each adjacent pair of vertices remains. A signed multi-graph M might have multiple underlying graphs; if so, these differ in the signs on certain edges (which were parallel in M). We say M is k-connected (or bipartite) if and only if its underlying graph is so. For a signed multi-graph M, let $f(M)$ be the number of all distinct odd cycles (of length at least three) in M.

be obtained from G_s by deleting the vertex z, adding two new edges xc, yc, and assigning the parities of xzc, yzc of G_s to xc, yc, respectively. In this case we will also rename C by $xycx$ in M_0.

Claim 1. *M_0 is a 3-connected non-bipartite signed graph with maximum degree at most $0.2T+1$ and there exists a non-separating induced odd cycle C in M_0 such that $|E_{M_0}(C, M_0-C)| \geq 4$, $t(M_0) = T_s$ and $f(G_s) \geq f(M_0)$. Moreover, the only possible vertex of degree $0.2T+1$ belongs to C.*

Proof. This is clear when $M_0 = G_s$. By the definition of M_0, we may assume that there exists an odd cycle $xyzx$ in G_s and $E^* = E(xyz, G_s - xyz)$ consists of three independent edges xa, yb, zc. By (1), $G_s \neq K_4$. If $G_s - xyz$ is not 2-connected, then $G_s - xyz$ either is an edge or has at least two end-blocks; in either case, it implies at least four edges in E^*, a contradiction. So $G_s - xyz$ is 2-connected. Now we see that the cycle $C = xycx$ is a non-separating induced odd cycle in M_0 with $|E(C, M_0 - C)| \geq 4$ (where the oddness follows by the parities of xc, yc). It is also easy to see that M_0 is 3-connected and non-bipartite with maximum degree at most $0.2T+1$ and $t(M_0) = t(G_s) = T_s$, where the only vertex possibly having degree $0.2T+1$ is the vertex $c \in V(C)$.

So it remains to show $f(G_s) \geq f(M_0)$. We prove this by showing an injection from odd cycles in M_0 to odd cycles in G_s. Let D be any odd cycle in M_0. If D contains none of xc, yc, then clearly D is also an odd cycle in G_s. If D only contains one of xc, yc (say xc), then replacing xc with xzc in D gives an odd cycle in G_s. Lastly D contains both xc, yc. Since the parity of xcy is the same as the parity of xzy, replacing xcy with xzy in D gives an odd cycle in G_s. This proves the claim. ∎

Adapting notation from Section 3, let $H = M_0 - C$, $t = t(H)$ and $m = |E_{M_0}(C, H)|$. By (1) and Claim 1, we have $m \geq 4$, $t(M_0) = T_s \geq 0.8T$, $f(M_0) \leq f(G_s) < 0.02T_s^2 \leq 0.02T^2$, and $\Delta(M_0) \leq 0.2T+1$, where the only possible vertex of degree $0.2T+1$ in M_0 belongs to C. Using Lemma 6.1, we can derive the following.

Claim 2. *$m \leq 0.2T$, $t \geq 0.6T$ and $\Delta(M_0) \leq 0.2T$. In particular, we have $M_0 \in \mathcal{G}_T$.*

Let $\mathcal{B}$ be the set of all blocks in H and $t_i = t(B_i)$ for each $B_i \in \mathcal{B}$. Let $\mathcal{T}$ be a fixed spanning tree in H. So the restriction of $\mathcal{T}$ to any block of H is also a tree. For $a, b \in V(H)$, the unique subpath $a\mathcal{T}b$ is called the *(a,b)-skeleton*, while any other (a,b)-path in H is called a *non-skeleton*.

Claim 3. *There exists a unique 2-connected block B_1 in H with $t_1 = t(B_1) > T/2$ and $t - t_1 < 0.1T$.*

Proof. This is clear if H is 2-connected by Claim 2. So assume H is not 2-connected. For any $B_i, B_j \in \mathcal{B}$, there exists a path P in the block structure of H between two end-blocks, say D_1, D_2 in H and passing through D_1, B_i, B_j, D_2 in order (it is possible that $D_1 = B_i$ and/or $D_2 = B_j$). Let the unique cut-vertex of H contained in D_ℓ be c_ℓ for $\ell \in [2]$, and let the two cut-vertices of H incident to B_i (respectively, to B_j) in P be α_i, β_i (respectively, α_j, β_j). Since M_0 is 3-connected, one can easily find two independent edges $x_\ell y_\ell \in E(C,H)$ with $x_\ell \in V(C)$ and $y_\ell \in V(D_\ell) - c_\ell$ for $\ell \in [2]$. By Lemma 2.4, for each $\ell \in \{i,j\}$ there exist t_ℓ non-skeleton $(\alpha_\ell, \beta_\ell)$-paths in B_ℓ. Using these non-skeletons, plus the (y_1, α_i)-, (β_i, α_j)- and (β_j, y_2)-skeletons, one can find $t_i t_j$ distinct (y_1, y_2)-paths in H, each of which yields a basic cycle. So $f(G) \geq f(G_s) \geq f(M_0) \geq \sum_{B_i, B_j \in \mathcal{B}} t_i t_j$. By Proposition 2.5, $t = \sum_{B_i \in \mathcal{B}} t_i \geq 0.6T$. Let t_1 be the maximum of the t_i's. If $t_1 < 0.2T$, then $\{t_i\}$ can be divided into two sets each of which has sum at least $0.2T$, implying that $f(G) \geq 0.04T^2$. So $t_1 \geq 0.2T$. If $t - t_1 \geq 0.1T$, then again $f(G) \geq t_1(t - t_1) \geq 0.02T^2$. This shows $t_1 > t - 0.1T \geq 0.5T$, proving the claim. ∎

Next, we define M_1 to be obtained from the signed subgraph $M_0[B_1 \cup C]$ by adding a new edge xb for every $xa \in E_{M_0}(C, H - B_1)$ with $x \in V(C)$, where $b \in V(B_1)$ is the unique cut-vertex separating a and B_1 in H. Moreover, for every such new edge xb, we let $P_{xb} := xa \cup a\mathcal{T}b$ and let the parity of xb be the parity of P_{xb}. We point out that M_1 is a multi-graph.

Claim 4. *M_1 is a 3-connected non-bipartite signed multi-graph such that $t(M_0) - t(M_1) = t - t_1$ and $f(M_0) - f(M_1) \geq t_1(t - t_1)$.*

Proof. Since M_0 is 3-connected, it is easy to verify that M_1 is 3-connected. By the definition of M_1, we have $|E_{M_1}(B_1, C)| = |E_{M_0}(H, C)| = m$, which, together with Proposition 2.5, implies that $t(M_0) - t(M_1) = t - t_1$. We now show that there exists an injection from odd cycles in M_1 to odd cycles in M_0. Consider any odd cycle D in M_1. If D does not contain any new edge in M_1, then obviously it is an odd cycle in M_0. Suppose D contains new edges in M_1. For a new edge xb which is not incident to any other new edges in D, then we can replace xb by the path P_{xb}. If there exists a pair of new edges xb, yb in D with $x, y \in V(C)$ and $b \in V(B_1)$, then we can replace xby by the symmetric difference of the paths P_{xb} and P_{yb}, which is an (x,y)-path in M_0 internally disjoint from $V(D)$ and has the same parity as xby in M_1. In this way, using the skeletons in H we obtain a unique odd cycle in M_0 from D. This gives the injection ϕ from odd cycles in M_1 to odd cycles in M_0.

Next we show that there are at least $t_1(t - t_1)$ odd cycles in M_0 which are distinct from the image of ϕ. Indeed, for any block $B_i \in \mathcal{B}$ with $i \neq 1$, the proof of Claim 3 provides at least $t_1 t_i$ odd cycles in M_0 which use non-skeleton paths in B_1, B_i and skeleton paths in other blocks. Summing over all such blocks B_i, we prove that $f(M_0) - f(M_1) \geq t_1(t - t_1)$. This finishes the proof of Claim 4. ∎

Let M_2 be obtained from M_1 by contracting the cycle C into a new vertex x^* and keeping all resulting multi-edges. Given a partition $V(C) = X \cup Y$, let $M_{X,Y}$ be obtained from M_1 by contracting X, Y into vertices x, y, respectively, adding one edge xy with parity 1 and keeping all other resulting multi-edges. Since C is induced, it is easy to see that $t(M_2) = t(M_{X,Y}) = t(M_1) - 1$.

Claim 5. *M_2 is 3-connected, and there exist some X, Y with $X \cup Y = V(C)$ such that $M_{X,Y}$ is 3-connected.*

Proof. Suppose that M_2 has a 2-cut $\{u, v\}$. Since M_1 is 3-connected, the only possibility is $x^* \in \{u, v\}$, but this contradicts the 2-connectivity of B_1. So M_2 is 3-connected.

Next we show that $M_{X,Y}$ is 3-connected if both x and y have at least two distinct neighbors in B_1. Suppose there is a 2-cut $\{u, v\}$ in such $M_{X,Y}$. Similarly the only possibility (by symmetry) is that $u \in V(B_1)$ and $v = x$. Since $B_1 - u$ is connected, it implies that y has no neighbor in $B_1 - u$. That is, all neighbors of y belong to $\{u, x\}$, a contradiction.

It suffices to show that there exist some X, Y with $X \cup Y = V(C)$ such that in $M_{X,Y}$ both x and y have at least two distinct neighbors in B_1. If H is not 2-connected, then as in the explanation after Lemma 3.2, one can define two staple edges for each end-block of H in M_0 (possibly including B_1) and thus H has at least four such edges. Using these four edges and by the definition of M_1, it is easy to find such a partition $X \cup Y$ of $V(C)$. Thus H is 2-connected. So $B_1 = H$ and $M_1 = M_0$. By Claim 1, we have $|E_{M_1}(C, B_1)| \geq 4$. In this case, again it is easy to find a desired partition $V(C) = X \cup Y$. This proves Claim 5. ∎

Let M_3 be a signed multi-graph as follows. If M_2 is non-bipartite, then let $M_3 = M_2$; otherwise let M_3 be some 3-connected $M_{X,Y}$ guaranteed by Claim 5. By the definition we see that M_3 is 3-connected with $t(M_3) = t(M_1) - 1$. Next we show that M_3 is also non-bipartite. It is enough to consider when $M_3 = M_{X,Y}$. In this case, M_2 is bipartite, so any cycle in M_2 passing through x^* is even. This also implies that any (x, y)-path in $M_3 = M_{X,Y}$ (except the edge xy) is even. Since the parity of xy in M_3 is one, we see that indeed M_3 is non-bipartite.

Finally, we define G_{s+1} to be an underlying signed graph of M_3 (that is, to form G_{s+1} from M_3, we keep only one edge of each adjacent pair of vertices) that contains at least one odd cycle. Such a non-bipartite underlying graph exists precisely because M_3 is non-bipartite. Let $\alpha = t(M_3) - t(G_{s+1})$, which is the number of edges deleted in this process. Clearly, each of the deleted edges corresponds to an edge in $E_{M_1}(C, B_1)$. Since $|E_{G_{s+1}}(B_1, V(G_{s+1}) \setminus B_1)| \geq 3$ and by Claim 2, we have $\alpha + 3 \leq |E_{M_1}(C, B_1)| = m \leq 0.2T$.

Claim 6. *G_{s+1} is a 3-connected non-bipartite signed graph such that $t(M_1) - t(G_{s+1}) = \alpha + 1$ and $f(M_1) - f(G_{s+1}) \geq t_1(\alpha + 1)$.*

Proof. By definition, it is clear that G_{s+1} is a 3-connected and non-bipartite signed graph such that $t(M_1) - t(G_{s+1}) = \alpha + 1$ and $t(G_{s+1}) \geq t(B_1) = t_1$ (since B_1 does not contain any parallel edges). So it suffices to show that $f(M_1) - f(G_{s+1}) \geq t_1(\alpha + 1)$. Let $\mathcal{F}$ be the family of odd cycles in M_1.

In the rest of this proof, for an edge $e \in E(G_{s+1})$ we say that the corresponding edge in M_1 is the preimage of e. The preimage of a subgraph G' of G_{s+1} is the subgraph of M_1 consisting of all preimages of the edges in G'. Let $u, v \in V(C)$ and P be an (u, v)-path in M_1 that is internally disjoint from C. Then there is a unique way to form an odd cycle in $\mathcal{F}$, by adding one of the two (u, v)-paths in C to P; such an odd cycle is denoted by D_P. For such a cycle D_P, we say its *feature* is 0 if the path P is even and 1 otherwise.

To show $f(M_1) - f(G_{s+1}) \geq t_1(\alpha+1)$, we first give an injection ϕ from the family of all odd cycles in G_{s+1} to $\mathcal{F}$. Let Q be any odd cycle in G_{s+1}. In the case $M_3 = M_2$, if $x^* \notin V(Q)$, then Q is also an odd cycle in M_1; otherwise $x^* \in V(Q)$, then the two edges in Q incident to x^* have the same end in C or different ones (say u, v). In the former case, Q also corresponds to an odd cycle in M_1 (we will view them as one cycle); in the latter case, we let P be the preimage of Q which is an odd path in M_1, and define $\phi(Q) = D_P \in \mathcal{F}$. Now consider the case $M_3 = M_{X,Y}$. Since M_2 is bipartite, all (x, y)-paths in $M_{X,Y}$ (except the edge xy) are even and any odd cycle Q in G_{s+1} must use x and y. In fact such Q must use xy (as otherwise one of the two (x, y)-paths in Q is odd, a contradiction). Then we let P' be the preimage of $Q - xy$ and define $\phi(Q) = D_{P'} \in \mathcal{F}$. This defines the injection ϕ, whose image $\mathrm{Im}(\phi)$ is a subset of $\mathcal{F}$ with $|\mathrm{Im}(\phi)| = f(G_{s+1})$. We point that for any $D \in \mathrm{Im}(\phi)$, either D is an odd cycle in G_{s+1}, or $D = D_P$ for some path P in M_1 which is the preimage of some subgraph (a path or cycle) in G_{s+1}. In the latter case, we also see that if $M_3 = M_2$, then the feature of D is always 1, and if $M_3 = M_{X,Y}$, then P is always a preimage of some (x, y)-path in G_{s+1}.

Now to finish this proof, it is enough to show $|\mathcal{F} \setminus \mathrm{Im}(\phi)| \geq t_1(\alpha+1)$. First we consider any edge $e \in E(M_3) \setminus E(G_{s+1})$, which corresponds to an edge

uv in $E_{M_1}(C, B_1)$ with $u \in V(C)$. Since M_1 is 3-connected, there exists an edge $u'v'$ in $E(M_1)$ with $u' \in V(C) - u$ and $v' \in V(B_1) - v$. We can choose $u'v'$ so that it corresponds to an edge in G_{s+1}. Since B_1 is 2-connected, by Lemma 2.4 there are at least t_1 distinct (v, v')-paths in B_1. For each of these paths, adding the edges $uv, u'v'$ and one of the two (u, u')-paths in C gives an odd cycle in M_1. There are α such edges e, which provides at least $t_1\alpha$ distinct odd cycles in $\mathcal{F}$. Clearly, these odd cycles (say D_P) are distinct from $\mathrm{Im}(\phi)$, because such a path P uses the edge uv and thus cannot be the preimage of any subgraph in G_{s+1}.

It remains to show there are another t_1 odd cycles in M_1 that are distinct from those above. We will prove this by considering the following three cases.

Suppose that the signed graph B_1 is non-bipartite. In this case $M_3 = M_2$. By Lemma 2.7, there exists a non-separating induced odd cycle D in G_{s+1} such that $x^* \notin V(D)$. Since M_1 is also 3-connected, there exist three disjoint paths from D to C in M_1, which yields three internally disjoint paths R_1, R_2, R_3 from D to x^* in G_{s+1}. To apply Lemma 5.1, we define an edge-coloring g, which assigns every edge x^*y in G_{s+1} the color $x_i \in V(C)$, where x_iy is the preimage of x^*y in M_1. Clearly, the three edges of R_1, R_2, R_3 incident to x^* have distinct colors assigned by this g. By Lemma 5.1 (with $G = G_{s+1}$), G_{s+1} contains at least $t(G_{s+1}) \geq t_1$ even cycles passing through x^* such that the two edges incident to x^* in every such cycle have different colors assigned by g. The preimage of every such cycle is an even path P in M_1 with two different ends in C. So we can get at least t_1 odd cycles D_P in M_1. Note that every such D_P has feature 0 (in the case of $M_3 = M_2$). This shows that these odd cycles are distinct from the odd cycles in M_1 found above. So in this case $f(M_1) - f(G_{s+1}) \geq t_1(\alpha + 1)$.

Now suppose that B_1 is bipartite but M_2 is non-bipartite. Again in this case we have $M_3 = M_2$. By Proposition 2.6, there exists a bipartition $V(B_1) = I \cup J$ such that each $e \in E(I, J)$ is odd and each $e \in E(B_1) \setminus E(I, J)$ is even. Since M_1 is 3-connected, there exist three independent edges say x_ia_i in $E_{M_1}(C, B_1)$ with $x_i \in V(C)$ for $i \in [3]$, which correspond to three edges x^*a_i in G_{s+1} for $i \in [3]$. Then we can find two vertices say a_1, a_2 such that either x^*a_1, x^*a_2 have the same parity and a_1, a_2 belong to the same part, or x^*a_1, x^*a_2 have the opposite parity and a_1, a_2 belong to different parts. Since B_1 is 2-connected, by Lemma 2.4 there are t_1 distinct (a_1, a_2)-paths in B_1. By our choice, these paths give at least t_1 even cycles in G_{s+1} passing through x^* (by adding x^*a_1, x^*a_2) and thus at least t_1 odd cycles in M_1 (by adding x_1a_1, x_2a_2 and the unique odd (x_1, x_2)-path of C), which always have feature 0. Again, these odd cycles in M_1 are distinct from those above. Thus $f(M_1) - f(G_{s+1}) \geq t_1(\alpha + 1)$ for this case.

Lastly we consider the case that M_2 is bipartite. Then $M_3 = M_{X,Y}$. As M_1 is 3-connected, there are three independent edges $x_i a_i$ in $E_{M_1}(C, B_1)$ for $i \in [3]$. Now two of them are incident with one of x, y; say they are $xa_1, xa_2 \in E(G_{s+1})$. By Lemma 2.4 there are at least t_1 distinct (a_1, a_2)-paths in B_1. Since M_2 is bipartite, adding xa_1, xa_2 to these paths result in at least t_1 even cycles in G_{s+1} passing through x. On the other hand, adding x_1a_1, x_2a_2 and the unique odd (x_1, x_2)-path in C will give at least t_1 odd cycles (say D_P) in M_1, where P is a preimage of some cycle through x in G_{s+1} in the case $M_3 = M_{X,Y}$. Therefore, these odd cycles are distinct from $\mathrm{Im}(\phi)$ as well as the odd cycles formed from edges in $E(M_3)\setminus E(G_{s+1})$. This completes the proof of Claim 6. ∎

To conclude this proof, we now show that G_{s+1} satisfies the properties (i) and (ii). Let $T_{s+1} = t(G_{s+1})$. Recall that we have $T_s = t(M_0)$ and $f(G_s) \geq f(M_0)$ from Claim 1, $t(M_0) - t(M_1) = t - t_1$ and $f(M_0) - f(M_1) \geq t_1(t - t_1)$ from Claim 4, and $t(M_1) - t(G_{s+1}) = \alpha + 1$ and $f(M_1) - f(G_{s+1}) \geq t_1(\alpha + 1)$ from Claim 6. Combining these together, we get

$$T_s - T_{s+1} = t - t_1 + \alpha + 1$$

and $$f(G_s) - f(G_{s+1}) \geq t_1(t - t_1 + \alpha + 1) = t_1(T_s - T_{s+1}).$$

By Claim 3, $t_1 > \frac{1}{2}T$ and $0 \leq t - t_1 < 0.1T$. We also proved $\alpha + 3 \leq m \leq 0.2T$. Thus it follows that $1 \leq T_s - T_{s+1} = t - t_1 + \alpha + 1 \leq 0.4T$ and $f(G_s) - f(G_{s+1}) \geq \frac{1}{2}T \cdot (T_s - T_{s+1})$. This proves (ii).

To prove (i), it suffices to show that the maximum degree $\Delta(G_{s+1})$ is at most $0.2T$. By Claim 2, $m \leq 0.2T$ and $\Delta(M_0) \leq 0.2T$. So each of the new vertices (either x^*, or x and y) has degree at most $m \leq 0.2T$ in G_{s+1}. In the case $M_3 = M_2$, suppose there exists some $u \in V(B_1)$ with $d_{G_{s+1}}(u) > |N_{M_0}(u) \cap (C \cup B_1)|$. Then u must be a cut-vertex in H and $d_{G_{s+1}}(u) = |N_{M_0}(u) \cap (C \cup B_1)| + 1 \leq d_{M_0}(u) \leq 0.2T$. This shows that $\Delta(G_{s+1}) \leq 0.2T$ when $M_3 = M_2$. Now let us assume $M_3 = M_{X,Y}$. By the above arguments, one can derive that $\Delta(G_{s+1}) \leq 0.2T{+}1$ and if $u \in V(G_{s+1})$ has degree $0.2T{+}1$ in G_{s+1}, then $u \in V(B_1)$ is adjacent to both x and y. Note that in this case M_2 is bipartite, so the parity of the path xuy is even. Since the parity of xy is 1 and B_1 is 2-connected, we see that u is contained in a non-separating induced odd cycle $C' = xuyx$ in G_{s+1}. By (1), we also have $f(G_{s+1}) \leq f(G_s) < 0.02T_s^2 \leq 0.02T^2$. Suppose that $t(G_{s+1}) = T_{s+1} < 0.8T$. Since now we have $f(G_j) - f(G_{j+1}) \geq \frac{1}{2}T(T_j - T_{j+1})$ for all $0 \leq j \leq s$, this implies that $f(G) \geq \frac{1}{2}T(T - T_{s+1}) + f(G_{s+1}) \geq \frac{1}{2}T(0.2T) \geq 0.02T^2$, a contradiction. Hence we may assume that $t(G_{s+1}) \geq 0.8T$. But this contradicts Lemma 6.1. So we can conclude that $\Delta(G_{s+1}) \leq 0.2T$ and thus G_{s+1} satisfies (i). This finishes the proof of Theorem 6.2. ∎

Now we are ready to prove Theorem 1.4.

Theorem 1.4 (Restated). *If G is a 3-connected non-bipartite graph, then $f(G)\geq 0.02t^2(G)$.*

Proof. Let G be a 3-connected non-bipartite graph. If $\Delta(G)\leq 0.2t(G)$, then by Theorem 6.2, we have $f(G)\geq 0.02t^2(G)$. So we may assume that there is a vertex x of degree $d(x)\geq 0.2t(G)+1$. Suppose there exists an odd cycle C in $G-x$. For any distinct $a,b\in N(x)$, as $G-x$ is 2-connected, there are two disjoint paths from $\{a,b\}$ to $u,v\in V(C)$, which together with one of the two (u,v)-paths in C give an odd (a,b)-path in $G-x$. Now adding edges ax and bx to this path provides an odd cycle, say $C(a,b)$ in G, which is distinct over all other pairs of $N(x)$. Also note that we have $d(x)\geq 0.2t(G)+1$. Thus $f(G)\geq \binom{d(x)}{2}\geq \frac{1}{2}(d(x)-1)^2\geq 0.02t^2(G)$.

Now we can assume that $G-x$ is bipartite with parts A,B. Let $T=t(G)$, $t=t(G-x)$, $d_1=|N(x)\cap A|$ and $d_2=|N(x)\cap B|$. Since G is 3-connected and non-bipartite, $G[A\cup B]=G-x$ is 2-connected and we may assume $d_1\geq d_2\geq 1$. This implies that $d_1\geq d(x)/2\geq 0.1T$. By Lemma 2.4 there are at least $t+1$ paths in $G-x$ between any vertex in $N(x)\cap A$ and any vertex in $N(x)\cap B$, all of which have odd lengths. Thus $f(G)\geq d_1d_2(t+1)\geq d_1(d_2+t)$. Note that we have $T+1=d_1+d_2+t$ and $d_1\geq 0.1T$. If $d_2+t\geq d_1$, then $f(G)\geq d_1(d_2+t)\geq 0.09T^2$, as desired. So we may assume that $d_1\geq d_2+t$. By the same analysis, we may further assume that $d_2+t\leq 0.1T$ and $d_1\geq 0.9T$.

So $n-1\geq d(x)\geq d_1\geq 0.9T$. Let B_i be the set of vertices in B of degree i in $G-x$ for $i\geq 2$. Since G is 3-connected, we have $d_2\geq |B_2|$ and $e(A,B)\geq 2|A|$. Also $e(A,B)=\sum_{i\geq 2} i|B_i|$, so

$$\begin{aligned} 2t &\geq 2(e(A,B)-|A|-|B|)\geq e(A,B)-2|B| \\ &=\sum_{i\geq 2} i|B_i| - 2\sum_{i\geq 2}|B_i| = \sum_{i\geq 3}(i-2)|B_i|. \end{aligned}$$

Thus using $2|A|\leq e(A,B)=\sum_{i\geq 2} i|B_i|$, we get $2(|A|-|B|)\leq \sum_{i\geq 3}(i-2)|B_i|\leq 2t$. Now we have

$$2d_2+4t\geq 2|B|=(|A|+|B|)-(|A|-|B|)\geq n-1-t\geq 0.9T-t,$$

which implies that $2d_2+5t\geq 0.9T$, a contradiction to $d_2+t\leq 0.1T$. This proves Theorem 1.4. ∎

7. Proof of Theorem 1.3

Theorem 1.3 (Restated). *If G is a 4-critical graph on n vertices and m edges, then $f(G)\geq 0.02t^2(G)$. Thus $f(G)\geq \Omega(m^2)\geq \Omega(n^2)$.*

Proof. We prove this by induction on the number of vertices. The base case $G = K_4$ is clear. Let G be a 4-critical graph. If G is 3-connected, then this follows by Theorem 1.4. So there exists some 2-cut $\{x,y\}$ in G. By Lemma 2.1, $xy \notin E(G)$ and there are unique proper induced subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{u,v\}$. We choose a 2-cut $\{x,y\}$ such that G_1 has the minimum order among all choices. By the minimality we see that $G_1 + xy$ is 3-connected. By Lemma 2.1 again either (1) $H_1 := G_1 + xy$ and $H_2 := G_2/\{x,y\}$ are 4-critical or (2) $H_1 := G_1/\{x,y\}$ and $H_2 := G_2 + xy$ are 4-critical. In either case, we have $t(H_i) = t(G_i) + 1$ for each $i \in [2]$ and $t(G) + 1 = t(H_1) + t(H_2)$. By induction, $f(H_i) \geq 0.02t^2(H_i)$ for each $i \in [2]$.

Suppose (1) occurs. Fix an (x,y)-path P_1 in G_1 of even length. Any odd cycle in H_2 becomes either an odd cycle or an odd (x,y)-path in G_2. In the latter case, concatenating with P_1 gives an odd cycle in G. So we get $0.02t^2(H_2)$ distinct odd cycles in G from H_2. Also fix an (x,y)-path P_2 in G_2 of odd length (such a path exists by Theorem 2.3). By similar augments, concatenating with P_2 if needed, we get $0.02t^2(H_1)$ odd cycles in G from H_1. Next we combine (x,y)-paths in G_1 and G_2 (but not using P_1, P_2) to get more odd cycles in G. Since $G_1 + xy$ is 3-connected and 4-critical, by Lemma 5.2, there are at least $t(G_1 + xy) - 1 = t(G_1)$ distinct (x,y)-paths (excluding the edge xy) of each parity in $G_1 + xy$ (thus in G_1). By Lemma 2.4, since $G_2 + xy$ is 2-connected, there are at least $t(G_2 + xy) = t(G_2) + 1$ distinct (x,y)-paths (excluding the edge xy) in G_2. Thus for every such path (except P_2) in G_2, there are at least $t(G_1) - 1$ distinct (x,y)-paths (excluding P_1) in G_1 of opposite parity. This yields at least $t(G_2)(t(G_1) - 1)$ odd cycles in G, all of which are distinct from the above ones derived from H_1 and H_2. For each $i \in [2]$, since H_i is 4-critical, we have $t(G_i) + 1 = t(H_i) \geq \frac{|V(H_i)|}{2} + 1 \geq 3$, which implies that $t(G_i) - 1 \geq t(H_i) - 2 \geq \frac{1}{3}t(H_i)$. Adding up all odd cycles we found, we derive that

$$\begin{aligned} f(G) &\geq 0.02t^2(H_1) + 0.02t^2(H_2) + t(G_2)(t(G_1) - 1) \\ &\geq 0.02t^2(H_1) + 0.02t^2(H_2) + \frac{1}{9}t(H_1)t(H_2) \\ &\geq 0.02 \cdot (t(H_1) + t(H_2))^2 \geq 0.02t^2(G), \end{aligned}$$

where the last inequality holds because $t(H_1) + t(H_2) = t(G) + 1$. Now suppose (2) occurs. In this case $H_1 = G_1/\{x,y\}$ is 4-critical. So both $(G_1 + xy) - x$ and $(G_1 + xy) - y$ are non-bipartite. Recall that $G_1 + xy$ is 3-connected. By Lemma 5.2, there are at least $t(G_1 + xy) - 1 = t(G_1)$ distinct (x,y)-paths (excluding the edge xy) of each parity in $G_1 + xy$. By similar analysis as above, we also can derive that $f(G) \geq 0.02t^2(H_1) + 0.02t^2(H_2) + t(G_2)(t(G_1) - 1) \geq 0.02t^2(G)$. This completes the proof of Theorem 1.3. ∎

8. Concluding remarks

In this paper we consider a problem of Gallai from 1984 which asks whether for $k \geq 4$ the number of distinct $(k-1)$-critical subgraphs in any k-critical graph is at least the order of the graph n. For general k, we improve a longstanding lower bound on this number proved by Abbott and Zhou [1], from 1995. In the case $k = 4$ – the main focus of this paper, we show this number is at least $\Omega(n^2)$, which is tight up to the constant factor by infinitely many 4-critical graphs.

Besides the original problem of Gallai, there are many related interesting problems one can ask. One may wonder if Theorem 1.4 can also be extended to the setting of signed graphs. However, unlike Theorem 6.2, the following example shows in negative.

Construction 8.1. *Assume that (A, B) is a bipartition of an even cycle C_{2n}. Let H be obtained from this C_{2n} by adding a vertex x and edges xu for all $u \in A \cup B$. Fix a vertex $b \in B$. Assign 0 to edges xu for all $u \in B - \{b\}$ and assign 1 to all edges in C_{2n} and edges xu for all $u \in A \cup \{b\}$.*

It is not hard to see that H is a 3-connected non-bipartite signed graph, every odd cycle in H passes through the edge xb and thus H contains at most $2t(H)$ odd cycles. This also illustrates that it is necessary to bound the maximum degree in Theorem 6.2.

In Theorem 1.3 we prove that $\min f_3(G) = \Theta(n^2)$, where the minimum is over all n-vertex 4-critical graphs G. This exceeds the original linear bound proposed by Gallai in the case $k = 4$. The following problem seems natural.

Problem 8.2. *Determine the order of magnitude of $\min f_{k-1}(G)$ over all n-vertex k-critical graphs G for all $k \geq 5$.*

It is of particular interest to consider the above minimum for all n-vertex 3-connected k-critical graphs. We are not sure if the additional 3-connectivity condition will change the magnitude of the minimum for $k \geq 5$, which would also be interesting to know. In the case of $k = 4$, we know the additional 3-connectivity condition does not change much, as there are 4-critical n-vertex graphs in both cases (3-connected or not) with $O(n^2)$ distinct odd cycles.

Let $k \geq 4$. We would like to emphasise that all results in this paper on 4-critical graphs can be easily extended to k-critical graphs. The reason is that the only structural property we used for 4-critical graphs is Lemma 2.1, which also holds for all k-critical graphs. For instance, Theorem 1.3 can be restated as: Any n-vertex k-critical graphs G has at least $0.02t^2(G) \geq \Omega(n^2)$ distinct odd cycles. We believe a better bound on the number of odd cycles should hold for $k \geq 5$.

Problem 8.3. *Determine the order of magnitude of the minimum number of distinct odd cycles over all n-vertex k-critical graphs for all $k \geq 5$.*

By considering the odd $(k-3)$-wheels $W(n, k-3)$, we see that this number is $O(n^{2(k-3)})$.

Lastly we point out that the lemmas in Sections 3 and 5 yield the same number of distinct even cycles in those graphs. Hence, one can derive the following for even cycles.

Theorem 8.4. *Let G be a graph which is either 4-critical or 3-connected. Then G contains at least $\Omega(t^2(G))$ distinct even cycles.*

We sketch a proof, as follows. If such G is bipartite, then it holds easily by a recursive use of Lemma 2.4 in any ear-decomposition of G. Otherwise, G is either 3-connected and non-bipartite or else 4-critical. Now the proofs are analogous to those of Theorems 1.4 and 1.3. This bound is also tight up to a constant factor, as shown by (even and odd) wheels $W(n, 1)$, which are 3-connected too.

One can ask for the analog of Problem 8.3 for even cycles as well. For more problems on k-critical graphs, we refer to Chapter 5 of the book [5] by Jensen and Toft.

Acknowledgement. We would like to thank Asaf Shapira for providing counterexamples to some conjectures we posed in an earlier version of this paper. We are grateful to the referees for their helpful comments and suggestions which improve the presentation.

References

[1] H. Abbott and B. Zhou: Some remarks on $(k-1)$-critical subgraphs of k-critical graphs, *Combinatorica* **15** (1995), 469–474.

[2] G. A. Dirac: The structure of k-chromatic graphs, *Fund. Math.* **40** (1953), 42–55.

[3] G. A. Dirac: On the structure of 5- and 6-chromatic abstract graphs, *J. Reine Angew. Math.* **214/215** (1964), 43–52.

[4] D. Hare: Tools for counting odd cycles in graph, *J. Combin. Theory Ser. B* **139** (2019), 163–192.

[5] T. R. Jensen and B. Toft: *Graph Coloring Problems*, Wiley-Intersci. Ser. Discrete Math. Optim., John Wiley & Sons, New York, 1995.

[6] K. Kawarabayashi, B. Reed and O. Lee: Removable cycles in non-bipartite graphs, *J. Combin. Theory Ser. B* **99** (2009), 30–38.
[7] A. Kostochka and M. Yancey: Ore's conjecture for $k=4$ and Grötzsch's theorem, *Combinatorica* **34** (2014), 323–329.
[8] U. Krusenstjerna-Hafstrøm and B.Toft: Special subdivisions of K_4 and 4-chromatic graphs, *Monatsh. Math.* **89** (1980), 101–110.
[9] A. Shapira and R. Thomas: Color-critical graphs have logarithmic circumference, *Adv. Math.* **227** (2011), 2309–2326.
[10] M. Stiebitz: Subgraphs of colour-critical graphs, *Combinatorica* **7** (1987), 303–312.

Jie Ma, Tianchi Yang

School of Mathematical Sciences
University of Science
and Technology of China
Hefei, Anhui 230026, China
{jiema,ytc}@ustc.edu.cn