Decomposing C_4-free graphs under degree constraints

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Abstract

A celebrated theorem of Stiebitz [12] asserts that any graph with minimum degree at least $s + t + 1$ can be partitioned into two parts which induce two subgraphs with minimum degree at least s and t, respectively. This resolved a conjecture of Thomassen. In this paper, we prove that for $s, t \geq 2$, if a graph G contains no cycle of length four and has minimum degree at least $s + t - 1$, then G can be partitioned into two parts which induce two subgraphs with minimum degree at least s and t, respectively. This improves the result of Diwan in [5], where he proved the same statement for graphs of girth at least five. Our proof also works for the case of variable functions, in which the bounds are sharp as showing by some polarity graphs. As a corollary, it follows that any graph containing no cycle of length four with minimum degree at least $k + 1$ contains k vertex-disjoint cycles.

Keywords: feasible partition, degree constraints, Stiebitz's Theorem, C_4-free graphs

1 Introduction

All graphs $G = (V, E)$ considered here are finite and simple. The degree of a vertex v in G is expressed as $d_G(v)$, and for a subset $A \subseteq V$, we denote by $d_A(v)$ the number of vertices in A that are adjacent to v in G. By a *partition* (A, B) of V, we mean that A, B are two disjoint non-empty sets with $A \cup B = V$.

Many problems raised in graph theory concern graph decompositions under certain constraints (for instance, graph coloring problems). Perhaps one of the earliest results regrading graph decompositions under degree constraints is due to Lovász [10] in 1966, who proved that any graph with maximum degree at most $s+t+1$ has a partition (A, B) such that the subgraphs induced on A and B have maximum degree at most s and t, respectively. This was generalized by Borodin and Kostochka [4] to the case of variable functions (the meaning of which will be clear from the contents later).

The counterpart of Lovász' theorem, i.e., graph decompositions under minimum degree constraints, also has received extensive research. Let $f(s, t)$ be the least function such that any graph with minimum degree at least $f(s, t)$ has a partition (A, B) so that the subgraphs induced on A and B have minimum degree at least s and t, respectively. The existence of $f(s, t)$ was proved by Thomassen [15] in 1983, and then this function was subsequently improved by Häggkvist, Alon, and Hajnal [7] (see the discussion in [16]). It was also conjectured by Thomassen [15, 16] that $f(s, t) = s+t+1$, and complete graphs show that this bound would be tight. Later, Stiebitz [12] resolved this conjecture completely. In fact he proved the following stronger result, in the setting of variable functions. Let $\mathbf{N}$ denote the set of non-negative integers.

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Theorem 1. *(Stiebitz [12]) Let G be a graph and $a, b : V(G) \to \mathbf{N}$ be two functions. If $d_G(x) \geq a(x) + b(x) + 1$ for every vertex $x \in V(G)$, then there is a partition (A, B) of $V(G)$ satisfying that $d_A(x) \geq a(x)$ for every $x \in A$, and $d_B(x) \geq b(x)$ for every $x \in B$.*

Kaneko [8] proved that any triangle-free graph with minimum degree at least $s + t$ can already force a partition (A, B) as above. The minimum degree condition was further sharpen by Diwan [5], when cycles of length four are also forbidden. To be precise, Diwan proved that, assuming $s, t \geq 2$, any graph of girth at least five with minimum degree at least $s+t-1$ has a partition (A, B) such that the subgraphs induced on A and B have minimum degree at least s and t, respectively. For related problems on graph decompositions with degree constraints or other variances, we refer readers to [1, 2, 3, 6, 9, 11, 13].

In this paper the following result is proved.

Theorem 2. *Let G be a graph containing no cycles of length four and $a, b : V(G) \to \mathbf{N}_{\geq 2}$ be two functions, where $\mathbf{N}_{\geq 2}$ denotes the set of integers at least two. If*

$$d_G(x) \geq a(x) + b(x) - 1$$

for every vertex $x \in V(G)$, then there is a partition (A, B) of $V(G)$ satisfying that $d_A(x) \geq a(x)$ for every $x \in A$, and $d_B(x) \geq b(x)$ for every $x \in B$.

This is tight in the following two perspectives. First, the ranges of the functions a, b cannot be relaxed to the set of integers at least one by the following example: Take any d-regular connected graph G and the constant functions $a = 1$ and $b = d$; then it is easy to see that none of the partitions (A, B) could satisfy the properties. Second, one also cannot lower the degree condition further by the following proposition.

Proposition 3. *There exist a graph G, which contains no cycle of length four, and two functions $a, b : V(G) \to \mathbf{N}_{\geq 2}$ such that $d_G(x) = a(x) + b(x) - 2$ for every vertex $x \in V(G)$ and moreover, for any partition (A, B) of $V(G)$, there is either a vertex $x \in A$ with $d_A(x) < a(x)$ or a vertex $x \in B$ with $d_B(x) < b(x)$.*

When choosing a, b as constants functions in Theorem 2, it strengthens Diwan's result to graphs containing no cycles of length four, instead of graphs with girth at least five. We state this in a more general version, namely for k-partitions where $k \geq 2$.

Corollary 4. *Let $s_1, ..., s_k \geq 2$ be integers. Any graph containing no cycles of length four with minimum degree at least $s_1 + ... + s_k - (k-1)$ can be partitioned into k parts such that the subgraphs induced on the k parts have minimum degree at least $s_1, ..., s_k$, respectively.*

This also can be used for finding vertex-disjoint cycles in graphs with high minimum degree. It is known (see [14]) that if a graph has girth at least five and minimum degree at least $k + 1$, then it contains k vertex-disjoint cycles. By choosing $s_1, ..., s_k$ to be two in the above statement, we can obtain the following.

Corollary 5. *Any graph containing no cycles of length four with minimum degree at least $k + 1$ contains k vertex-disjoint cycles.*

The rest of the paper is organized as follows. In Section 2, we introduce some notations and show several propositions (including Proposition 3). In Section 3 we complete the proof of Theorem 2.

2 Notations and propositions

Let G be a graph and $f : V(G) \to \mathbf{N}$ be a function. We say that G is *f-degenerate* if for every subgraph H of G there is a vertex u such that $d_H(u) \leq f(v)$. For a subset $A \subseteq V(G)$, we say that A is *f-degenerate* if $G[A]$ is f-degenerate, and it is *f-good* if for every vertex $u \in A$, $d_A(u) \geq f(u)$. A vertex u is called an *f-vertex in A* if $u \in A$ and $d_A(u) = f(u)$. It is immediate from the definitions that

Proposition 6. *A subset A of $V(G)$ does not contain any f-good subset if and only if it is $(f-1)$-degenerate.*

We point out that this fact will be repeatedly used in the coming proofs.

Let $a, b : V(G) \to \mathbf{N}$ be two functions. We call a pair (A, B) of disjoint subsets of $V(G)$ as a *feasible pair* (with respect to a, b) if A is a-good and B is b-good. If in addition (A, B) is a partition of $V(G)$, then we also call it a *feasible partition*. The following nice property was first proved in [12]. We give a proof for the completeness.

Proposition 7. *([12]) Assume that for any $x \in V(G)$, $d_G(x) \geq a(x) + b(x) - 1$. If G has a feasible pair, then G also has a feasible partition.*

Proof. Choose a feasible pair (A, B) in G such that $A \cup B$ is maximal. We show that (A, B) must be a partition. Suppose $C = V(G) \backslash (A \cup B)$ is non-empty. Then any $x \in C$ satisfies that $d_A(x) \leq a(x) - 1$, as otherwise $(A \cup \{x\}, B)$ is also feasible. This implies that for any $x \in C$, $d_{B \cup C}(x) \geq b(x)$ and thus $B \cup C$ is b-good, completing the proof. □

We now prove Proposition 3.

Proof of Proposition 3. One such example is a triangle with constant functions $a = b = 2$. We provide other examples by considering Erdős-Renyi polarity graphs. Let q be a prime and V be a 3-dimensional vector space over $\mathbb{F}_q$. The Erdős-Renyi polarity graph ER_q is a simple graph whose vertices are the 1-dimensional subspaces $[\vec{v}]$ in V, where two vertices $[\vec{v}]$ and $[\vec{w}]$ are adjacent if and only if the vectors $\vec{v}$ and $\vec{w}$ are orthogonal. It is well-known that ER_q contains no cycles of length four.

Let $q = 3$. Then the graph ER_3 has $q^2 + q + 1 = 13$ vertices as well as the following properties. If we let S be the set of vertices of degree 3 in ER_3, then S is an independent set of size 4 and $T := V(ER_3) \backslash S$ consists of all vertices of degree 4. Moreover, the induced subgraph on T is connected and its edges can be partitioned into four edge-disjoint triangles. Choose functions a, b such that a is the constant function three and $b(x) = 2$ if $x \in S$ and $b(x) = 3$ if $x \in T$. Then for every vertex x in ER_3 it holds that $d_{ER_3}(x) = a(x) + b(x) - 2$. It remains to show that there is no feasible partition in ER_3. Suppose for a contradiction that there exists a feasible partition (A, B). We claim that the vertices of any triangle uvw in T must belong to the same part. Indeed, if u is in one part and v, w are in another part, then $u \in T$ has at most two neighbors in its own part, contradicting that (A, B) is feasible with respect to the functions a, b. Since the induced subgraph on T is connected, this implies that T is contained in one part, say A. Then B is just a subset of S, which is an independent set, a contradiction. This completes the proof that ER_3 and the so-defined functions a, b serve as an example to this proposition.

Using similar arguments, one can show that ER_2 (plus some proper functions a, b) is also an example for this proposition. □

In the coming proof, we sometime adopt the notations $u \sim v$ and $u \not\sim v$ to express the situation that the vertices u, v are adjacent or not, respectively.

3 The proof of Theorem 2

Throughout this section, let G be a graph which contains no cycles of length four and $a, b : V(G) \to \mathbf{N}_{\geq 2}$ be two functions such that for any $x \in V(G)$, $d_G(x) \geq a(x) + b(x) - 1$. Suppose for a contradiction that G contains no feasible partitions. By Proposition 7, we have the fact that

$$\text{there is no feasible pairs in } G. \tag{1}$$

Our proof proceeds with a sequence of claims.

Claim 1. It suffices to assume that for any $x \in V(G)$, $d_G(x) = a(x) + b(x) - 1$.

Proof. Indeed we may increase a, b to get functions a', b' such that $a' \geq a$, $b' \geq b$ and $d_G(x) = a'(x) + b'(x) - 1$ for all $x \in V(G)$. Now suppose Theorem 2 holds under the assumption of these inequalities. As $a' \geq a$ and $b' \geq b$, any feasible partition of $V(G)$ with respect to a', b' is also a feasible partition with respect to a, b. This proves Claim 1. □

Definition 1. A partition (A, B) of $V(G)$ is an (a, b)-*partition* if A is $(a-1)$-degenerate and B is $(b-1)$-degenerate.

Claim 2. There exist (a, b)-partitions in G.

Proof. Consider a minimal a-good subset $A \subseteq V(G)$ (note that such subsets exist, as $V(G)$ is one). So $|A| \geq 2$. Let $B = V \backslash A$. Clearly B contains no b-good subsets, as otherwise there exist feasible pairs, contradicting (1). So, by Proposition 6, B is $(b-1)$-degenerate. By the minimality of A, there exists a vertex $x \in A$ with $d_A(x) = a(x)$. By Claim 1, $d_B(x) = b(x) - 1$ and thus $B \cup \{x\}$ is also $(b-1)$-degenerate. On the other hand, $A \backslash \{x\}$ is non-empty and $(a-1)$-degenerate (again by the minimality of A). So $(A \backslash \{x\}, B \cup \{x\})$ gives an (a, b)-partition. □

For any partition (A, B) in G, we define a weight function as following:

$$w(A, B) := E(G[A]) + E(G[B]) + \sum_{x \in A} b(x) + \sum_{x \in B} a(x). \tag{2}$$

Claim 3. For a partition (A, B), let $u \in A, v \in B$ be two vertices such that $d_A(u) = a(u) - \alpha$ and $d_B(v) = b(v) - \beta$. Let $\delta = 1$ if u, v are adjacent and $\delta = 0$ otherwise. Then

$$w(A \backslash \{u\}, B \cup \{u\}) - w(A, B) = 2\alpha - 1,$$

$$w(A \cup \{v\} \backslash \{u\}, B \cup \{u\} \backslash \{v\}) - w(A, B) = 2(\alpha + \beta - 1 - \delta).$$

Proof. This follows directly from the definition and Claim 1. We only show the second identity. Its left hand side equals

$$-d_A(u) - d_B(v) - b(u) - a(v) + (d_B(u) - \delta) + (d_A(v) - \delta) + b(v) + a(u).$$

After simplifying, this gives $2(\alpha + \beta - 1 - \delta)$. □

Definition 2. Let $\mathscr{P}$ be the family consisting of all (a, b)-partitions (A, B), which attain the maximum weight $w(A, B)$ among all (a, b)-partitions in G. For any $(A, B) \in \mathscr{P}$, define

$$A^* = \{x \in A \mid d_A(x) \leq a(x) - 1\} \text{ and } B^* = \{x \in B \mid d_B(x) \leq b(x) - 1\}.$$

It is easy to see that both A^* and B^* are non-empty. So for any $x \in B^*$, we have $|A| \geq d_A(x) \geq a(x) \geq 2$. Hence, we see that both A and B contain at least two vertices.

Claim 4. For any $(A, B) \in \mathscr{P}$, every vertex in A^* is adjacent to every vertex in B^*.

Proof. Suppose that there exist non-adjacent vertices $u \in A^*$ and $v \in B^*$. Let $d_A(u) = a(u) - \alpha$ and $d_B(v) = b(v) - \beta$. So $\alpha, \beta \geq 1$.

First consider $u \in A$. We have $|A| \geq 2$, so $A\backslash\{u\}$ is non-empty. By Claim 3, $w(A\backslash\{u\}, B \cup \{u\}) - w(A, B) = 2\alpha - 1 \geq 1$, thus $(A\backslash\{u\}, B \cup \{u\})$ cannot be an (a, b)-partition. Since $A\backslash\{u\}$ is $(a-1)$-degenerate, this implies that $B \cup \{u\}$ cannot be $(b-1)$-degenerate. Therefore there exists a b-good subset $B' \subseteq B \cup \{u\}$. As u, v are not adjacent, $d_{B\cup\{u\}}(v) = d_B(v) \leq b(v) - 1$, so $B' \subseteq B \cup \{u\}\backslash\{v\}$.

By considering $v \in B$, similarly we can find an a-good subset $A' \subseteq A \cup \{v\}\backslash\{u\}$. Then, (A', B') forms a feasible pair, a contradiction to (1). The proof of Claim 4 is completed. □

Claim 5. For any $(A, B) \in \mathscr{P}$, either A^* or B^* consists of exactly one vertex. Moreover, every vertex in $V(G)\backslash A^*$ is adjacent to at most one vertex in B^* and every vertex in $V(G)\backslash B^*$ is adjacent to at most one vertex in A^*.

Proof. Otherwise there is some C_4 by Claim 4 (note that both A^*, B^* are non-empty). □

Claim 6. For any $(A, B) \in \mathscr{P}$, $u \in A^*$ and $v \in B^*$, we have $d_A(u) = a(u) - 1, d_B(v) = b(v) - 1$ and $(A \cup \{v\}\backslash\{u\}, B \cup \{u\}\backslash\{v\}) \in \mathscr{P}$.

Proof. Let $d_A(u) = a(u) - \alpha$ and $d_B(v) = b(v) - \beta$, where $\alpha, \beta \geq 1$.

We first show that $(A \cup \{v\}\backslash\{u\}, B \cup \{u\}\backslash\{v\})$ is an (a, b)-partition. Suppose not. Without loss of generality, we may assume that there exists a b-good subset $B' \subseteq B \cup \{u\}\backslash\{v\}$. Then we must have $u \in B'$. If $A \cup \{v\}$ is $(a-1)$-degenerate, then $(A \cup \{v\}, B\backslash\{v\})$ is an (a, b)-partition and by Claim 3, $w(A \cup \{v\}, B\backslash\{v\}) - w(A, B) = 2\beta - 1 \geq 1$, a contradiction. Therefore there exists an a-good subset $A' \subseteq A \cup \{v\}$. If $u \notin A'$, then (A', B') is a feasible pair, a contradiction. So $u \in A'$. Then the only possibility is that $d_A(u) = a(u) - 1$. This also shows $d_B(u) = b(u)$ and thus $d_{B\cup\{u\}\backslash\{v\}}(u) = b(u) - 1$, contradicting with $u \in B'$. So indeed $(A \cup \{v\}\backslash\{u\}, B \cup \{u\}\backslash\{v\})$ is an (a, b)-partition.

By Claim 3 again, $w(A \cup \{v\}\backslash\{u\}, B \cup \{u\}\backslash\{v\}) - w(A, B) = 2(\alpha + \beta - 2) \geq 0$. By the maximality of $w(A, B)$, $\alpha = \beta = 1$. So $(A \cup \{v\}\backslash\{u\}, B \cup \{u\}\backslash\{v\}) \in \mathscr{P}$. □

Claim 7. For any $(A, B) \in \mathscr{P}$, $|A\backslash A^*| \geq 2$ and $|B\backslash B^*| \geq 2$.

Proof. By Claims 5 and 6, we may assume $B^* = \{v\}$ and $d_B(v) = b(v) - 1 \geq 1$. Choose any $v_1 \in B\backslash B^*$. Since $d_B(v_1) \geq b(v_1) \geq 2$, there exists a neighbor of v_1 in $B\backslash B^*$. So $|B\backslash B^*| \geq 2$. Similarly, if $|A^*| = 1$, then we also have $|A\backslash A^*| \geq 2$. Assume $|A^*| \geq 2$.

If there exists some vertex $u_1 \in A\backslash A^*$, then $d_A(u_1) \geq a(u_1) \geq 2$. By Claim 5, u_1 has at most one neighbor in A^* and thus at least one neighbor in $A\backslash A^*$, therefore $|A\backslash A^*| \geq 2$.

So we may assume $A = A^* = \{u_1, ..., u_\ell\}$. By Claim 6, $d_A(u_i) = a(u_i) - 1 \geq 1$. This, together with Claim 5, shows that in fact any u_i has exact one neighbour in A and $a(u_i) = 2$. Since all vertices in A are adjacent to v, we see that $A \cup \{v\}$ induces a union of triangles which pairwise intersect at v. As $d_A(v) = a(v)$, $A \cup \{v\}$ is a-good. For any $x \in B\backslash\{v\}$, there is at most one neighbor of x in $A \cup \{v\}$, as otherwise there is a C_4. So $d_{B\backslash\{v\}}(x) \geq a(x) + b(x) - 2 \geq b(x)$. We then find a feasible partition $(A \cup \{v\}, B\backslash\{v\})$. □

Definition 3. For any $(A, B) \in \mathscr{P}$, we define

$$A^\diamond = \{u \in A\backslash A^* \mid d_{A\backslash A^*}(u) \leq a(u) - 1\} \text{ and } B^\diamond = \{v \in B\backslash B^* \mid d_{B\backslash B^*}(v) \leq a(v) - 1\}.$$

Claim 8. For any $(A, B) \in \mathscr{P}$, the subsets $A^\diamond$ and $B^\diamond$ are non-empty. And any $u \in A^\diamond$ has exactly one neighbor in A^* and $d_A(u) = a(u)$; similarly, any $v \in B^\diamond$ has exactly one neighbor in B^* and $d_B(v) = b(v)$.

Proof. Claim 7 shows that $A\backslash A^*$ and $B\backslash B^*$ induce two non-empty subgraphs, which are $(a-1)$-degenerate and $(b-1)$-degenerate, respectively. So $A^\diamond$ and $B^\diamond$ are non-empty. It suffices to consider $u \in A^\diamond$. By Claim 5, u has at most one neighbor in A^* and thus $d_A(u) \le a(u)$. But $u \notin A^*$, which means $d_A(u) \ge a(u)$. This shows that $d_A(u) = a(u)$ and u has exactly one neighbor in A^*. $\square$

Claim 9. For any $(A, B) \in \mathscr{P}$, there exists one of the following five configurations in A (see Figure 1):

(A1) two a-vertices u_1, u_2 in A are adjacent to the same vertex $u \in A^*$,

(A2) two a-vertices u_1, u_2 in A are adjacent to $u, u' \in A^*$, respectively,

(A3) there exist two a-vertices u_1, u_2 in A and a vertex $u \in A^*$ such that $u_1 \sim u_2$, $u_1 \sim u$ and $u_2 \not\sim u$,

(A4) there exist an a-vertex u_1 in A, an $(a+1)$-vertex u_2 in A and a vertex $u \in A^*$ such that u_1, u_2, u form a triangle, and

(A5) there exist an a-vertex u_1 in A, an $(a+1)$-vertex u_2 in A and two vertices $u, u' \in A^*$ such that $u_1 \sim u_2$, $u_1 \sim u$ and $u_2 \sim u'$.

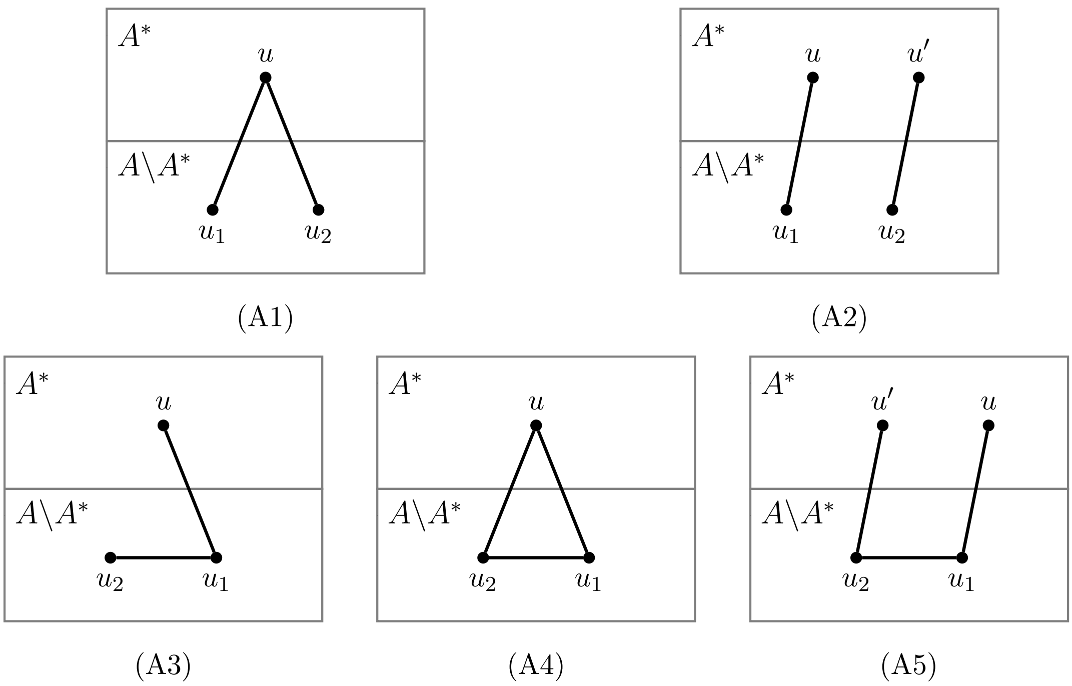


Figure 1: The five configurations in A

And the analog also holds for B (call the five configurations as (B1)-(B5), respectively).

Proof. If $A^\diamond$ has at least two vertices (say u_1, u_2), then by Claim 8, each of u_1, u_2 has exactly one neighbor in A^*. This leads to the configuration (A1) or (A2).

If $A^\diamond$ has exactly one vertex (say u_1), then by Claim 7, $A\backslash(A^* \cup \{u_1\})$ is non-empty and also $(a-1)$-degenerate. Then u_1 has a neighbour $u_2 \in A\backslash(A^* \cup \{u_1\})$ satisfying that $d_{A\backslash A^*}(u_2) = a(u_2)$. This leads to three possible configurations: (A3) when u_2 has no neighbour in A^*, (A4) when u_1, u_2 have the same neighbour in A^*, and (A5) when u_1, u_2 have different neighbours in A^*. This proves Claim 9. $\square$

Definition 4. For any $(A, B) \in \mathscr{P}$, a path $u_1 \sim u \sim v \sim v_1$ is called a *special path*, if $u \in A^*$, $v \in B^*$, u_1 is an a-vertex in A, and v_1 is a b-vertex in B.

Claim 10. For any special path $u_1 \sim u \sim v \sim v_1$, either $u_1 v \in E(G)$ or $v_1 u \in E(G)$.

Proof. Suppose that $u_1 v, v_1 u \notin E(G)$. Let (A', B') be the new partition obtained from (A, B) by exchanging u and v. By Claim 6, $(A', B') \in \mathscr{P}$. Also u_1 becomes an $(a-1)$-vertex in A' and v_1 becomes a $(b-1)$-vertex in B'. Then by Claim 4, we have $u_1 v_1 \in E(G)$. So u_1, u, v, v_1 form a cycle of length four, a contradiction. $\square$

Now let us fix a partition $(A, B) \in \mathscr{P}$. So by Claim 9, there exist two configurations, say (Ai) in A and (Bj) in B. In what follows, we will finish the proof by showing that any combination of (Ai) and (Bj) for all $1 \leq i, j \leq 5$ will derive some contradiction (either finding a cycle of length four or contradicting the above claims).

Take the vertex $u \in A^*$ and the a-vertex u_1 in A from Claim 9; and call the analogous vertices of u, u_1 in B as v, v_1, respectively. Note that in any situation, we have that $u \in A^*, v \in B^*$, u_1 is an a-vertex in A and v_1 is a b-vertex in B. Therefore, $u_1 \sim u \sim v \sim v_1$ is a special path for $(A, B) \in \mathscr{P}$. By Claim 10, we have either $u_1 v \in E(G)$ or $v_1 u \in E(G)$. Without loss of generality,

$$\text{we assume that } u_1 v \in E(G) \text{ and } v_1 u \notin E(G). \tag{3}$$

If the configuration (A4) or (A5) occurs, then (3) will force a C_4, a contradiction. Therefore, there are only 3 configurations left (under the assumption (3)), namely (A1), (A2) or (A3). We distinguish among these three cases.

Case 1: Configuration (A1) occurs.

We see that $u_2 \sim u \sim v \sim v_1$ is a special path. By Claim 10, either $u_2 v \in E(G)$ or $u v_1 \in E(G)$. If $u_2 v \in E(G)$, then u_1, u_2, u, v form a C_4 and if $u v_1 \in E(G)$, then u_1, v_1, u, v form a C_4. This shows that under the assumption (3), (A1) does not occurs.

Case 2: Configuration (A2) occurs.

In this case, we will show that either there exists a C_4 or this can be reduced to the configuration (A3). Note that we have $|A^*| \geq 2$. So $B^* = \{v\}$. So only the configurations (B1), (B3), and (B4) can occur in B.

First suppose that (B3) occurs.[1] Then there is another b-vertex v_2 (other than v_1) in B such that $v_2 \sim v_1$ and $v_2 \not\sim v$. Let (A', B') be the partition obtained from (A, B) by exchanging u and v_1. We may easily infer that $uv_1, uv_2, u_1 v_1, u_1 v_2 \notin E(G)$ (as otherwise there is a C_4). So u_1, v_1 are $(a-1)$-vertices in A', v, v_2 are $(b-1)$-vertices in B', and u is a b-vertex in B'. We claim that $(A', B') \in \mathscr{P}$. We first observe that A' is $(a-1)$-degenerate; otherwise, as A contains no a-good subsets, there must exist an a-good subset in A' which contains v_1, but v_1 is an $(a-1)$-vertex in A', a contradiction. If there exists a b-good subset $B'' \subseteq B'$, then similarly $u \in B''$ and so $d_{B''}(u) = d_{B'}(u) = b(u)$, which shows that all neighbors of u in B' should also belong to B''. But the neighbor v of u is a $(b-1)$-vertex in B', a contradiction. So B' is $(b-1)$-degenerate and thus (A', B') is an (a, b)-partition. By Claim 3, we also have $w(A', B') - w(A, B) = 0$. This proves that $(A', B') \in \mathscr{P}$. Then by Claim 4, u_1, v_1, v, v_2 give a C_4. This shows that (B3) does not occur.

Now we consider when (B1) or (B4) occurs. We claim that all vertices in $A^\diamond$ are adjacent to v. Consider any vertex $w \in A^\diamond \backslash \{u_1\}$ and assume $wv \notin E(G)$. By Claim 8, w is an a-vertex in A and adjacent to exactly one vertex in A^* (say w'). If $w' = u$, then the

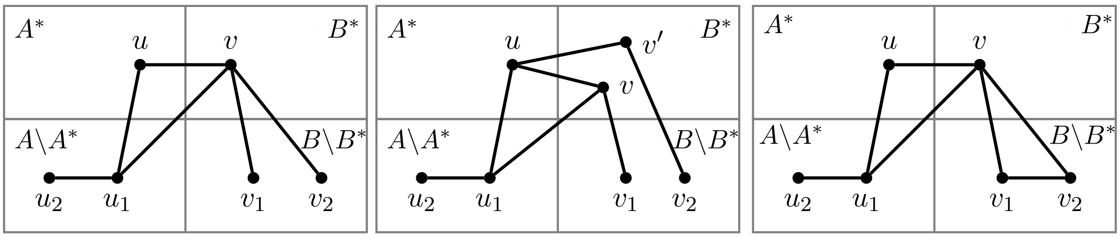
[1] In this paragraph we will only use the vertices $u_1, u \in A$, so this shows that under the assumption (3), (B3) does not occur no matter which configuration is in A.

configuration (A1) occurs. So we have $w' \neq u$. Then the special path $w \sim w' \sim v \sim v_1$ forces either $wv \in E(G)$ or $w'v_1 \in E(G)$. So $w'v_1 \in E(G)$. In (B4), w', v, v_1, v_2 will form a C_4. Now let us consider (B1), where v_2 is a b-vertex in B and $v_2v \in E(G)$. Then $w \sim w' \sim v \sim v_2$ is also a special path. As $wv \notin E(G)$, we must have $w'v_2 \in E(G)$, which again gives a C_4 (with vertices w', v, v_1, v_2). This proves the claim.

We see that all vertices in $A^* \cup A^\diamond$ are adjacent to v and thus any vertex in A has at most one neighbor in $A^* \cup A^\diamond$ (otherwise, there is a C_4). This implies that $A \backslash (A^* \cup A^\diamond) \neq \emptyset$, as otherwise any vertex $w \in A^\diamond$ has $d_{A^* \cup A^\diamond}(w) = d_A(w) = a(w) \geq 2$, a contradiction. Thus, there exists a vertex $x \in A \backslash (A^* \cup A^\diamond)$ with $d_{A \backslash (A^* \cup A^\diamond)}(x) \leq a(x) - 1$. But $d_A(x) \geq a(x)$ (as $x \notin A^*$) and x has at most one neighbor in $A^* \cup A^\diamond$. This shows that x is an a-vertex in A and has exactly one neighbor (say x') in $A^* \cup A^\diamond$. Also as $x \notin A^\diamond$, we have $d_{A \backslash A^*}(x) \geq a(x)$, which shows that $x' \in A^\diamond$. Let x'' be the unique neighbor of x' in A^* (by Claim 8). Now the three vertices x, x', x'' give the configuration (A3) in A. Note that we also have $x'v \in E(G)$ and $v_1x \notin E(G)$ (i.e., the equivalent assumption as (3)). Therefore, it suffices to consider the following case.

Case 3: Configuration (A3) occurs.

There are 5 configurations in B to consider.



(A3)+(B1) (A3)+(B2) (A3)+(B4)

Figure 2: Case 3

Suppose (B1) occurs. So there exists a b-vertex v_2 in B adjacent to v such that $v_2 \neq v_1$. Let (A_1, B_1) be obtained from (A, B) by exchanging u and v_1. One can easily see that $u_1v_1, u_2v_1, u_2u, v_2u \notin E(G)$ (as otherwise there is a C_4). So v_1, u_1 are $(a-1)$-vertices in A_1, u_2 is an a-vertex in A_1, v is a $(b-1)$-vertex in B_1, and u is a b-vertex in B_1. It is worth noting that v_2 may be a $(b-1)$-vertex or b-vertex in B_1, depending on whether $v_1v_2 \in E(G)$ or not, respectively. We claim that $(A_1, B_1) \in \mathscr{P}$. Indeed, A_1 is $(a-1)$-degenerate and B_1 is $(b-1)$-degenerate (via the same argument as in the second paragraph of Case 2). By Claim 3, $w(A_1, B_1) = w(A, B)$, which shows $(A_1, B_1) \in \mathscr{P}$. If v_2 is a $(b-1)$-vertex in B_1, then by Claim 4, u_1, v_1, v, v_2 form a C_4. Otherwise v_2 is a b-vertex in B_1. Then we get a special path $u_2 \sim u_1 \sim v \sim v_2$, implying that u_1v_2 or $u_2v \in E(G)$. In either case, there is a C_4. This shows that (B1) cannot occur.

Suppose (B2) occurs. Then there exist a $(b-1)$-vertex v' and two b-vertices v_1, v_2 in B such that $v' \neq v$ and $v_1v, v_2v' \in E(G)$. Let (A_2, B_2) be obtained from (A, B) by exchanging u_1 and v'. It is easy to see that $u_1v', u_2v', vv', v_1v', v_1u_1, v_2u_1 \notin E(G)$. So u, u_2 are $(a-1)$-vertices in A_2, v' is an a-vertex in A_2, u_1, v_2 are $(b-1)$-vertices in B_2, and v, v_1 are b-vertices in B_2. By Claim 3, we have $w(A_2, B_2) = w(A, B)$. We also see that B_2 is $(b-1)$-degenerate (as any b-good subset of B_2 must contain u_1 but u_1 is a $(b-1)$-vertex in B_2), and A_2 is $(a-1)$-degenerate (because any a-good subset of A_2 must contain v' and all neighbors of v' in A_2, but u, as a neighbor of v', is an $(a-1)$-vertex in A_2, a contradiction). Therefore, $(A_2, B_2) \in \mathscr{P}$. Then by Claim 4, u, u_2, u_1, v_2 form a C_4. This shows that (B2) cannot occur.

By the footnote of Case 2, we have seen that (B3) cannot occur.

Suppose (B4) occurs. There is a $(b+1)$-vertex v_2 in B such that $v_2v, v_2v_1 \in E(G)$. Let (A_4, B_4) be obtained from (A, B) by exchanging u and v and exchanging u_1 and v_1. Since $uv_1, u_1v_1, u_2u, u_2v, u_2v_1, uv_2, u_1v_2 \notin E(G)$, we see that v, u_2 are $(a-1)$-vertices in A_4, v_1 is an a-vertex in A_4, u is a b-vertex in B_4, and u_1, v_2 are $(b-1)$-vertices in B_4. By applying Claim 3 twice, we have $w(A_4, B_4) = w(A, B)$. We claim that $(A_4, B_4) \in \mathscr{P}$. We first show that A_4 is $(a-1)$-degenerate. Suppose not, then A_4 has an a-good subset A' which contains at least one of the new vertices v, v_1. Since $d_{A_4}(v) = a(v) - 1$, this implies that $v_1 \in A'$ and moreover all neighbors of v_1 in A_4 are in A', but this is a contradiction as $v \sim v_1$. Similarly, one can show that B_4 is $(b-1)$-degenerate. Thus $(A_4, B_4) \in \mathscr{P}$. By Claim 4, v, u_2, u_1, v_2 form a C_4. Therefore, (B4) cannot occur.

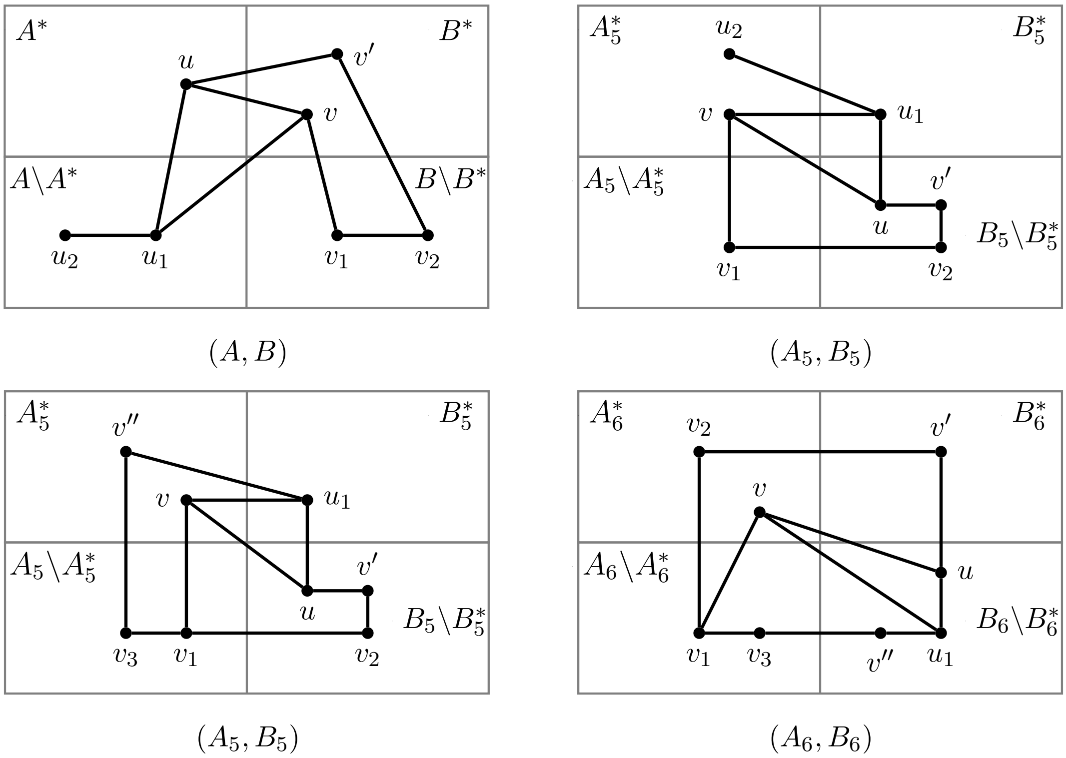


Figure 3: (A3)+(B5)

Finally we assume that (B5) occurs. Then there exist a $(b-1)$-vertex v', a b-vertex v_1, and a $(b+1)$-vertex v_2 in B such that $v_1 \sim v_2, v_1 \sim v$ and $v_2 \sim v'$. Let (A_5, B_5) be obtained from (A, B) by exchanging u and v and exchanging u_1 and v_1. As $u_2u, u_2v, u_2v_1, v'u_1, v'v, v'v_1, v_2u, v_2u_1, v_2v \notin E(G)$, we find that v, u_2 is an $(a-1)$-vertex in A_5, v_1 is an a-vertex in A_5, u_1 is a $(b-1)$-vertex in B_5, and u, v', v_2 are b-vertices in B_5. And by Claim 3, $w(A_5, B_5) = w(A, B)$. Similarly as before, one can show that $(A_5, B_5) \in \mathscr{P}$.

Let us observe that u_1, u, v' give a configuration (B3) in B_5. As $uv \in E(G)$ (where $v \in A_5^*$), by the above proof of Case 3, (A1)-(A4) cannot occur in A_5 (by the symmetry between the functions a and b, here we may view B_5, A_5 as the new parts A, B, respectively). So the configuration (A5) must occur in A_5. Following the proof of Claim 7, we show that the vertices v, v_1 must be involved in this configuration. Indeed, if $A_5^\diamond$ has at least two vertices, then (A1) or (A2) occurs, a contradiction. Thus $A^\diamond$ has exactly one vertex, that is v_1. Then v_1 has a neighbour say v_3 in $A_5 \backslash (A_5^* \cup \{v_1\})$ such that $d_{A_5 \backslash A_5^*}(v_3) = a(v_3)$. Since neither (A3) nor (A4) occur in A_5, v_3 must have a neighbour, say v'', in A_5^* which is distinct from v. Note that v_3 is an $(a+1)$-vertex in A_5 (see Figure 3).

Let (A_6, B_6) be obtained from (A_5, B_5) by exchanging v_2 and v''. Since there is no

C_4 in G, we see that $v_2v'', v_2u, v_2u_1, v_2v, v''u, v''v, v''v', v''v_1 \notin E(G)$. So v' is a $(b-1)$-vertex in B_6, u, u_1, v'' are b-vertices in B_6, v, v_2 are $(a-1)$-vertices in A_6, and v_1 is an $(a+1)$-vertex in A_6. Clearly A_6 is $(a-1)$-degenerate. If B_6 contains a b-good subset S, then $v'' \in S$ and thus $u_1, u, v' \in S$, contradicting that v' is a $(b-1)$-vertex in B_6. So B_6 is $(b-1)$-degenerate. This, together with $w(A_6, B_6) = w(A_5, B_5)$ (by Claim 3), shows that $(A_6, B_6) \in \mathscr{P}$. Then by Claim 4, $v'v_2 \in E(G)$ and thus v', v, u_1, u form a C_4. This contradiction completes the proof of Theorem 2. $\square$

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