A short proof on stability of 4-cycles

Jialin He   Jie Ma   Tianchi Yang

November 2019

In this note we give a self-contained proof of the following result. For more results on related topics, we direct interested readers to [3].

Theorem. Let $q \geq 10^9$ be even and G be a C_4-free (q^2+q+1)-vertex graph with more than $\frac{1}{2}q(q+1)^2 - 0.2q$ edges. Then there exists a unique polarity graph of order q containing G as a subgraph.

Proof. The proof stems from the celebrated work of Füredi and will be completed in three steps.

Step 1. We may assume $\Delta(G) = q+1$. First suppose on the contrary that $\Delta(G) = d(v_1) \geq q+3$. Let T be the number of 2-paths in G with none of its endpoints in $N(v_1)$. Since any two vertices have at most one common neighbor and any two vertices in $N(v_i)$ are contained in a 2-path, we have

$$\binom{q^2+q+1-\Delta}{2} = \binom{n-\Delta}{2} \geq T \geq \sum_{i=2}^{n} \binom{|N(v_i) \setminus N(v_1)|}{2}.$$

Since G is C_4-free, we see $|N(v_i) \setminus N(v_1)| = d(v_i) - d(v_i, v_1) \geq d(v_i) - 1$ for $2 \leq i \leq n$. As $e(G) > \frac{1}{2}q(q+1)^2 - 0.2q$, we have that $\sum_{i=2}^{n} |N(v_i) \setminus N(v_1)| \geq 2e(G) - \Delta - (n-1) \geq q^3 + q^2 - 0.4q - \Delta$. Using Jensen's inequality, we have

$$\binom{q^2+q+1-\Delta}{2} \geq \sum_{i=2}^{n} \binom{|N(v_i) \setminus N(v_1)|}{2} \geq (q^2+q)\binom{\frac{q^3+q^2-0.4q-\Delta}{q^2+q}}{2}.$$

This is equivalent to $(q^2+q)(q^2+q+1-\Delta)(q^2+q-\Delta) \geq (q^3+q^2-0.4q-\Delta)(q^3-1.4q-\Delta)$. As q is large, the above inequality does not hold for $q+3 \leq \Delta \leq q^2+q$. This shows $\Delta(G) \leq q+2$.

Let S_i be the set of all vertices of degree i. Next we show that $|S_{q+2}| \leq 1$. Suppose on the contrary that there are at least two vertices v_1 and v_2 of degree $q+2$. First suppose $N(v_1) \cap N(v_2) = \emptyset$. Then for $2 < i \leq n$, since G is C_4-free, we have

$$|N(v_i) \setminus (N(v_1) \cup N(v_2))| = d(v_i) - d(v_i, v_1) - d(v_i, v_2) \geq d(v_i) - 2.$$

Similarly, by double counting the number of 2-paths with none of its endpoints in $N(v_1) \cup N(v_2)$ and using Jensen's inequality, we have

$$\binom{n-2\Delta}{2} \geq \sum_{i=3}^{n} \binom{|N(v_i) \setminus (N(v_1) \cup N(v_2))|}{2} \geq (n-2)\binom{\frac{\sum_{i=3}^{n}(d(v_i)-2)}{n-2}}{2} = (n-2)\binom{\frac{2e(G)-2\Delta-2n+4}{n-2}}{2}.$$

This is a contradiction as this inequality does not hold. Therefore we may assume $N(v_1) \cap N(v_2) \neq \emptyset$. Let $N(v_1) \cap N(v_2) = \{v_3\}$ and let $A = N(v_3) \setminus \{v_1, v_2\}$. Then we have $|N(v_i) \setminus (N(v_1) \cup N(v_2))| \geq d(v_i) - 1$ for $v_i \in A$, and $|N(v_i) \setminus (N(v_1) \cup N(v_2))| \geq d(v_i) - 2$ for $v_i \notin N(v_3)$. Thus we have

$$\sum_{i=3}^{n} \binom{|N(v_i) \setminus (N(v_1) \cup N(v_2))|}{2} \geq \binom{d(v_3)-2}{2} + \sum_{v_i \in A} \binom{d(v_i)-1}{2} + \sum_{v_j \notin N[v_3]} \binom{d(v_j)-2}{2}$$

By similarly arguments as above, we have

$$\binom{n-2\Delta+1}{2} \geq \sum_{i=3}^{n} \binom{|N(v_i) \setminus (N(v_1) \cup N(v_2))|}{2} \geq \binom{d(v_3)-2}{2} + (n-3)\binom{\frac{2e(G)-2\Delta-2n+4}{n-3}}{2}.$$

The minimum value of the equation on the right is taken when $d(v_3) = 2$, which is still greater than the left hand side, a contradiction. This proves $|S_{q+2}| \le 1$.

Now assume the following holds: If G_0 is C_4-free graphs on q^2+q+1 vertices with at least $\frac{1}{2}q(q+1)^2 - 0.2q$ edges and $\Delta(G_0) = q+1$, then there exists a unique polarity graph of order q containing G_0 as a subgraph. Since $|S_{q+2}| \le 1$, we can delete at most one edge from G to get a subgraph G' with maximum degree $q+1$ and $e(G') \ge e(G) - 1 \ge \frac{1}{2}q(q+1)^2 - 0.2q$. By the above assumption there exists a unique polarity graph H containing G' as a subgraph. Let e be the possible edge in $E(G)\backslash E(G')$. If e does not exist, then $G = G'$ is a subgraph of H. So $e = xy \notin E(H)$. By properties on polarity graphs, $H \cup \{e\}$ contains at least $q-1$ copies of C_4, all of which contain e and are edge-disjoint otherwise. Consider $G' \cup \{e\}$, which is a subgraph of G and thus is C_4-free. Any of these $q-1$ copies of C_4 in $H \cup \{e\}$ has an edge not in $G' \cup \{e\}$, which are distinct. This shows that $e(G') \le e(H) - (q-1) \le \frac{1}{2}q(q+1)^2 - (q-1)$, which contradicts $e(G') \ge \frac{1}{2}q(q+1)^2 - 0.2q$. This proves Step 1.

Therefore in the remaining proof, it is enough to assume that $e(G) \ge \frac{1}{2}q(q+1)^2 - 0.2q$ and $\Delta(G) = q+1$. We let $B = \{x \in V : |N(x) \cap S| \ge 0.1q\}$ and $A = S_{q+1}\backslash B$. Let $\mathcal{R} = \{N(x) : x \in A\}$.

Step 2. We show that $\mathcal{R}$ can be embedded into a projective plane of order q uniquely. Let S be the set of all vertices of degree at most q. We claim that if q is even and $\Delta(G) = q+1$, then any vertex in S_{q+1} has a neighbor in S and moreover, $|S| \ge q+1$. Indeed, suppose on the contrary that there exists some $v \in S_{q+1}$ and all its neighbors have degree $q+1$. Let m be the number of edges contained in $G[N(v)]$. Clearly these edges form a matching (as otherwise there is a C_4) and since q is even, we have $m \le \frac{q}{2}$. We count the number M of edges between $N(v)$ and $V\backslash N(v)$. As G is C_4-free, every vertex in $V\backslash N[v]$ has at most one neighbor in $N(v)$. Hence, we have that $q^2+q = n-(q+2)+(q+1) \ge M = \sum_{x\in N(v)} d(x) - 2m \ge (q+1)^2 - q$, a contradiction. Thus $n - |S| = |S_{q+1}| \le e(S, S_{q+1}) \le \sum_{x\in S} d(x) \le q|S|$. So $(q+1)|S| \ge n = q^2+q+1$, implying that $|S| \ge q+1$. This proves the claim. Moreover, we have

$$q+1 \le |S| \le \sum_{i=0}^{q}(i+1)|S_{q-i}| = f(V) = (q+1)n - 2e(G) \le 1.4q+1 \tag{1}$$

and thus $q^2 - 0.4q \le |S_{q+1}| \le q^2$. For any $T \subseteq S$, it holds that $1.4q+1 \ge f(V) \ge f(T) + (|S| - |T|) \ge f(T) + (q+1-|T|)$. This implies that $f(T) \le |T| + 0.4q$ for any $T \subseteq S$ and in particular,

$$d(x) \ge 0.6q \quad \text{and} \quad d(x)+d(y) \ge 1.6q \quad \text{for any } x, y \in V. \tag{2}$$

Now we show $|B| \le 14$ and $|A| \ge q^2 - 0.4q - 14$. To see this, let t be the number of adjacent ordered pairs (b, v) with $b \in B$ and $v \in S$. We have $|B| \cdot 0.1q \le t \le |S| \cdot q \le 2q^2$, implying that $|B| \le 20q$. Consider the subgraph G_0 of G induced by the set $B \cup S$, where $|B \cup S| \le 22q$. Since G_0 is C_4-free, by Reiman's Theorem, we derive that $\frac{1}{2}|B| \cdot 0.1q \le e(G_0) \le \frac{22q}{4} \cdot 10q^{\frac{1}{2}} = 55q^{3/2}$ and thus $|B| \le 1100\sqrt{q}$. For any $b, b' \in B$, we have $|N_S(b) \cap N_S(b')| \le 1$. By (1) and the inclusion-exclusion principle,

$$1.4q+1 \ge |S| \ge |\cup_{b\in B} N_S(b)| \ge \sum_{b\in B}|N_S(b)| - \sum_{b,b'\in B}|N_S(b) \cap N_S(b')| \ge |B| \cdot 0.1q - \binom{|B|}{2}.$$

Since $|B| \le 1100\sqrt{q}$ and q is large, we further derive $|B| \le 14$. So $|A| \ge |S_{q+1}| - |B| \ge q^2 - 0.4q - 14$.

Next we investigate properties on special vertices of degree $q+1$ defined as follows and then show that $\mathcal{R}$ is 1-intersecting. We say a vertex $v \in V$ has **property 1**, if $v \in S_{q+1}$ satisfies that $|N(v) \cap S_{q+1}| = q$ and $|N(v) \cap S_q| = 1$. Let V_1 denote the set of all vertices of property 1 in G.

We claim $|V_1| \ge 0.6q^2 - 1.8q$. Indeed, for $uv \in E(G)$ with $u \in S$ and $v \in S_{q+1}$, we assign a weight $w(uv)$ to be the deficiency $f(u)$. Let W denote the sum of the weights of these edges. We note that any vertex in V_1 contributes one to the sum W, while any vertex in $S_{q+1}\backslash V_1$ contributes at least two. Hence, by (1) we can derive that

$$|V_1| + 2(|S_{q+1}| - |V_1|) \le W \le \sum_{i=0}^{q}(q-i)(i+1)|S_{q-i}| \le q \cdot f(V) \le q(1.4q+1).$$

Since $|S_{q+1}| \geq q^2 - 0.4q$, we have $|V_1| \geq 2|S_{q+1}| - q(1.4q+1) \geq 0.6q^2 - 1.8q$.

Next we describe the structure of the neighborhood of a vertex in V_1. Suppose $v \in V_1$ has $N(v) = \{v_1, ..., v_{q+1}\}$. Let $N_i = N(v_i) \setminus N[v]$ for $i \in [q+1]$. We show that for $v \in V_1$, the sets $N_1, ..., N_{q+1}$ form a partition of $V \setminus N[v]$, and $G[N(v)]$ consists of a matching of size $\frac{q}{2}$ plus an isolated vertex of degree q. Indeed, assume that the induced graph $G[N(v)]$ contains m edges, which clearly form a matching. Since G is C_4-free, any $x \in V \setminus N[v]$ has at most one neighbor in $N(v)$. Hence

$$(q^2+q+1) - (q+2) + (q+1) \geq \sum_{x \in N(v)} d(x) - 2m = q(q+1) + q - 2m, \tag{3}$$

implying that $m \geq \frac{q}{2}$. Since q is even, we derive that $m = \frac{q}{2}$ and moreover, (3) must be an equality. This further shows that $N_1, ..., N_{q+1}$ form a partition of $V \setminus N[v]$.

Suppose $d(v_{q+1}) = q$. It remains to show that v_{q+1} is an isolated vertex in $G[N(v)]$. Suppose for a contradiction that the edge set of $G[N(v)]$ is $\{v_2v_3, ..., v_qv_{q+1}\}$. Then $|N_1| = q$, $|N_{q+1}| = q-2$ and $|N_i| = q-1$ for $2 \leq i \leq q$. Since G is C_4-free, every $G[N_i]$ contains at most $\lfloor |N_i|/2 \rfloor$ edges and there is no edge between N_{2i} and N_{2i+1} for $1 \leq i \leq \frac{q}{2}$. Also, there are at most $\min\{|N_i|, |N_j|\}$ edges between N_i and N_j for $i, j \in [q+1]$. Thus we have

$$\sum_{x \in N_2} d(x) \leq |N_2| + 2\lfloor |N_2|/2 \rfloor + \min\{|N_1|, |N_2|\} + \sum_{4 \leq i \leq q+1} \min\{|N_2|, |N_i|\} = (q+1)|N_2| - 2.$$

So $f(N_2) = (q+1)|N_2| - \sum_{x \in N_2} d(x) \geq 2$. Similarly, we have $f(N_i) \geq 2$ for all $2 \leq i \leq q-1$. Together with (1), we can obtained $1.4q + 1 \geq f(V) \geq \sum_{2 \leq i \leq q-1} f(N_i) \geq 2q - 4$ which is a contradiction.

The following is a key for constructing a large $(q+1)$-uniform 1-interesting hypergraph. Suppose $v \in V_1$ has $N(v) = \{v_1, ..., v_{q+1}\}$. If $u \in S_{q+1} \setminus N[v]$ is adjacent to $S_{q+1} \cap N(v)$, then we have $|N(u) \cap N(v_i)| = 1$ for all $i \in [q+1]$. (We denote this property by $(\star)$.) To see this, by the above analysis, we assume that $uv_1, v_1v_2 \in E(G)$ for some $v_1, v_2 \in S_{q+1}$. Then u has exactly one neighbor in $N[v]$, no neighbors in N_2 and $N_1, ..., N_{q+1}$ form a partition of $V \setminus N[v]$. Since $u \in S_{q+1}$ has at most one neighbor in each N_i for $i \neq 2$, it follows that u must have exactly one neighbor in each N_i for $i \neq 2$. Since $N(u) \cap N(v_2) = \{v_1\}$, we see that indeed $|N(u) \cap N(v_i)| = 1$ holds for all $i \in [q+1]$.

We then show that the neighborhood of any vertex in A contains many vertices of property 1. To do so, for any $x \in A$ we define

$$S_x = N(x) \cap S \quad \text{and} \quad S_x^* = S_x \cup (N(S_x) \cap N(x)). \tag{4}$$

Since $x \in A$, we have $|S_x| \leq 0.1q$. Every vertex in S_x has at most one neighbor in $N(x)$, so $|S_x^* \setminus S_x| \leq |S_x|$ and thus $|S_x^*| \leq 2|S_x| \leq 0.2q$.

We conclude that for $x \in A$, there are at least $0.3q + 1$ vertices of property 1 in $N(x) \setminus S_x^*$. To show this, let $N(x) = \{x_1, ..., x_{q+1}\}$ and $N_i = N(x_i) \setminus N[x]$ for $i \in [q+1]$. We assert that $f(N_i) \geq 1$ for any $x_i \in N(x) \setminus S_x^*$. Indeed by definition, such $x_i \in S_{q+1}$ and every neighbor of x_i in S must lie outside of $N[x]$ (that is in N_i). Also by the above analysis, x_i has at least one neighbor in S which belongs to N_i. So we have $f(N_i) \geq 1$. From this argument, we also see that $x_i \in N(x) \setminus S_x^*$ has $f(N_i \cup \{x_i\}) = 1$ if and only if $x_i \in V_1$. If we let m be the number of vertices of property 1 in $N(x) \setminus S_x^*$, then we have

$$m + 2(d(x) - |S_x^*| - m) + |S_x| \leq \sum_{i \in [q+1]} f(N_i \cup \{x_i\}) \leq f(V) \leq 1.4q + 1.$$

Using $d(x) = q + 1$ and $2|S_x^*| - |S_x| \leq 0.3q$, we can derive that $m \geq 0.3q + 1$.

Now we are ready to prove that $\mathcal{R}$ is a 1-intersecting $(q+1)$-hypergraph with $|\mathcal{R}| \geq q^2 - 0.4q - 14$. It is clear that $\mathcal{R}$ is $(q+1)$-uniform and $|\mathcal{R}| = |A| \geq q^2 - 0.4q - 14$. So it is enough to show that $\mathcal{R}$ is 1-interesting. Suppose that there exist some $x, y \in A$ with no common neighbor. First consider the case $xy \in E(G)$. By the above analysis, there exists some $z \in N(x) \cap V_1 - \{y\}$. Clearly we have $yz \notin E(G)$. Applying $(\star)$ by viewing z as the vertex v therein, since $y \in S_{q+1} \setminus N[z]$ is adjacent to

$x \in S_{q+1} \cap N(z)$, we can conclude that $|N(y) \cap N(x)| = 1$, a contradiction. Assume that $xy \notin E(G)$. Let $N(x) = \{x_1, ..., x_{q+1}\}$. Let $N_i = N(x_i)\backslash N[x]$ for $i \in [q+1]$ and $Y = V\backslash(N[x] \cup N_1 \cup ... \cup N_{q+1})$. So we have $y \in Y$. Since each x_i has at most one neighbor in $N(x)$, we get that $|Y| \leq n - (q+2) - \sum_{i=1}^{q+1}(d(x_i) - 2) = \sum_{i=1}^{q+1} f(x_i)$. Let $N_1(x)$ be the set of vertices in $N(x)\backslash S_x^*$ of property 1. By above analysis, $|N_1(x)| \geq 0.3q + 1$. Further let $N_2(x) = N(x)\backslash(N_1(x) \cup S_x^*)$. Then, we have $f(N_i) = 1$ for each $x_i \in N_1(x)$ and $f(N_j) \geq 2$ for each $x_j \in N_2(x)$. Thus, we can derive that

$$|Y| \leq \sum_{i=1}^{q+1} f(x_i) = \sum_{x_i \in S_x} f(x_i) \leq 1.4q + 1 - |N_1(x)| - 2|N_2(x)|.$$

Since $N(x) = N_1(x) \cup N_2(x) \cup S_x^*$, we see that the number of neighbors of y in those N_i's with $x_i \in N_1(x)$ is at least $d(y) - (|Y| - 1) - |S_x^*| - |N_2(x)|$, which is at least

$$(q+2) - (1.4q+1) + |N_1(x)| + |N_2(x)| - |S_x^*| \geq 0.2q + 2 \geq 0.1q,$$

where we used the above estimation on $|Y|$ and the facts that $q+1 = |N_1(x)| + |N_2(x)| + |S_x^*|$, $|S_x^*| \leq 0.2q$. Since $|N(y) \cap S| < 0.1q$, among those neighbors of y, there is a vertex $z \in N(y) \cap S_{q+1}$. Suppose that $z \in N_j$ for some $x_j \in N_1(x) \subseteq V_1$. Applying ($\star$) by viewing x_j as the vertex v, since $y \in S_{q+1}\backslash N[x_j]$ is adjacent to $z \in N(x_j) \cap S_{q+1}$, we can derive that y and $x \in N(x_j)$ have a common neighbor. Since G is C_4-free, x and y have exactly one common neighbor. Thus $\mathcal{R}$ is 1-intersecting.

By an embedding result in [4], $\mathcal{R}$ can be embedded into a projective plane $\mathcal{P}$ of order q. Moreover by a result in [1], this embedding is unique. This proves Step 2.

Step 3. There exists a unique polarity graph of order q containing G as a subgraph. Let $\mathcal{R}^c = \mathcal{P}\backslash\mathcal{R}$. We say $v \in V$ is *feasible*, if there exists a line $L \in \mathcal{P}$ with $N(v) \subseteq L$; otherwise, we say v is *non-feasible*. For non-feasible v, we say it is *near-feasible*, if there exist a line $L \in \mathcal{R}^c$ and a subset $K_v \subseteq N(v)$ such that $N(v)\backslash K_v \subseteq L$ and $|K_v| \leq 50\sqrt{q}$. In both definitions, we say v and L are *associated* with each other. For feasible v, we let $K_v = \emptyset$. By (2) and since G is C_4-free, for any two feasible or near-feasible vertices u and v, we have

$$|(N(u)\backslash K_u) \cup (N(v)\backslash K_v)| \geq \big(d(u) - 50\sqrt{q}\big) + \big(d(v) - 50\sqrt{q}\big) - 1 \geq 1.6q - 100\sqrt{q} - 1 > q + 1. \quad (5)$$

This implies that each line in $\mathcal{P}$ is associated with at most one feasible or near-feasible vertex. On the other hand, if there are two lines in $\mathcal{P}$ associated with the same feasible or near-feasible vertex v, as $d(v) \geq 0.6q$ by (2), then it is easy to see that these two lines will intersect with more than two vertices, a contradiction. So each feasible or near-feasible vertex is associated with a unique line in $\mathcal{P}$.

Next we study some properties on non-feasible vertices $v \in V$. Let $N(v) = \{v_1, ..., v_d\}$. Since v is non-feasible, we see $N(v) \not\subseteq L$ for any $L \in \mathcal{P}$ and thus $v \notin A$. Then any pair $\{v_i, v_j\}$ for $i, j \in [d]$ is not contained in any line $N(u) \in \mathcal{R}$. This is because that otherwise, we see that $v_i u v_j v v_i$ forms a C_4 in G, a contradiction. So every such pair $\{v_i, v_j\}$ is contained in a unique line $L \in \mathcal{R}^c$. Let $\mathcal{L}_v$ be the family of lines $L \in \mathcal{P}$ which contains at least two vertices of $N(v)$. Then we have $\mathcal{L}_v \subseteq \mathcal{R}^c$ and thus

$$|\mathcal{L}_v| \leq |\mathcal{R}^c| = |\mathcal{P}| - |\mathcal{R}| \leq 1.4q + 15. \quad (6)$$

We also point out that any vertex in $N(v)$ appears in at least two lines of $\mathcal{L}_v$.

We process to show that all non-feasible vertices are near-feasible in the following. First we show that any vertex $v \in V$ has a neighbor v_j with $d_{\mathcal{R}}(v_j) = |N(v_j) \cap A| \geq q - 16$. In addition, if $v \notin B$ has degree at least $0.9q+43$, then v has a neighbor v_j with $d_{\mathcal{R}}(v_j) \geq q-1$. To see this, let $N(v) = \{v_1, ..., v_d\}$. By (2), we have $d = d(v) \geq 0.6q$. Let $N_i = N(v_i)\backslash N[v]$ for $i \in [d]$. Since the sets $N_i \cup \{v_i\}$ are disjoint over $i \in [d]$, we have $1.4q + 1 \geq f(V) \geq \sum_{i\in[d]} f(N_i \cup \{v_i\}) + f(v) = \sum_{i\in[d]} f(N_i \cup \{v_i\}) + (q+1-d)$. By averaging, there is some $j \in [d]$ with $f(N_j \cup \{v_j\}) \leq \frac{0.4q}{d} + 1 \leq \frac{5}{3}$. By the definition of f, there is some $j \in [d]$ with $f(N_j \cup \{v_j\}) \leq 1$. Therefore, $d_{\mathcal{R}}(v_j) = |N_(v_j) \cap A| \geq |N_j \cap A| \geq |N_j| - |N_j \cap S| - |B| \geq \big(d(v_j) - 2\big) - f(N_j) - 14 = \big(q - 1 - f(v_j)\big) - f(N_j) - 14 \geq q - 16$, as desired.

Next we consider vertices $v \notin B$ with $d = d(v) \geq 0.9q + 43$. Let $B_v = N(v) \cap (S \cup B)$ and $B_v^* = B_v \cup (N(B_v) \cap N(v))$. Then we have $|B_v| \leq |N(v) \cap S| + |B| \leq 0.1q + 14$. Since G is C_4-free, every vertex in B_v has at most one neighbor in $N(v)$, implying that $|B_v^*| \leq 2|B_v|$. Let $T = \{v_i \in N(v)\backslash B_v^* : N_i \cap B = \emptyset\}$. Since N_i's are disjoint and there are at most $|B|$ many N_i's containing some vertex in B, we get $|T| \geq |N(v)\backslash B_v^*| - |B| \geq d - 0.2q - 42$. If $f(N_j) \geq 2$ for all $v_j \in T$, then $1.4q + 1 \geq f(V) \geq 2|T| \geq 2(d - 0.2q - 42) \geq 1.4q + 2$, a contradiction. Therefore, there exists a vertex $v_j \in T$ such that $f(N_j) \leq 1$. By the definition of T, we can see that $d_{\mathcal{R}}(v_j) = d(v_j) - 1 - f(N_j) \geq q - 1$.

Partition V into three disjoint sets $U_1 \cup U_2 \cup U_3$, where U_1 consists of all feasible vertices and U_2 consists of non-feasible vertices $v \notin B$ with $d(v) \geq 0.9q + 43$.

We claim that there exists one vertex $w \in V$ such that all $v \in U_2$ are near-feasible with $K_v = \{w\}$. Indeed, for any $v \in U_2$, by the above property, there is a neighbor v_j of v with $d_{\mathcal{R}}(v_j) \geq q - 1$. By the foregoing discussion, v_j appears in at least two lines in $\mathcal{L}_v \subseteq \mathcal{R}^c$. If $d_{\mathcal{R}}(v_j) \geq q$, then $d_{\mathcal{P}}(v_j) \geq q + 2$, a contradiction. So $d_{\mathcal{R}}(v_j) = q - 1$ and there are exactly two lines, say L_1 and L_2, in $\mathcal{L}_v \subseteq \mathcal{R}^c$ containing v_j. Let $N_1 = L_1 \cap N(v)$ and $N_2 = L_2 \cap N(v)$. Then we have $N_1 \cap N_2 = \{v_j\}$ and $N_1 \cup N_2 = N(v)$. Consider any other line $L_i \in \mathcal{L}_v\backslash\{L_1, L_2\}$ for $i \geq 3$. Set $N_i = L_i \cap N(v)$. We see that for any $i \geq 3$ and $j \in \{1, 2\}$, $|N_i \cap N_j| \leq 1$ and $|N_i \cap N_1| + |N_i \cap N_2| \geq |N_i \cap (N_1 \cup N_2)| = |N_i| \geq 2$. This shows that for any $i \geq 3$, N_i consists of two vertices, one from $N_1\backslash\{v_j\}$ and the other from $N_2\backslash\{v_j\}$. Hence, $|\mathcal{L}_v| = (|N_1| - 1)(|N_2| - 1) + 2$.

Let $d = d(v)$. We may assume that $d - 1 \geq |N_1| \geq |N_2| \geq 2$. If $|N_2| \geq 3$, then we have $|\mathcal{L}_v| = (|N_1| - 1)(|N_2| - 1) + 2 \geq 2(d - 3) + 2 = 2d - 4 \geq 1.8q + 82 > 1.4q + 15 \geq |\mathcal{L}_v|$, where the last inequality holds by (6), a contradiction. Thus, $|N_1| = d - 1$ and $|N_2| = 2$, implying $|\mathcal{L}_v| = d$. Suppose that $N_2 = \{v_j, w\}$. Then every N_i for $2 \leq i \leq d$ contains the vertex w. Also $N(v)\backslash\{w\} \subseteq L_1 \in \mathcal{R}^c$, implying that $v \in U_2$ is near-feasible with $K_v = \{w\}$.

Assume there is another non-feasible vertex $v' \in U_2$ with $K_{v'} = \{w'\}$, where $w' \neq w$. Let $d = d(v)$ and $d' = d(v')$. By the above arguments, we see w and w' appear in $d - 1$ and $d' - 1$ lines in $\mathcal{R}^c$, respectively. By (6), we have $|\mathcal{R}^c| + 2 \leq 1.4q + 17 \leq 1.8q + 84 \leq (d - 1) + (d' - 1)$, which shows that w and w' appear in at least two lines of $\mathcal{R}^c$ in common. This contradicts that $\mathcal{P}$ is a projective plane.

Next we show that all non-feasible vertices are near-feasible. To see this, let $v \in V$ be any non-feasible vertex. We have $d(v) \geq 0.6q$. By the previous property, v has a neighbor u with $d_{\mathcal{R}}(u) = q + 1 - m$, where $m \leq 17$. Let $\mathcal{U} = \{L \in \mathcal{L}_v : u \in L \cap N(v)\}$. We have $|\mathcal{U}| \leq m$ and $\cup_{L \in \mathcal{U}} N_L = N(v)$, where $N_L := L \cap N(v)$. We assert that for all but at most one $L \in \mathcal{U}$, the size of N_L is at most $2\sqrt{q}$. Suppose on the contrary that there are $L_1, L_2 \in \mathcal{U}$ with $|N_{L_1}| \geq 2\sqrt{q} + 1$ and $|N_{L_2}| \geq 2\sqrt{q} + 1$. Then all pairs (x, y) with $x \in N_{L_1}\backslash\{u\}$ and $y \in N_{L_2}\backslash\{u\}$ should appear in distinct lines in $\mathcal{L}_v$. By (6), this shows that $1.4q + 15 \geq |\mathcal{L}_v| \geq (|N_{L_1}| - 1)(|N_{L_2}| - 1) \geq 4q$, a contradiction.

Let L_1 be the line in $\mathcal{U}$ with the maximum N_{L_1} and let $K_v = \bigcup_{L \in \mathcal{U}\backslash\{L_1\}}(N_L\backslash\{u\})$. Then $N(v)\backslash K_v \subseteq L_1 \in \mathcal{R}^c$ with $|K_v| \leq \sum_{L \in \mathcal{U}\backslash\{L_1\}}(|N_L| - 1) \leq (m-1) \cdot 2\sqrt{q} \leq 32\sqrt{q} \leq 50\sqrt{q}$. Therefore, v is near-feasible.

We express $V = \{v_1, ..., v_n\}$ such that $U_1 = \{v_1, ..., v_a\}$, $U_2 = \{v_{a+1}, ..., v_b\}$ and $U_3 = \{v_{b+1}, ..., v_n\}$ for $1 \leq a < b \leq n$. Since all vertices in G are feasible or near-feasible, by the discussion after (5), we can conclude that each $v_i \in V$ is associated with a unique line denoted by L_i in $\mathcal{P}$.

Let $\pi : V \leftrightarrow \mathcal{P}$ be a function which maps $v_i \leftrightarrow L_i$ for every $i \in [n]$. Let $\mathcal{M} = (m_{ij})$ be the incidence matrix of $\mathcal{P}$ with respect to π.

Let $s := |U_3|$. We point out that any $v \in U_3$ either is in B or has $d(v) \leq 0.9q + 42$. In the latter case, we have the deficiency $f(v) = q + 1 - d(v) \geq 0.1q - 41$. Hence by (1), we have

$$s \leq |B| + \frac{f(V)}{0.1q - 41} \leq 14 + \frac{1.4q + 1}{0.1q - 41} \leq 29.$$

Let K be the union of K_v's over all $v \in V$. By the above analysis, we know that $K_v = \emptyset$ for $v \in U_1$, $K_v = \{w\}$ for $v \in U_2$ and $|K_v| \leq 32\sqrt{q}$ for $v \in U_3$. Hence $|K| \leq 1 + s \cdot 32\sqrt{q} \leq 929\sqrt{q}$.

Finally we show that $\mathcal{M}$ is symmetric. Indeed, we assert that if $v_i \in A\backslash K$, then $m_{ij} = m_{ji}$ for all $j \in [n]$. If $m_{ij} = 1$, then as $v_i \in A$, we have $v_j \in L_i = N(v_i) \in \mathcal{R}$. Since $v_i \notin K$, we see $v_i \in N(v_j)\backslash K \subseteq N(v_j)\backslash K_{v_j} \subseteq L_j$, which shows that $m_{ji} = 1 = m_{ij}$. Now we observe that as $v_i \in A$, the i'th column and the i'th row of $\mathcal{M}$ have exactly $q + 1$ many 1-entries, and all these 1-entries are in the symmetric positions. This shows that the i'th column and the i'th row are symmetric, proving the assertion. Since $|A\backslash K| \geq |A| - |K| \geq (q^2 - 0.4q - 14) - 929\sqrt{q} \geq q^2 - q + 3$, by a lemma in [2] (its Lemma 3.7), the whole matrix $\mathcal{M}$ is symmetric.

Hence we see that the above function $\pi : V \leftrightarrow \mathcal{P}$ is a polarity of the projective plane $\mathcal{P}$. Let H be the polarity graph of π. For any $k \times \ell$ matrices $\mathcal{X} = (x_{ij})$ and $\mathcal{Y} = (y_{ij})$, we say $\mathcal{X}$ *is at most* $\mathcal{Y}$ if $x_{ij} \leq y_{ij}$ for all i, j and we express this by $\mathcal{X} \leq \mathcal{Y}$.

Now we are going to finish the proof by showing that G is a subgraph of H. Let $\mathcal{A} = (a_{ij})$ be the adjacent matrix of the graph G. It suffices to shows that $\mathcal{A} \leq \mathcal{M}$. We call these (i, j)-entries with $a_{ij} = 1$ and $m_{ij} = 0$ *problematic*. Since both $\mathcal{A}$ and $\mathcal{M}$ are 0/1 matrices, it is equivalent for us to show that there is no problematic entries.

For every $v_i \in U_1$, as it is feasible, we see that $N(v_i) \subseteq L_i$ and thus the i'th row of $\mathcal{A}$ is at most the i'th row of $\mathcal{M}$. Since both $\mathcal{A}$ and $\mathcal{M}$ are symmetric, the i'th column of $\mathcal{A}$ is also at most the i'th column of $\mathcal{M}$, whenever $v_i \in U_1$. Now consider vertices $v_i \in U_2$. By the above discusion, $N(v_i)\backslash\{w\} \subseteq L_i$, where $w = v_\ell$ is fixed. Consider $a_{ij} = 1$ for possible j which is not ℓ. Then we have $v_j \in N(v_i)\backslash\{w\} \subseteq L_i$. This shows that the i'th row of $\mathcal{A}$ is at most the i'th row of $\mathcal{M}$, except the (i, ℓ)-entry. By symmetry, we see that for all $v_i \in U_2$, the i'th column of $\mathcal{A}$ is at most the i'th column of $\mathcal{M}$, except the possible (ℓ, i)-entry. We also know w is feasible or near-feasible. So $|K_w| \leq 50\sqrt{q}$ and the number of problematic (ℓ, i)-entries is clearly at most $|K_w| \leq 50\sqrt{q}$. This further shows that the number of problematic (i, j)- or (j, i)-entries for all $v_i \in U_2$ is at most $100\sqrt{q}$. Note that $|U_3| = s$ is at most 29. Putting all the above together, we see that the number of problematic (i, j)-entries for $i, j \in [n]$ is at most $100\sqrt{q} + 29^2 \leq 101\sqrt{q}$.

Let E_0 be the set of v_iv_j for all problematic (i, j)-entries. It is easy to see that $E_0 = E(G)\backslash E(H)$ and $|E_0| \leq 101\sqrt{q}$. Suppose that there is some edge say $e = v_iv_j \in E_0$. By the polarity lemma , $H \cup \{e\}$ contains at least $q - 1$ copies of C_4, all of which contain the edge e and are edge-disjoint otherwise. Hence in order to turn $H \cup \{e\}$ into a subgraph of G containing e (which is C_4-free), one needs to delete at least $q - 1$ edges in $H \cup \{e\}$. On the other hand, since H is a polarity graph, we have $e(H) \leq \frac{1}{2}q(q+1)^2$ and $|E(H)\backslash E(G)| - |E_0| = e(H) - e(G) \leq 0.2q$. So one can delete $|E(H)\backslash E(G)| \leq 0.2q + |E_0| \leq 0.2q + 101\sqrt{q} < q - 1$ edges to turn $H \cup \{e\}$ into a subgraph of G while preserving the edge e. This is a contradiction. Therefore, $E_0 = \emptyset$ and G is a subgraph of H.

It only remains to show that the polarity graph H is unique. Recall that the projective plane $\mathcal{P}$ containing $\mathcal{R}$ has been shown to be unique. So it is equivalent to show that the polarity π is unique. Suppose for a contradiction that there exists another polarity $\pi' : V \leftrightarrow \mathcal{P}$, where $\pi' : v_i \leftrightarrow L_{\sigma(i)}$ for some permutation σ on $[n]$. Let $\mathcal{M}' = (m'_{ij})$ be the incidence matrix of $\mathcal{P}$ with respect to π'. By the same proof as above, we can deduce that $\mathcal{A} \leq \mathcal{M}'$. By (2), we see that any vertex $v_i \in V$ has degree at least $0.6q \geq 2$. Choose any pair $\{x_i, y_i\} \subseteq N(v_i)$. Since the i'th row of $\mathcal{A}$ is at most the i'th row of $\mathcal{M}'$, we see $\{x_i, y_i\} \subseteq N(v_i) \subseteq L_{\sigma(i)} \in \mathcal{P}$. Also we have $\{x_i, y_i\} \subseteq N(v_i) \subseteq L_i \in \mathcal{P}$. Since $\mathcal{P}$ is a projective plane, it is clear that $L_{\sigma(i)} = L_i$ for all $i \in [n]$. This shows that $\pi = \pi'$ and indeed the polarity graph H is unique. The proof of the result in this note is completed. □

References

[1] S. Dow, An improved bound for extending partial projective planes, *Discrete Math.* **45** (1983), 199–207.

[2] Z. Füredi, Quadrilateral-free graphs with maximum number of edges, preprint 1988, `http://www.math.uiuc.edu/~z-furedi/PUBS/furedi_C4from1988.pdf`

[3] J. He, J. Ma and T. Yang, Stability and supersaturation of 4-cycles, arXiv.1912.00986.

[4] K. Metsch, Linear Spaces with Few Lines, *Springer Lecture Notes in Math.*, 1991.