Non-repeated cycle lengths and Sidon sequences

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Abstract

We prove a conjecture of Boros, Caro, Füredi and Yuster on the maximum number of edges in a 2-connected graph without repeated cycle lengths, which is a restricted version of a longstanding problem of Erdős. Our proof together with the matched lower bound construction of Boros, Caro, Füredi and Yuster show that this problem can be conceptually reduced to the seminal problem of finding the maximum Sidon sequences in number theory.

1 Introduction

An old problem of Erdős since 1975 (see [1], p. 247, Problem 11) asks to determine the maximum number $n + f(n)$ of edges in an n-vertex graph in which no two cycles have the same length. An early result of Shi [9] gives that $f(n) \geq \lfloor(\sqrt{8n-15}-3)/2\rfloor$, with equality for $2 \leq n \leq 16$. Since then Lai has obtained a series of sequential improvements on the lower bound (see [2, 6] for details), including the current record [6] that $f(n) \geq \sqrt{238n/99} \approx 1.55\sqrt{n}$. For the upper bound, Lai [5] proved $f(n) = O(\sqrt{n \log n})$, which was later reproved in [3]. In a breakthrough result, Boros, Caro, Füredi and Yuster [2] deduced $f(n) \leq 1.98\sqrt{n}$ from the minimum cover of non-uniform hypergraphs, and thus established the order of the magnitude of $f(n)$ to be $\Theta(\sqrt{n})$. It remains open to determine $f(n)$, even asymptotically.

Another interesting problem is to consider the restricted version of Erdős' problem for 2-connected graphs. Following the notation in [2], let $n + f_2(n)$ be the maximum number of edges in an n-vertex 2-connected graph in which no two cycles have the same length. In 1988, employing the standard ear-decomposition of 2-connected graphs, Shi [9] proved $f_2(n) \leq \sqrt{2n} + o(\sqrt{n})$. In [3] Chen, Lehel, Jacobson and Shreve revisited this upper bound and used it to derive $f(n) = O(\sqrt{n \log n})$. Using Sidon sequences in number theory, they [3] also showed that $f_2(n) \geq \sqrt{n/2} - o(\sqrt{n})$. A sequence of integers $a_1, a_2, ..., a_k$ is called a *Sidon sequence* if all pairwise sums $a_i + a_j$ for $1 \leq i \leq j \leq k$ are distinct. Let $b_2(n)$ denote the maximum size of a Sidon subsequence of $\{1, 2, ..., n\}$. It is well known that $b_2(n) = \sqrt{n} + o(\sqrt{n})$, where the upper bound was proved by Erdős and Turán in their celebrated paper [4] (later simplified in [7]) and the lower bound was provided by Singer [10]. Boros, Caro, Füredi and Yuster [2] refined the use of Sidon sequences and made a significant improvement on the lower bound of [3] by showing that

$$f_2(n) \geq \sqrt{n} - o(\sqrt{n}). \tag{1}$$

To illustrate this somehow surprised relation between $f_2(n)$ and Sidon sequences, we now give a sketch for the proof of (1). Utilizing the result of Singer [10] (together with Erdős-Turán Theorem [4]), it is demonstrated in [2] that for any integer $n > 0$, there exist integers $a_1 = 1 < a_2 < ... < a_k = n-1$ such

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that $k = \sqrt{n} - O(n^{9/20})$ and all differences $a_j - a_i$ for $1 \leq i < j \leq k$ are pairwise distinct. Construct an n-vertex 2-connected graph G as follows: let $V(G) = \{v_0, v_1, ..., v_{n-1}\}$ and $E(G)$ consist of the edges in a Hamilton cycle $C = v_0 v_1 ... v_{n-1} v_0$ and the edges $v_0 v_{a_i}$ for all $1 < i < k$. It is easy to see that each cycle in G contains two edges incident to v_0 (say $v_0 v_{a_i}$ and $v_0 v_{a_j}$) and the subpath of C between v_{a_i} and v_{a_j} not containing v_0. So all cycle lengths in G are of the form $a_j - a_i + 2$ for $1 \leq i < j \leq k$, which are pairwise distinct. This proves (1) that $f_2(n) \geq e(G) - n = k - 2 = \sqrt{n} - o(\sqrt{n})$.

The authors of [2] further conjectured that the lower bound (1) is asymptotically tight.

Conjecture 1.1 (Boros, Caro, Füredi and Yuster, Conjecture 5.3 in [2])**.**

$$\lim_{n \to \infty} f_2(n)/\sqrt{n} = 1.$$

As remarked in [2], this would imply "the (difficult) upper bound in the Erdős Turán Theorem" on Sidon sequences. A weaker question was raised in [8] to determine the maximum number of edges in a hamiltonian graph with no two cycles of the same length.[1]

Our main result in this paper is to give a proof of Conjecture 1.1 by the following.

Theorem 1.2. *Any n-vertex 2-connected graph with no two cycles of the same length contains at most $n + \sqrt{n} + o(\sqrt{n})$ edges.*

We introduce some notation. Let G be a graph. For a subset A of edges (or vertices) in G, let $G \backslash A$ be the graph obtained from G by deleting the elements in A. Let P be a path with endpoints x, y. We say that P is an (x, y)-path and any vertex or edge in $P \backslash \{x, y\}$ is *inner*. For a tree T with $x, y \in V(T)$, we denote xTy to be the unique subpath in T between x and y. Suppose F, F' are subgraphs of G. By $F \triangle F'$ we denote the subgraph consisting of the edges which appears in exactly one of F and F', and by $F \backslash F'$ we denote the subgraph consisting of the edges in F but not in F'. An F-*ear* in G is a path in G whose two endpoints lie in F but whose inner vertices do not. An *ear-decomposition* of G is a nested sequence $(G_0, G_1, ..., G_s)$ of subgraphs of G such that G_0 is a cycle, $G_{i+1} = G_i \cup P_{i+1}$ where P_{i+1} is a G_i-ear in G for $0 \leq i < s$, and $G_s = G$. It is well known that a graph G is 2-connected if and only of it has an ear-decomposition. Let us point out that any ear-decomposition of G has $s = |E(G)| - |V(G)|$. Throughout this paper, let $[n] = \{1, 2, ..., n\}$ and all logarithms in this paper are binary (with base 2).

We organize this paper as follows. In Section 2, we set up our proof environment by defining an ear-decomposition associated with a special linear ordering of vertices and then using it to construct a family $\mathcal{F}$ of paths which will serve as the building blocks to generate cycles later. In Section 3, we prove some preliminary propositions on the paths in $\mathcal{F}$ and classify all pairs of $\mathcal{F}$ in three types. In Section 4, we prove Lemma 4.1, which gives a structural description on $\mathcal{F}$ (very loosely speaking, it shows that almost all pairs of $\mathcal{F}$ form a similar local structure). In Section 5, we prove Lemma 5.1, which roughly says that one can reorder the paths of $\mathcal{F}$ in a nice way such that for almost every edge e, the paths containing e are listed almost consecutively. In Section 6, we complete the proof of Theorem 1.2. In the final section, we conclude this paper by mentioning some remarks.

2 Basic setting

Throughout the rest of the paper, let G be an n-vertex 2-connected graph with $n + s$ edges, where n is sufficiently large and $s \geq (1 + o(1))\sqrt{n}$. Our ultimate goal is to show that G contains two cycles of

[1] Note that a hamiltonian graph is naturally 2-connected.

the same length. To this end, we assume in the rest of the paper that G contains at most one cycle of length i for each $3 \leq i \leq n$ and thus in particular,

$$G \text{ contain at most } n-2 \text{ cycles.} \tag{2}$$

To begin with, we define an ear-decomposition $(G_0, G_1, ..., G_s)$ of G and a linear order $\prec$ of $V(G)$ using the following iterated procedure. (This will be crucial for all the coming proofs.)

(i) Fix an edge $uv \in E(G)$ and let G_0 be any cycle in G containing uv. Let $P_0 = ux_1 \cdots x_a v$ be the path $G_0 \backslash \{uv\}$. We define a linear order on $V(G_0)$ by letting $u \prec x_1 \prec \cdots \prec x_a \prec v$.

(ii) Now suppose we have defined G_{i-1} and a linear order $\prec$ on $V(G_{i-1})$ for some $1 \leq i \leq s$. Among all choices of G_{i-1}-ears in G, let P_i be a G_{i-1}-ear with endpoints $\ell_i, r_i \in V(G_i)$ such that ℓ_i is minimum under $\prec$ of $V(G_{i-1})$ and subject to this, r_i is minimum under $\prec$ of $V(G_{i-1})$.[2] Let $G_i = G_{i-1} \cup P_i$. Write $P_i = \ell_i y_1 \cdots y_b r_i$ and let ℓ_i^+ be the vertex of G_{i-1} that succeeds ℓ_i immediately in the linear order $\prec$. We extend the linear order $\prec$ on $V(G_{i-1})$ to $V(G_i)$ by inserting all vertices y_j with $j \in [b]$ between ℓ_i and ℓ_i^+ such that $\ell_i \prec y_1 \prec \cdots \prec y_b \prec \ell_i^+$.

Using this ear-decomposition, we define

$$L = P_0 \cup \big(\cup_{i\in[s]} P_i \backslash \{r_i\}\big) \quad \text{ and } \quad R = P_0 \cup \big(\cup_{i\in[s]} P_i \backslash \{\ell_i\}\big). \tag{3}$$

It is easy to see that L and R are two spanning trees in G, and we will view u as the root of L and v as the root of R. Now we define a family of (u, v)-paths as following:

$$\text{Let } f_0 = P_0 \text{ and for } i \in [s], \text{ let } L_i = uL\ell_i,\ R_i = r_i R v \text{ and } f_i = L_i \cup P_i \cup R_i.$$

Let $\mathcal{F} = \{f_i : 0 \leq i \leq s\}$. These paths will be used to generate cycles in coming proofs.

3 Preliminaries on $\mathcal{F}$

In this section, we prove some basic propositions about the paths in $\mathcal{F}$. The first one can be derived directly from the above definitions.

Proposition 1. *Let $x, y \in V(G)$ and let i, j be the minimum indices such that $x \in V(P_i)$ and $y \in V(P_j)$. If $x \in uLy$ or $x \in yRv$, then $i \leq j$.*

Proposition 2. *For any $i, j \in [s]$, $L_i \cup R_j$ does not contain cycles.*

Proof. Suppose there is a cycle C in $L_i \cup R_j$. Then there must exist two vertices a, b such that $C = aL_ib \cup aR_jb$. Let $a \prec b$ and let k and ℓ be the minimum indices satisfying $a \in V(P_k)$ and $b \in V(P_\ell)$. Clearly we have $a \in uLb$ and then Proposition 1 implies $k \leq \ell$. But we also have $b \in aRv$ and by Proposition 1 again, we derive $\ell \leq k$. So $k = \ell$ and a, b are two inner vertices of P_k. This shows that $aL_ib = aLb = aP_kb = aRb = aR_jb$, a contradiction. □

Proposition 3. *For distinct $i, j \in \{0, 1, ..., s\}$, we have $P_i \not\subseteq f_j$.*

Proof. Suppose that P_i is a subpath of f_j for some $i \neq j$. Clearly we may assume $i, j \in [s]$. Since P_i, P_j have no common edges, it follows that either $P_i \subseteq L_j$ or $P_i \subseteq R_j$. Now we note that P_i is not a subpath of the tree L (or respectively R), but L_j (or respectively R_j) is, a contradiction. □

[2]Note that by this choice, we have $\ell_i \prec r_i$ for each $i \in [s]$.

The following proposition is also straightforward to see (we omit its proof here).

Proposition 4. *For any $i \in \{0, 1, ..., s\}$, f_i is a (u, v)-path, whose vertices, as traversing from u to v, increase in the linear order $\prec$.*[3]

Let $i \neq j \in \{0, 1, ..., s\}$. A vertex in $f_i \cup f_j$ is called **splitting** if it has at least three neighbors in $f_i \cup f_j$. By Proposition 4, there exists some integer $t \geq 1$ such that $f_i \triangle f_j$ consists of t cycles $a_\ell P_i b_\ell \cup a_\ell P_j b_\ell$ for $\ell \in [t]$, where $a_1 \prec b_1 \preceq a_2 \prec \cdots \prec b_{t-1} \preceq a_t \prec b_t$ are all splitting vertices of $f_i \cup f_j$.

Proposition 5. *For distinct $i, j \in \{0, 1, ..., s\}$, $f_i \triangle f_j$ consists of one or two cycles such that each cycle shares edges with $P_i \cup P_j$ and each of P_i and P_j shares edges with at most one cycle in $f_i \triangle f_j$.*

Proof. First we show that each cycle C in $f_i \triangle f_j$ contains some edges in $P_i \cup P_j$. Suppose not. Then we have $E(C) \subseteq E(L_i \cup R_i \cup L_j \cup R_j)$. Let us assume $i < j$ here. By the choice of ears, we see $\ell_i \preceq \ell_j$. If there exists $x \in V(L_i) \cap V(R_j)$, then $\ell_j \prec r_j \preceq x \preceq \ell_i$, a contradiction. So $V(L_i) \cap V(R_j) = \emptyset$. If C has an edge in L_i, then L_i has at least two splitting vertices of $f_i \cup f_j$. But $L_i \cup L_j$ has at most one splitting vertex, while L_i and $R_i \cup R_j$ have no common vertex, a contradiction. So C has no edge in L_i and similarly we can show C has no edge in R_j. Thus C is contained in $R_i \cup L_j$, a contradiction to Proposition 2.

Next we show each of P_i and P_j shares edges with at most one cycle in $f_i \triangle f_j$. Suppose for a contradiction that P_i shares edges with two cycles in $f_i \triangle f_j$. Then there exists a common vertex x in f_i and f_j between these two cycles. We see that x is an inner vertex of P_i and thus cannot be an inner vertex of P_j. So $x \in V(L_j) \cup V(R_j)$. If $x \in V(L_j)$, then both of uf_ix and uL_jx are in L and thus cannot contain cycles, a contradiction. If $x \in V(R_j)$, then $xf_iv \cup xR_jv \subseteq R$, a contradiction. Combining with these two facts, it is easy to see that $f_i \triangle f_j$ consists of at most two cycles. □

Proposition 6. *Suppose $f_i \triangle f_j$ consists of two cycles, where $i < j$. Let $a \prec b \preceq c \prec d$ be all splitting vertices of $f_i \cup f_j$. Then $P_i \subseteq af_ib$, $P_j \subseteq cf_jd$ and there exists some $k < i$ such that $f_k = uf_jc \cup cf_iv \in \mathcal{F}$ and b, c are inner vertices of P_k.*

Proof. Let k be the minimum index such that $c \in V(P_k)$. So c is an inner vertex of P_k. By Proposition 5, we have $E(P_i) \cap E(af_ib) \neq \emptyset$ and $E(P_j) \cap E(cf_jd) \neq \emptyset$ (this is because, otherwise, $E(P_i) \cap E(af_ib) = \emptyset$ and $E(P_j) \cap E(af_jb) \neq \emptyset$, which would imply that $\ell_j \prec b \preceq \ell_i$, contradicting that $i < j$). So $c \in V(L_j \cup P_j \backslash r_j)$. This, together with the fact c is an inner vertex of P_k, show that $uf_kc = uLc = uf_jc$. We also have $E(P_i) \cap E(cf_id) = \emptyset$ and thus $c \in V(R_i)$, implying $k < i$ and $cf_kv = cRv = cf_iv$. Hence $f_k = uf_jc \cup cf_iv \in \mathcal{F}$.

Now we see $f_i \triangle f_k = af_ib \cup af_jb$ is a cycle and since $k < i$, this cycle must contain the entire P_i and some edge in P_k. Similarly since $k < j$, $f_j \triangle f_k = cf_id \cup cf_jd$ is a cycle which contains the entire P_j and some edge in P_k. Therefore, we see $P_i \subseteq af_ib$, $P_j \subseteq cf_jd$, and P_k contains some edge in af_jb and some edge in cf_id, which shows that b, c are inner vertices of P_k. □

Let i, j be distinct. By Proposition 5, we see that $f_i \backslash f_j$ consists of one or two subpaths, exactly one of which contains edges in P_i. By the **primary segment** $\mathrm{ps}(i, j)$ of $f_i \backslash f_j$, we denote the unique subpath in $f_i \backslash f_j$ containing edges in P_i. Note that $\mathrm{ps}(i, j)$ and $\mathrm{ps}(j, i)$ are distinct.

We now classify all pairs of $\mathcal{F}$ in the following three types:

[3] It will be convenient for us to picture that each f_i has an imagined orientation from u to v, as to capture the linear ordering $\prec$ on its vertices.

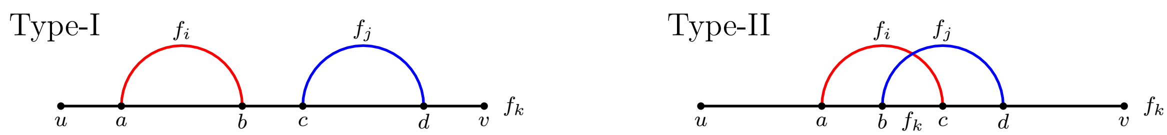


Figure 1

- A pair $\{f_i, f_j\}$ in $\mathcal{F}$ is called **type-I**, if $f_i \triangle f_j$ consists of two cycles. In this case, we call the path $f_k \in \mathcal{F}$ guaranteed in Proposition 6 as the **base** of $\{f_i, f_j\}$. See Figure 1 for an illustration.
- A pair $\{f_i, f_j\}$ in $\mathcal{F}$ is called **type-II**, if it is not type-I and there exists some $f_\ell \in \mathcal{F}$ such that $\mathrm{ps}(i, \ell) = af_ic$ and $\mathrm{ps}(j, \ell) = bf_jd$ where $a \prec b \prec c \prec d$ lie in f_ℓ. Such a path f_ℓ is called a **crossing path** of $\{f_i, f_j\}$, and the crossing path f_ℓ with minimum ℓ is called the **base** of $\{f_i, f_j\}$. See Figure 1.
- Finally, a pair $\{f_i, f_j\}$ in $\mathcal{F}$ is **normal**, if it is neither type-I nor type-II.

Proposition 7. *Let i, j, ℓ be distinct. If $P_\ell \subseteq f_i \cup f_j$, then $\{f_i, f_j\}$ is type-I with base f_ℓ.*

Proof. By Proposition 3, we see $P_\ell \not\subseteq f_i$ and $P_\ell \not\subseteq f_j$. So $P_\ell \cap f_i \neq \emptyset$ and $P_\ell \cap f_j \neq \emptyset$. By Proposition 6, $\{f_i, f_\ell\}$ and $\{f_j, f_\ell\}$ are not type-I. Let $a \prec b$ be the splitting vertices in $f_i \cup f_\ell$ and $c \prec d$ be the splitting vertices in $f_j \cup f_\ell$. We may assume $a \preceq c$. If $c \prec b$, since $E(cf_\ell b) \cap E(f_i \cup f_j) = \emptyset$, we see $E(cf_\ell b) \cap E(P_\ell) = \emptyset$. This implies that either $E(P_\ell) \cap E(af_\ell b) = \emptyset$ or $E(P_\ell) \cap E(cf_\ell d) = \emptyset$, a contradiction. Hence, $a \prec b \preceq c \prec d$ lie in f_ℓ. If there exists some $z \in V(af_ib) \cap V(cf_jd)$, then we have a contradiction that $c \prec z \prec b$. So af_ib and cf_jd are internally disjoint. Now we see that $\{f_i, f_j\}$ is a type-I pair with base f_ℓ. $\square$

For paths R_1, R_2 in G, we write $R_1 \preceq R_2$ (resp., $R_1 \prec R_2$) if any $s \in V(R_1)$ and $t \in V(R_2)$ satisfy $s \preceq t$ (resp., $s \prec t$).

Proposition 8. *Let i, j, k, ℓ be distinct. If $P_\ell \subseteq f_i \cup f_j \cup f_k$, then there exist $\alpha, \beta \in \{i, j, k\}$ such that $P_\ell \subseteq f_\alpha \cup f_\beta$ and thus $\{f_\alpha, f_\beta\}$ is type-I with base f_ℓ.*

Proof. If there are $\alpha, \beta \in \{i, j, k\}$ such that $P_\ell \subseteq f_\alpha \cup f_\beta$, then the conclusion follows by Proposition 7. So we may assume that there exist edges e_i, e_j, e_k in P_ℓ such that $e_i \in f_i \backslash (f_j \cup f_k)$, $e_j \in f_j \backslash (f_i \cup f_k)$, and $e_k \in f_k \backslash (f_i \cup f_j)$. Without loss of generality, we may assume that $e_i \preceq e_j \preceq e_k$ lie in f_ℓ. Clearly $\{f_j, f_\ell\}$ is not type-I (as otherwise, $E(P_\ell) \cap E(f_j) = \emptyset$ by Proposition 6). We have $e_j \in E(f_j \cap f_\ell)$. So f_j contains either the subpath of f_ℓ from u to e_j or the subpath of f_ℓ from e_j to v, which implies either $e_i \in E(f_j)$ or $e_k \in E(f_j)$, a contradiction. This completes the proof. $\square$

4 Almost all pairs are normal

For any subset $\mathcal{A}$ of $\mathcal{F}$, we define $G(\mathcal{A})$ to be the subgraph of G consisting of all edges e, which appears in some path of $\mathcal{A}$ but not in every path of $\mathcal{A}$. Note that if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{F}$, then $G(\mathcal{A})$ is a subgraph of $G(\mathcal{B})$. We say $\{x, y\} \subseteq \bigcap_{f_i \in \mathcal{A}} V(f_i)$ is the **separator** of $\mathcal{A}$, if $G(\mathcal{A}) \subseteq \bigcup_{f_i \in \mathcal{A}} xf_iy$ and subject to this, $\bigcup_{f_i \in \mathcal{A}} xf_iy$ is minimal.

The main result of this section is the following lemma.

Lemma 4.1. *There exist disjoint subsets $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ in $\mathcal{F}$ such that $\sum_{i\in[4]} |\mathcal{F}_i| \geq s - 90\sqrt{n}/\log n$, all $G(\mathcal{F}_i)$'s are edge-disjoint, and each $\mathcal{F}_i$ contains at most $2\sqrt{n}\log^2 n$ pairs of type-I and type-II.*

We show that the proof of Lemma 4.1 can be reduced to two lemmas in below.

Lemma 4.2. *There exist two disjoint subsets $\mathcal{A}_1, \mathcal{A}_2$ in $\mathcal{F}$ such that $|\mathcal{A}_1| + |\mathcal{A}_2| > s - 30\sqrt{n}/\log n$, $G(\mathcal{A}_1)$ and $G(\mathcal{A}_2)$ are edge-disjoint, and each $\mathcal{A}_i$ contains at most $\sqrt{n}\log^2 n$ type-I pairs.*

Lemma 4.3. *There exist two disjoint subsets $\mathcal{B}_1, \mathcal{B}_2$ in $\mathcal{F}$ such that $|\mathcal{B}_1| + |\mathcal{B}_2| > s - 60\sqrt{n}/\log n$, $G(\mathcal{B}_1)$ and $G(\mathcal{B}_2)$ are edge-disjoint, and each $\mathcal{B}_i$ contains at most $\sqrt{n}\log^2 n$ type-II pairs.*

Proof of Lemma 4.1 (Assume Lemmas 4.2 and 4.3). Let $\mathcal{A}_1, \mathcal{A}_2$ be obtained from Lemma 4.2 and $\mathcal{B}_1, \mathcal{B}_2$ be obtained from Lemma 4.3. For $1 \leq i, j \leq 2$, let $\mathcal{C}_{ij} = \mathcal{A}_i \cap \mathcal{B}_j$. Following the properties of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$, it is easy to see that the four subsets $\mathcal{C}_{ij}$'s are disjoint, each $\mathcal{C}_{ij}$ contains at most $2\sqrt{n}\log^2 n$ pairs of type-I and type-II, and

$$\sum_{1\leq i,j\leq 2} |\mathcal{C}_{ij}| = |(\mathcal{A}_1 \cup \mathcal{A}_2) \cap (\mathcal{B}_1 \cup \mathcal{B}_2)| \geq |\mathcal{A}_1 \cup \mathcal{A}_2| + |\mathcal{B}_1 \cup \mathcal{B}_2| - |\mathcal{F}| \geq s - 90\sqrt{n}/\log n.$$

It remains to show that all $G(\mathcal{C}_{ij})$'s are edge-disjoint. Fix $i \in [2]$. Since $\mathcal{C}_{ij} \subseteq \mathcal{B}_j$ for each $j \in [2]$, by the observation before Lemma 4.1, we have $G(\mathcal{C}_{ij}) \subseteq G(\mathcal{B}_j)$ for each $j \in [2]$. We also see from Lemma 4.3 that $G(\mathcal{B}_1)$ and $G(\mathcal{B}_2)$ are edge-disjoint, so it is clear that $G(\mathcal{C}_{i1})$ and $G(\mathcal{C}_{i2})$ are edge-disjoint for all $i \in [2]$. Similarly, $G(\mathcal{C}_{1j})$ and $G(\mathcal{C}_{2j})$ are edge-disjoint for all $j \in [2]$, finishing the proof. □

For the proofs of Lemmas 4.2 and 4.3 (also for the proof in Section 5), we need to introduce some notation on collections of paths in $\mathcal{F}$, which are used to generate cycles of some special characters.

- A triple $\{f_i, f_j, f_\ell\}$ in $\mathcal{F}$ is called **feasible**, if $\{uv\} \cup f_i \cup f_j \cup f_\ell$ contains a cycle $C(i, j, \ell)$ such that for all possible $\{\alpha, \beta, \gamma\} = \{i, j, \ell\}$, either $P_\alpha \backslash (f_\beta \cup f_\gamma) = \emptyset$ or $C(i, j, \ell)$ contains some edge in $P_\alpha \backslash (f_\beta \cup f_\gamma)$. Such a cycle $C(i, j, \ell)$ is called **3-feasible**.
- A quadruple $\{f_i, f_j, f_k, f_\ell\}$ in $\mathcal{F}$ is called **feasible**, if $\{uv\} \cup f_i \cup f_j \cup f_k \cup f_\ell$ contains a cycle $C(i, j, k, \ell)$ such that for all possible $\{\alpha, \beta, \gamma, \theta\} = \{i, j, k, \ell\}$, either $P_\alpha \backslash (f_\beta \cup f_\gamma \cup f_\theta) = \emptyset$ or $C(i, j, k, \ell)$ contains some edge in $P_\alpha \backslash (f_\beta \cup f_\gamma \cup f_\theta)$. Such a cycle $C(i, j, k, \ell)$ is called **4-feasible**.

4.1 The number of feasible tuples

In the following proposition, we estimate the number of feasible triples and quadruples in $\mathcal{F}$.

Proposition 9. *There are at most n feasible triples and at most $4n$ feasible quadruples in $\mathcal{F}$.*

Proof. A feasible triple $\{f_i, f_j, f_k\}$ is called *degenerate*, if there exists some $\{\alpha, \beta, \gamma\} = \{i, j, k\}$ with $P_\alpha \backslash (f_\beta \cup f_\gamma) = \emptyset$. By Propositions 6 and 7, $\{f_\beta, f_\gamma\}$ is type-I with base f_α, where $\alpha < \min\{\beta, \gamma\}$. Thus, each degenerate feasible triple consists of a unique type-I pair and its base.[4] Next we claim that for any two distinct non-degenerate feasible triples $\{f_i, f_j, f_k\}$ and $\{f_{i'}, f_{j'}, f_{k'}\}$, their 3-feasible cycles $C(i, j, k)$ and $C(i', j', k')$ are distinct. Suppose $C(i, j, k) = C(i', j', k')$. Let $i < j < k$ and $i' < j' < k'$. By symmetry let us assume $k \geq k'$. If $k > k'$, then we have $E(P_k) \cap E(f_{i'} \cup f_{j'} \cup f_{k'}) = \emptyset$, which contradicts that $E(P_k) \cap E(C(i, j, k)) \neq \emptyset$. Thus it must be $k = k'$. Now assume $j \geq j'$. If $j > j'$, then $E(P_j) \cap E(f_{i'} \cup f_{j'}) = \emptyset$, implying that $C(i', j', k)$ does not contain edges in $P_j \backslash f_k$.

[4]Reversely, each type-I pair determines a degenerate feasible triple.

But $C(i,j,k)$ does contain edges in $P_j\backslash f_k$, a contradiction. Thus $j = j'$. Finally we may assume $i \geq i'$. If $i > i'$, then $E(P_i) \cap E(f_{i'}) = \emptyset$ and thus $C(i',j,k)$ does not contain edges in $P_i\backslash(f_k \cup f_j)$. However, $C(i,j,k)$ does contain such edges. This gives a contradiction that $\{i,j,k\} = \{i',j',k'\}$, proving the claim. It is straightforward to see that each non-degenerate feasible triple and each degenerate feasible triple have different 3-feasible cycles. Hence, each feasible triple contributes a unique 3-feasible cycle. By (2), we see that $\mathcal{F}$ has at most n feasible triples.

Similarly, a feasible quadruple $\{f_i, f_j, f_k, f_\ell\}$ is called *degenerate* if there exists some $\{\alpha,\beta,\gamma,\theta\} = \{i,j,k,\ell\}$ with $P_\alpha\backslash(f_\beta \cup f_\gamma \cup f_\theta) = \emptyset$. It is analogous to show that each non-degenerate feasible quadruple contributes a unique 4-feasible cycle. So $\mathcal{F}$ has at most n non-degenerate feasible quadruples.

Now consider a degenerate feasible quadruple $\{f_i, f_j, f_k, f_\ell\}$ with $P_i \subseteq f_j \cup f_k \cup f_\ell$.

We claim that $P_\alpha\backslash(f_\beta \cup f_\gamma \cup f_i) \neq \emptyset$ for any $\{\alpha,\beta,\gamma\} = \{j,k,\ell\}$. Suppose for a contradiction that $P_j \subseteq f_i \cup f_k \cup f_\ell$. Since both of f_i, f_j cannot be the base of $\{f_k, f_\ell\}$ at the same time, by the symmetries between f_i and f_j and between f_k and f_ℓ, using Proposition 8 we may assume that f_i is the base of the type-I pair $\{f_j, f_k\}$. We then see $i < \min\{j,k\}$ from Proposition 6, and this in turn implies that $\{f_k, f_\ell\}$ is type-I with base f_j. But then $\{f_j, f_k\}$ is not type-I, a contradiction.

By Proposition 8, we may assume that $P_i \subseteq f_j \cup f_k$ and $\{f_j, f_k\}$ is type-I with base f_i. This yields $f_i \subseteq f_j \cup f_k$ and thus the 4-feasible cycle $C(i,j,k,\ell)$ is contained in $\{uv\} \cup f_j \cup f_k \cup f_\ell$. Using the previous claim, $C(i,j,k,\ell)$ contains some edge in $P_\alpha\backslash(f_\beta \cup f_\gamma \cup f_i) \subseteq P_\alpha\backslash(f_\beta \cup f_\gamma)$ for any $\{\alpha,\beta,\gamma\} = \{j,k,\ell\}$. Thus this cycle is also 3-feasible for $\{f_j, f_k, f_\ell\}$, which now is known to be a feasible triple. Note that f_i is the base of $\{f_j, f_k\}$. So the number of degenerate feasible quadruples is at most three times the number of feasible triples, that is at most $3n$. Hence in total there are at most $4n$ feasible quadruples in $\mathcal{F}$. This completes the proof. $\square$

For any pair $\{i,j\} \subseteq \{0\} \cup [s]$, let $\mathcal{W}_{ij}$ be the set of all paths $f_\ell \in \mathcal{F}$ such that the triple $\{f_i, f_j, f_\ell\}$ is either feasible or contained in a feasible quadruple. By Proposition 9, we see that

$$\sum_{\text{all pairs } \{i,j\}} |\mathcal{W}_{ij}| \leq n\binom{3}{2} + 4n\binom{4}{2} \cdot 2 = 51n. \tag{4}$$

4.2 Proof of Lemma 4.2

To show Lemma 4.2, we will first establish some properties on type-I pairs. In this subsection, unless otherwise specified we assume $i < j$ and $\{f_i, f_j\}$ is a type-I pair in $\mathcal{F}$ with base f_k. Let $a \prec b \preceq c \prec d$ be all splitting vertices in $f_i \cup f_j$. For any $\ell \in [s]\backslash\{i,j,k\}$, by Proposition 7 we have $P_\ell \nsubseteq f_i \cup f_j$.

Let $\mathcal{M}_{ij}$ consist of paths $f_\ell \in \mathcal{F}$ with $\ell \notin \{i,j,k\}$ such that $f_\ell\backslash(f_i \cup f_j)$ is a path $x_\ell f_\ell y_\ell$ with $x_\ell \prec y_\ell \preceq c$ and $cf_\ell v = cf_i v$, where either $x_\ell \preceq a \prec b \preceq y_\ell \preceq c$ or both x_ℓ, y_ℓ lie in one of af_ib and af_jb. Let $\mathcal{N}_{ij}$ consist of paths $f_\ell \in \mathcal{F}$ with $\ell \notin \{i,j,k\}$ such that $f_\ell\backslash(f_i \cup f_j)$ is a path $x_\ell f_\ell y_\ell$ with $b \preceq x_\ell \prec y_\ell$ and $uf_\ell b = uf_j b$, where either $b \preceq x_\ell \preceq c \prec d \preceq y_\ell$ or both x_ℓ, y_ℓ lie in one of cf_id and cf_jd.

Proposition 10. *Let $\ell \notin \{i,j,k\}$. If the triple $\{f_i, f_j, f_\ell\}$ is not feasible, then $f_\ell \in \mathcal{M}_{ij} \cup \mathcal{N}_{ij}$.*

Proof. Fix $\ell \notin \{i,j,k\}$ such that $\{f_i, f_j, f_\ell\}$ is a non-feasible triple. We point out that by Proposition 7 and the fact that $\{f_i, f_j\}$ is type-I, $P_\alpha\backslash(f_\beta \cup f_\gamma) \neq \emptyset$ for any $\{\alpha,\beta,\gamma\} = \{i,j,\ell\}$. Take any subpath $xf_\ell y$ in $f_\ell\backslash(f_i \cup f_j)$ which contains some edge of P_ℓ, where we assume $x \prec y$. By Proposition 6, since $i < j$, we see that $P_i \subseteq af_ib$, $P_j \subseteq cf_jd$, and $f_i \cup f_j$ can be partitioned into edge-disjoint paths af_ib, cf_jd and f_k. We will proceed by considering whether x, y lie in af_ib, cf_jd or f_k. In the coming proof, we will make use of the symmetry between x and y and the symmetry between f_i and f_j.

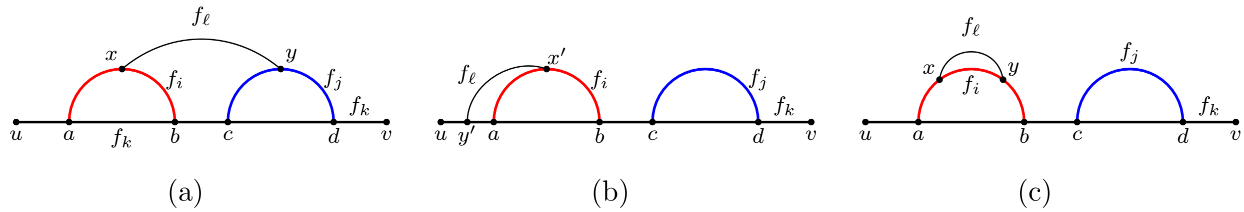


Figure 2

First, suppose that x, y are not both in af_ib, cf_jd or f_k. If x lies in af_ib and y lies in cf_jd (see Figure 2-a), then as $P_i \subseteq af_ib$ and $P_j \subseteq cf_jd$, one can see that $\{uv\} \cup f_i \cup f_j \cup xf_\ell y$ always contains a 3-feasible cycle $C(i, j, \ell)$ and thus $\{f_i, f_j, f_\ell\}$ is feasible, a contradiction. By symmetry between af_ib and cf_jd, it suffices to consider the case that one of $\{x, y\}$ (say x') lies in $af_ib \backslash \{a, b\}$ and the other y' lies in $f_k \backslash \{a, b\}$ (see Figure 2-b), where $\{x, y\} = \{x', y'\}$ and y' may lies in any of $uf_ia, af_jb, bf_ic, cf_id$ and df_iv. However, it is not hard to see that in any possible location of y', one can always find a 3-feasible cycle $C(i, j, \ell)$ in $\{uv\} \cup f_i \cup f_j \cup xf_\ell y$, a contradiction. For example, when $y' \in uf_ia\backslash\{a\}$, if $E(P_i) \cap af_ix' \neq \emptyset$, then $y'f_\ell x' \cup x'f_ia \cup af_jv \cup \{vu\} \cup uf_iy'$ forms a 3-feasible cycle for $\{f_i, f_j, f_\ell\}$; otherwise $E(P_i) \cap x'f_ib \neq \emptyset$, then $y'f_\ell x' \cup x'f_ib \cup bf_jv \cup \{vu\} \cup uf_iy'$ forms a 3-feasible cycle for $\{f_i, f_j, f_\ell\}$.

Next, suppose that both x, y lie in one of the paths af_ib and cf_jd. We first assume that x, y lie in af_ib (see Figure 2-c). Then $f_i \triangle f_\ell$ contains the cycle $xf_\ell y \cup xf_iy$. If $\{f_i, f_\ell\}$ is type-I, then by Proposition 6, we see $E(xf_iy) \cap E(P_i) = \emptyset$, which implies that $P_i \subseteq af_ix$ or $P_i \subseteq xf_ib$. In this case one can easily find a 3-feasible cycle for $\{f_i, f_j, f_\ell\}$, a contradiction. Therefore, $\{f_i, f_\ell\}$ is not type-I and $f_i \triangle f_\ell$ consists of the cycle $xf_\ell y \cup xf_iy$. This shows that $f_\ell = (f_i \backslash xf_iy) \cup xf_\ell y \in \mathcal{M}_{ij}$. By symmetry, if x, y lie in cf_jd, then one can show that $f_\ell \in \mathcal{N}_{ij}$.

It remains to consider that both x, y lie in f_k. Recall that $x \prec y$. We discuss according to the location of x. Assume that $x \prec a$. Then we have $b \preceq y \preceq c$ (as otherwise, one can always find a 3-feasible cycle in $\{uv\} \cup f_i \cup f_j \cup xf_\ell y$ for $\{f_i, f_j, f_\ell\}$, a contradiction). In this case, since $xf_\ell y$ contains some edge of P_ℓ, we see that $uf_\ell x = uLx = uf_ix$ and $yf_\ell v = yRv = yf_iv$, implying that $f_\ell \in \mathcal{M}_{ij}$.

Next assume that $a \preceq x \prec b$. We can derive that if $x = a$ then $a \prec y \preceq c$, and if $a \prec x \prec b$ then $a \prec x \prec y \preceq b$ (as otherwise there is a 3-feasible cycle for $\{f_i, f_j, f_\ell\}$). In both cases, we see that the first endpoint of P_ℓ precedes c in the linear order $\prec$. Suppose (f_i, f_ℓ) is type-I. Since c is a vertex in f_i with $P_i \prec c$, by Proposition 6, there exists a (u, c)-path containing P_i and P_ℓ, whose vertices increase in $\prec$ as traversing from u to c. Then this path can be easily extended to a cycle containing $cf_jd \supseteq P_j$, which is 3-feasible for $\{f_i, f_j, f_\ell\}$, a contradiction. Hence (f_i, f_ℓ) is not type-I. Suppose that (f_k, f_ℓ) is type-I. Then $f_\ell \backslash f_k$ consists of two paths $xf_\ell y$ and $x'f_\ell y'$. Since b, c are inner vertices of P_k, Proposition 6 shows that $y \preceq x' \prec b \preceq c \prec y'$, where x' lies in $yf_kb\backslash\{b\}$ and y' lies in $cf_kv\backslash\{c\}$, and $xf_\ell y'$ is internally disjoint from af_iy'. If $x'f_\ell y'$ is internally disjoint from cf_jd, then we can find 3-feasible cycle $af_jx \cup xf_\ell y' \cup y'f_id \cup df_jc \cup cf_ia$ for $\{f_i, f_j, f_\ell\}$, a contradiction. So $x'f_\ell y'$ intersects with cf_jd. Let x, y, x', z be all splitting vertices in $f_j \cup f_\ell$, where $z \in V(cf_jd)\backslash\{c, d\}$. Then we have $d = y'$ and $x'f_\ell y' = x'f_\ell z \cup zf_jd$. This shows that (f_j, f_ℓ) is type-I and by Proposition 6, $P_j \subseteq cf_jz$. Then $af_ic \cup cf_jz \cup zf_\ell a$ is a 3-feasible cycle for $\{f_i, f_j, f_\ell\}$, a contradiction. Summarizing, we have that both (f_i, f_ℓ) and (f_k, f_ℓ) are not type-I. Then in either case of $a \preceq x \prec y \preceq b$ and $x = a \prec b \prec y \preceq c$, we see that $xf_\ell y$ is the unique path in $f_\ell \backslash (f_i \cup f_j)$ and $cf_\ell v = cf_iv$, implying that $f_\ell \in \mathcal{M}_{ij}$.

Putting the above together, we infer $b \preceq x$. By the symmetry between x and y, one can also infer that $y \preceq c$. That is $b \preceq x \prec y \preceq c$. Then $uf_ix \cup xf_\ell y \cup yf_jv \cup vu$ forms a 3-feasible cycle for $\{f_i, f_j, f_\ell\}$.

This final contradiction completes the proof of Proposition 10. ☐

Recall the definition of $\mathcal{W}_{ij}$ in the end of Subsection 4.1.

Proposition 11. *Let $f_p \in \mathcal{M}_{ij}\backslash\mathcal{W}_{ij}$ and $f_q \in \mathcal{N}_{ij}\backslash\mathcal{W}_{ij}$. Then the paths $x_pf_py_p$ and $x_qf_qy_q$ are internally disjoint, where $x_p \prec y_p \preceq x_q \prec y_q$.*

Proof. Suppose for a contradiction that $x_pf_py_p$ and $x_qf_qy_q$ share a common inner vertex z. If $y_p \preceq x_q$, then we have $x_p \prec z \prec y_p \preceq x_q \prec z \prec y_q$, a contradiction. So we have $x_q \prec y_p$, which implies that $x_p \preceq a \prec b \preceq x_q \prec z \prec y_p \preceq c \prec d \preceq y_q$. We see $\{f_p, f_q\}$ is type-I. By Proposition 6, one of $x_pf_pz \cup zf_qy_q$ and $x_qf_qz \cup zf_py_p$ contains some edge in P_p and in P_q, respectively. In either case, we can find a 4-feasible cycle for $\{f_i, f_j, f_p, f_q\}$ and thus $\{f_i, f_j, f_p\}$ is contained in a feasible quadruple, which shows that $f_p \in \mathcal{W}_{ij}$, a contradiction. Hence the paths $x_pf_py_p$ and $x_qf_qy_q$ are internally disjoint. Suppose that $x_q \prec y_p$. Then we have $x_p \preceq a \prec b \preceq x_q \prec y_p \preceq c \prec d \preceq y_q$. In this case, one can derive the same contradiction by finding a 4-feasible cycle for $\{f_i, f_j, f_p, f_q\}$. This shows that $y_p \preceq x_q$. ☐

Proposition 12. *Let $\mathcal{S}$ be any subset of $\mathcal{F}$ with separator $\{x, y\}$ such that $|\mathcal{S}| \geq s - \frac{30\sqrt{n}}{\log n}$ and $x \prec y$. Assume that there do not exist two disjoint subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{S}$ such that $|\mathcal{S}_1| + |\mathcal{S}_2| \geq |\mathcal{S}| - \frac{52\sqrt{n}}{\log^2 n}$, $G(\mathcal{S}_1)$ and $G(\mathcal{S}_2)$ are edge-disjoint, and each $\mathcal{S}_i$ contains at most $\sqrt{n}\log^2 n$ type-I pairs for $i \in [2]$.*[5]

Then there exists $\mathcal{S}' \subseteq \mathcal{S}$ with separator $\{x', y'\}$ such that $|\mathcal{S}'| \geq |\mathcal{S}| - \frac{53\sqrt{n}}{\log^2 n}$, $x \preceq x' \prec y' \preceq y$, and each (x', y')-path in $G(\mathcal{S}')$ can be extended to two distinct (x, y)-paths in $G(\mathcal{S})$.

Proof. If every type-I pair $\{f_i, f_j\}$ in $\mathcal{S}$ has $|\mathcal{W}_{ij}| \geq 51\sqrt{n}/\log^2 n$, then by (4), one can infer that $\mathcal{S}$ itself contains at most $\sqrt{n}\log^2 n$ type-I pairs, a contradiction to the assumption (as we can just take $\mathcal{S}_1 = \mathcal{S}$ and $\mathcal{S}_2 = \emptyset$). So we may assume that there is a type-I pair $\{f_i, f_j\}$ in $\mathcal{S}$ with $|\mathcal{W}_{ij}| < 51\sqrt{n}/\log^2 n$, where $i < j$. Let f_k be the base of $\{f_i, f_j\}$ and $a \prec b \preceq c \prec d$ be all splitting vertices in $f_i \cup f_j$, where $x \preceq a$ and $d \preceq y$.

By Proposition 10, any $f_\ell \in \mathcal{S}\backslash(\mathcal{W}_{ij} \cup \{f_i, f_j, f_k\})$ belongs to $\mathcal{M}_{ij} \cup \mathcal{N}_{ij}$. Let $\mathcal{S}_1 = (\mathcal{S} \cap \mathcal{M}_{ij})\backslash(\mathcal{W}_{ij} \cup \{f_i, f_j, f_k\})$ and $\mathcal{S}_2 = (\mathcal{S} \cap \mathcal{N}_{ij})\backslash(\mathcal{W}_{ij} \cup \{f_i, f_j, f_k\})$. Then we have

$$|\mathcal{S}_1| + |\mathcal{S}_2| = |\mathcal{S}\backslash(\mathcal{W}_{ij} \cup \{f_i, f_j, f_k\})| \geq |\mathcal{S}| - 52\sqrt{n}/\log^2 n.$$

By Proposition 11, there exists some vertex z in bf_ic such that any $f_p \in \mathcal{S}_1$ and $f_q \in \mathcal{S}_2$ satisfy $V(x_pf_py_p) \preceq z \preceq V(x_qf_qy_q)$. So $G(\mathcal{S}_1)$ and $G(\mathcal{S}_2)$ are edge-disjoint. Now we claim that for any $\ell \in \{1, 2\}$, any type-I pair $\{f_\alpha, f_\beta\}$ in $\mathcal{S}_\ell$ together with any path f_γ in $\mathcal{S}_{3-\ell}$ form a feasible triple. Without loss of generality, let $\ell = 1$. Then every vertex in the two cycles of $f_\alpha \triangle f_\beta$ precedes z in $\prec$, so $f_\gamma \notin \mathcal{M}_{\alpha\beta} \cup \mathcal{N}_{\alpha\beta}$. Then Proposition 10 shows that $\{f_\alpha, f_\beta, f_\gamma\}$ is feasible, proving the claim.

Suppose that every $\mathcal{S}_\ell$ for $\ell \in [2]$ contains at least $\sqrt{n}/\log^2 n$ paths. If for some $t \in [2]$, $\mathcal{S}_t$ contains more than $\sqrt{n}\log^2 n$ type-I pairs, then the above claim shows that there are more than n feasible triples, a contradiction to Proposition 9. So we may assume that each of $\mathcal{S}_1$ and $\mathcal{S}_2$ contains at most $\sqrt{n}\log^2 n$ type-I pairs, but then such $\mathcal{S}_1$ and $\mathcal{S}_2$ contradict our assumption. Therefore, there exists some $\ell \in [2]$ such that $|\mathcal{S}_\ell| < \sqrt{n}/\log^2 n$ and thus $|\mathcal{S}_{3-\ell}| \geq |\mathcal{S}| - 53\sqrt{n}/\log^2 n$. We now explain that such $\mathcal{S}_{3-\ell}$ is the desired $\mathcal{S}'$. Without loss of generality, let $\ell = 1$ and let the separator of $\mathcal{S}_2$ be $\{x', y'\}$. Then we can derive $x \preceq a \prec b \preceq z \preceq x' \prec y' \preceq y$. It is clear that af_ib and af_jb can be extended to two paths in $G(\mathcal{S})$ from x to x' and internally disjoint from $G(\mathcal{S}_2)$. This completes the proof. ☐

We are ready to prove Lemma 4.2.

[5]Here it is possible that one $\mathcal{S}_i$ is an empty set (if so, $G(\mathcal{S}_i)$ is an empty graph).

Proof of Lemma 4.2. Assume on the contrary that there do not exist two disjoint subsets $\mathcal{S}_1, \mathcal{S}_2$ in $\mathcal{F}$ satisfying that $|\mathcal{S}_1| + |\mathcal{S}_2| > s - 30\sqrt{n}/\log n$, $G(\mathcal{S}_1)$ and $G(\mathcal{S}_2)$ are edge-disjoint, and each $\mathcal{S}_i$ contains at most $\sqrt{n}\log^2 n$ type-I pairs.

Let $\mathcal{F}_0 = \mathcal{F}$ have separator $\{x_0, y_0\}$ with $x_0 \prec y_0$. So $|\mathcal{F}_0| = s + 1$. Suppose that we have defined $\mathcal{F}_i \subseteq \mathcal{F}$ for some $i \geq 0$ with separator $\{x_i, y_i\}$ such that $x_i \prec y_i$ and $|\mathcal{F}_i| \geq s - \frac{30\sqrt{n}}{\log n} + \frac{52\sqrt{n}}{\log^2 n}$. One can easily derive from our assumption that such $\mathcal{F}_i$ satisfies the conditions of Proposition 12 (for the $\mathcal{S}$ therein). So by Proposition 12, there exists $\mathcal{F}_{i+1} \subseteq \mathcal{F}_i$ with separator $\{x_{i+1}, y_{i+1}\}$ such that $|\mathcal{F}_{i+1}| \geq |\mathcal{F}_i| - \frac{53\sqrt{n}}{\log^2 n}$, $x_i \preceq x_{i+1} \prec y_{i+1} \preceq y_i$, and each (x_{i+1}, y_{i+1})-path in $G(\mathcal{F}_{i+1})$ can be extended to two different (x_i, y_i)-paths in $G(\mathcal{F}_i)$. We repeat this process until the first subset (say $\mathcal{F}_t$) of size less than $s - \frac{30\sqrt{n}}{\log n} + \frac{52\sqrt{n}}{\log^2 n}$ appears. Since $|\mathcal{F}_i| - |\mathcal{F}_{i+1}| \leq \frac{53\sqrt{n}}{\log^2 n}$ for $0 \leq i < t$ and n is sufficiently large, we have $t \geq \frac{30}{53}\log n - 1$.

Note that for each $0 \leq i < t$, every (x_{i+1}, y_{i+1})-path in $G(\mathcal{F}_{i+1})$ can be extended to two distinct (x_i, y_i)-paths in $G(\mathcal{F}_i)$. Also $G(\mathcal{F}_t)$ contains at least $|\mathcal{F}_t|$ many (x_t, y_t)-paths, where $|\mathcal{F}_t| \geq s - \frac{30\sqrt{n}}{\log n} - \frac{\sqrt{n}}{\log^2 n} \geq \sqrt{n}/2$. Hence, there exist at least $2^t|\mathcal{F}_t| \geq 2^t\sqrt{n}/2 \geq n$ distinct (x_0, y_0)-paths in $G(\mathcal{F}) = G(\mathcal{F}_0)$, which can be extended to at least n distinct (u, v)-paths and thus at least n cycles in G. This contradicts (2) and completes the proof of Lemma 4.2. □

4.3 Proof of Lemma 4.3

In this subsection we prove Lemma 4.3. Throughout we assume $i < j$ and $\{f_i, f_j\}$ is a type-II pair with base f_k. Let $a \prec b \prec c \prec d$ be vertices in f_k such that $\mathrm{ps}(i,k) = af_ic$ and $\mathrm{ps}(j,k) = bf_jd$.

Before we show some structural properties on type-II pairs, we point out that af_ic and bf_jd are disjoint. This is because otherwise there exists some $z \in V(af_ic) \cap V(bf_jd)$ and thus $f_i \triangle f_j$ contains at least two cycles, from which Proposition 5 infers that $\{f_i, f_j\}$ is type-I, a contradiction.

Proposition 13. *$f_i = uf_ka \cup af_ic \cup cf_kv$, $f_j = uf_kb \cup bf_jd \cup df_kv$, and $E(P_k) \cap E(bf_kc) \neq \emptyset$.*

Proof. We first show that both $\{f_i, f_k\}$ and $\{f_j, f_k\}$ are not type-I.

Suppose for a contradiction that $\{f_i, f_k\}$ is a type-I pair. By Proposition 6, $\{f_i, f_k\}$ has a base path f_ℓ for some $\ell < \min\{i, k\}$. Our plan is to show that f_ℓ is a crossing path of $\{f_i, f_j\}$, which together with $\ell < k$ would contradict that f_k is the base of $\{f_i, f_j\}$. To see this, first note that $af_ic \cup af_kc$ is a cycle in $f_i \triangle f_k$. So $af_kc = af_\ell c$ and $af_ic = \mathrm{ps}(i, \ell)$. Also b is a splitting vertex of $f_j \cup f_\ell$ and obviously there is no other splitting vertex of $f_j \cup f_\ell$ in $bf_\ell c$. By Proposition 6 we also have $cf_\ell v = cf_iv$. Since $\{f_i, f_j\}$ is not type-I, there is at most one splitting vertex of $f_i \cup f_j$ in cf_iv; on the other hand, as $c \notin f_j$ there should be some splitting vertex of $f_i \cup f_j$ in cf_iv. Let d' be the unique splitting vertex of $f_i \cup f_j$ in cf_iv. Since $cf_\ell v = cf_iv$, d' is also a splitting vertex of $f_j \cup f_\ell$. If $d \in V(bf_jd')$, as $E(P_j) \cap E(bf_jd) \neq \emptyset$, we have $\mathrm{ps}(j, \ell) = bf_jd'$. Since $a \prec b \prec c \prec d'$ lie in f_ℓ with $af_ic = \mathrm{ps}(i, \ell)$ and $\mathrm{ps}(j, \ell) = bf_jd'$, this shows that f_ℓ is a crossing path of $\{f_i, f_j\}$, a contradiction. Hence, we have $d' \in V(bf_jd)$. If $E(P_j) \cap E(bf_jd') \neq \emptyset$, then again we have $\mathrm{ps}(j, \ell) = bf_jd'$, which also shows that f_ℓ is a crossing path of $\{f_i, f_j\}$. So we may assume that $E(P_j) \cap E(bf_jd') = \emptyset$ and thus $P_j \subseteq d'f_jv$. However in this case, since $\{f_i . f_j\}$ is type-II, the cycle $f_i \triangle f_j$ belongs to $uf_id' \cup uf_jd'$, which cannot contain any edge in P_j, a contradiction to Proposition 5. This proves that $\{f_i, f_k\}$ is not type-I and thus $f_i = uf_ka \cup af_ic \cup cf_kv$. Similarly, we can show $f_j = uf_kb \cup bf_jd \cup df_kv$.

By Proposition 3, each of $\mathrm{ps}(k, i) = af_kc$ and $\mathrm{ps}(k, j) = bf_kd$ contains some edge in P_k. Therefore, we see that bf_kc contains some edge in P_k, completing the proof. □

Let $\mathcal{A}_{ij}$ consist of all paths $f_\ell \in \mathcal{F}$ with $\ell \notin \{i,j,k\}$ satisfying that $f_\ell = uf_ix_\ell \cup x_\ell f_\ell y_\ell \cup y_\ell f_i v$, where $x_\ell, y_\ell \in V(af_ic)$ are splitting vertices in $f_\ell \cup f_i$ and $x_\ell f_\ell y_\ell$ is disjoint from $f_k \cup f_j$. Let $\mathcal{B}_{ij}$ consist of all paths $f_\ell \in \mathcal{F}$ with $\ell \notin \{i,j,k\}$ satisfying that $f_\ell = uf_jx_\ell \cup x_\ell f_\ell y_\ell \cup y_\ell f_j v$, where $x_\ell, y_\ell \in V(bf_jd)$ are splitting vertices in $f_\ell \cup f_j$ and $x_\ell f_\ell y_\ell$ is disjoint from $f_k \cup f_i$.

Proposition 14. *Let $\ell \notin \{i,j,k\}$. If $f_\ell \notin \mathcal{W}_{ij}$, then $f_\ell \in \mathcal{A}_{ij} \cup \mathcal{B}_{ij}$.*

Proof. Fix some $\ell \notin \{i,j,k\}$ with $f_\ell \notin \mathcal{W}_{ij}$. Our general proof strategy is, using the symmetry between x and y and the symmetry between af_ic and bf_jd, either to show $f_\ell \in \mathcal{A}_{ij} \cup \mathcal{B}_{ij}$, or to find a 3-feasible cycle $C(i,j,\ell)$, or to find a 4-feasible cycle $C(i,j,k,\ell)$. Each of the latter two cases implies that $f_\ell \in \mathcal{W}_{ij}$ and thus reaches a contradiction.

We first show that there exists some path say $xf_\ell y$ in $f_\ell \backslash (f_i \cup f_j \cup f_k)$, which contains some edge in P_ℓ. Otherwise, $P_\ell \subseteq f_i \cup f_j \cup f_k$, then by Proposition 8 f_ℓ is the base of some type-I pair in $\{f_i, f_j, f_k\}$; however, by Proposition 13 none of the pair in $\{f_i, f_j, f_k\}$ is type-I, a contradiction.

We shall divide the coming proof into several cases by considering the locations of x, y. Note that by Proposition 13, each of x, y lies in af_ic, bf_jd or f_k. Before we proceed, it will be convenient for later use to collect some properties about af_ic, bf_jd, bf_kc and $xf_\ell y$. Given $\alpha \in I \subseteq \{0\} \cup [s]$, we write $\cup I \backslash \alpha$ for the union of paths f_β for all $\beta \in I \backslash \{\alpha\}$. A path Q is called *α-unique* for I, if either $P_\alpha \backslash (\cup I \backslash \alpha) = \emptyset$ or $E(Q) \cap E(P_\alpha \backslash (\cup I \backslash \alpha)) \neq \emptyset$. We claim that

$$af_ic \text{ is } i\text{-unique, } bf_jd \text{ is } j\text{-unique, } bf_kc \text{ is } k\text{-unique, and } xf_\ell y \text{ is } \ell\text{-unique for } \{i,j,k,\ell\}. \quad (5)$$

This is clear for $xf_\ell y$. Considering bf_kc, we have either $E(bf_kc) \cap E(P_k \backslash (f_i \cup f_j \cup f_\ell)) \neq \emptyset$ or $E(bf_kc) \cap E(P_k) \subseteq E(f_\ell)$, the latter of which implies that $P_k \subseteq f_i \cup f_j \cup f_\ell$ (using the structure of Proposition 13). Hence, bf_kc is k-unique for $\{i,j,k,\ell\}$. Similarly, one can see that af_ic is i-unique and bf_jd is j-unique for $\{i,j,k,\ell\}$, completing the proof of (5). Also we claim that

$$af_ic \text{ is } i\text{-unique, } bf_jd \text{ is } j\text{-unique, and } xf_\ell y \text{ is } \ell\text{-unique for } \{i,j,\ell\}. \quad (6)$$

This is also clear for $xf_\ell y$. Now suppose $E(af_ic) \cap E(P_i) \subseteq E(f_\ell)$, which implies $P_i \subseteq f_k \cup f_\ell$. By Proposition 7, $\{f_k, f_\ell\}$ is type-I with base f_i and thus $f_i \subseteq f_k \cup f_\ell$. Note that the splitting vertices a, c in $f_i \cup f_k$ are also splitting vertices in $f_k \cup f_\ell$. Since b, d are the splitting vertices in $f_j \cup f_k$ such that $a \prec b \prec c \prec d$ lie in f_k, we can conclude that $f_j \notin \mathcal{M}_{k\ell} \cup \mathcal{N}_{k\ell}$ and thus Proposition 10 implies that $\{f_j, f_k, f_\ell\}$ is feasible. Using the fact that $f_i \subseteq f_k \cup f_\ell$, one can verify that any 3-feasible cycle for $\{f_j, f_k, f_\ell\}$ is a 4-feasible cycle for $\{f_i, f_j, f_k, f_\ell\}$. This shows that $f_\ell \in \mathcal{W}_{ij}$, a contradiction. Hence $E(af_ic) \cap E(P_i) \not\subseteq E(f_\ell)$, which implies that af_ic is i-unique for $\{i,j,\ell\}$. Analogously, $E(bf_jd) \cap E(P_j) \not\subseteq E(f_\ell)$ and thus bf_jd is j-unique for $\{i,j,\ell\}$, proving (6).

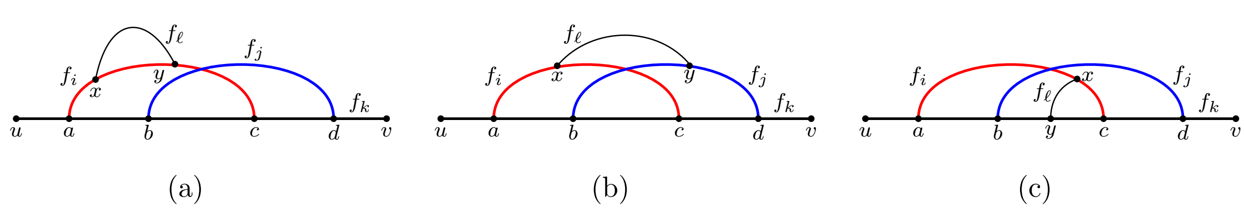


Figure 3

Now consider the case that both of x, y lie in af_ic or bf_jd. By symmetry, let us assume $x, y \in V(af_ic)$ with $x \prec y$ (see Figure 3-a). If $\{f_i, f_\ell\}$ is not type-I, then it is easy to see that $f_\ell \in \mathcal{A}_{ij}$. Now suppose $\{f_i, f_\ell\}$ is type-I. By Proposition 6, $P_\ell \subseteq xf_\ell y$ and $E(xf_iy) \cap E(P_i) = \emptyset$. As $\mathrm{ps}(i,k) = af_ic$, we see that $R = af_ix \cup xf_\ell y \cup yf_ic$ is an (a,c)-path, which is both i-unique and ℓ-unique for $\{i,j,\ell\}$. So by (6), $R \cup cf_id \cup df_ja$ forms a 3-feasible cycle $C(i,j,\ell)$.

Suppose that $x \in V(af_ic)\backslash\{a,c\}$ and $y \in V(bf_jd)\backslash\{b,d\}$ (see Figure 3-b). By (6), it is easy to see that one of the following four cycles can form a 3-feasible cycle $C(i,j,\ell)$. These include: (a) $xf_\ell y \cup yf_ja \cup af_ix$, (b) $xf_\ell y \cup yf_jv \cup \{uv\} \cup uf_ix$, (c) $xf_\ell y \cup yf_jd \cup df_ix$, and (d) $xf_\ell y \cup yf_ju \cup \{uv\} \cup vf_ix$.

Next suppose that both x, y lie in $f_k\backslash\{a,d\}$. Let $x \prec y$. If $x, y \notin V(bf_kc)$, by (5) and (6) one can always find either a 3-feasible cycle $C(i,j,\ell)$ containing $af_ic \cup bf_jd \cup xf_\ell y$ or a 4-feasible cycle $C(i,j,k,\ell)$ containing $af_ic \cup bf_jd \cup bf_kc \cup xf_\ell y$. So by symmetry between x and y, we may assume that $y \in V(bf_kc)$. If $x \prec a$, then by (5), one of the cycles $xf_\ell y \cup yf_kb \cup bf_jd \cup df_ix$ and $xf_\ell y \cup yf_kc \cup cf_ia \cup af_jv \cup \{uv\} \cup uf_kx$ gives a 4-feasible cycle $C(i,j,k,\ell)$. If $a \prec x \prec b$, then one of the cycles $xf_\ell y \cup yf_kb \cup bf_jd \cup df_ia \cup af_kx$ and $xf_\ell y \cup yf_kc \cup cf_iu \cup \{uv\} \cup vf_jx$ gives a 4-feasible cycle $C(i,j,k,\ell)$. Then it remains to consider $b \preceq x \prec y \preceq c$. If $E(P_k) \cap E(xf_ky) \neq \emptyset$, then $\{f_k, f_\ell\}$ is not type-I and thus $f_\ell = uf_kx \cup xf_\ell y \cup yf_kv$, which implies that $uf_ic \cup cf_\ell b \cup bf_jv \cup \{vu\}$ forms a 3-feasible cycle $C(i,j,\ell)$. So $E(P_k) \cap E(xf_ky) = \emptyset$. By (5), one of bf_kx and yf_kc is k-unique for $\{i,j,k,\ell\}$. Therefore, $xf_\ell y \cup yf_kc \cup cf_iu \cup \{uv\} \cup vf_jb \cup bf_kx$ forms a 4-feasible $C(i,j,k,\ell)$, proving this case.

The only possible case left is that one of x, y is in $V(af_ic \cup bf_jd)\backslash\{b,c\}$ and the other is in $V(f_k)$. By symmetry, we may assume that $x \in V(af_ic)\backslash\{c\}$ and $y \in V(f_k)\backslash\{a,c\}$ (see Figure 3-c). Recall that in the proof of (6), we proved $E(af_ic) \cap E(P_i) \not\subseteq E(f_\ell)$. So either (a) $E(af_ix) \cap E(P_i) \not\subseteq E(f_\ell)$, or (b) $E(xf_ic) \cap E(P_i) \not\subseteq E(f_\ell)$. First suppose (a) occurs. If $y \in V(uf_kb \cup df_kv)\backslash\{a,c\}$, then there is a (y,a)-path Q in the cycle $\{uv\} \cup f_j$ containing bf_jd. In this case, we see that $Q \cup af_ix \cup xf_\ell y$ forms a 3-feasible cycle $C(i,j,\ell)$.p So we may assume $y \in V(bf_kd)\backslash\{c\}$. Then one of the cycles $uf_ix \cup xf_\ell y \cup yf_kb \cup bf_jv \cup \{uv\}$ and $af_ix \cup xf_\ell y \cup yf_kd \cup df_ja$ is a 4-feasible cycle for $\{f_i, f_j, f_k, f_\ell\}$.

Therefore (b) occurs. If $y \preceq b$, then $yf_\ell x \cup xf_id \cup df_jy$ forms a 3-feasible cycle $C(i,j,\ell)$. If $c \prec y$, then $yf_\ell x \cup xf_ic \cup cf_kb \cup bf_jd \cup df_ky$ forms a 4-feasible cycle $C(i,j,k,\ell)$. So it only remains to consider $y \in V(bf_kc)\backslash\{b,c\}$. If bf_ky is k-unique for $\{i,j,k,\ell\}$, then $bf_ky \cup yf_\ell x \cup xf_id \cup df_jb$ forms a 4-feasible cycle $C(i,j,k,\ell)$, a contradiction. By (5) we have $E(yf_kc) \cap E(P_k\backslash f_\ell) \neq \emptyset$. In particular, $yf_kc \not\subseteq f_\ell$. Also we have $bf_ky \not\subseteq f_\ell$ (as otherwise $bf_\ell x \cup xf_id \cup df_jb$ is a 3-feasible cycle $C(i,j,\ell)$). Note that y is a splitting vertex in $f_k \cup f_\ell$, so $\{f_k, f_\ell\}$ must be type-I, with base say $f_{k'}$. Since $E(yf_kc) \cap E(P_k\backslash f_\ell) \neq \emptyset$ and $bf_ky \not\subseteq f_\ell$, by Proposition 6, y can not be the first or the last splitting vertex in $f_k \cup f_\ell$. So $x \prec y$ (as otherwise, y must be the first splitting vertex in $f_k \cup f_\ell$) and y is an inner vertex of $P_{k'}$. This further implies that $f_{k'} = uf_ky \cup yf_\ell v$, and bf_ky is k'-unique and $xf_\ell y$ is ℓ-unique for $\{i,j,k',\ell\}$. Because of (b) and $E(bf_jd) \cap E(P_j) \not\subseteq E(f_\ell)$ (proved in the proof of (6)), one can derive that xf_ic is i-unique and bf_jd is j-unique for $\{i,j,k',\ell\}$. Then $xf_\ell y \cup yf_kb \cup bf_jd \cup df_ix$ forms a 4-feasible cycle for $\{f_i, f_j, f_{k'}, f_\ell\}$, so $f_\ell \in \mathcal{W}_{ij}$. This final contradiction finishes the proof of Proposition 14. □

Proposition 15. *For $f_\alpha \in \mathcal{A}_{ij}\backslash\mathcal{W}_{ij}$ and $f_\beta \in \mathcal{B}_{ij}\backslash\mathcal{W}_{ij}$, the paths $x_\alpha f_\alpha y_\alpha$ and $x_\beta f_\beta y_\beta$ are disjoint.*

Proof. By Proposition 13, we see that a, d are two splitting vertices in $f_\alpha \cup f_\beta$. Suppose for a contradiction that $x_\alpha f_\alpha y_\alpha$ and $x_\beta f_\beta y_\beta$ have a common vertex z. It is clear that z is a splitting vertex in $f_\alpha \cup f_\beta$ with $a \prec z \prec d$. So $\{f_\alpha, f_\beta\}$ is type-I. By Proposition 6, exactly one of $Q_1 = x_\alpha f_\alpha z \cup zf_\beta y_\beta$ and $Q_2 = x_\beta f_\beta z \cup zf_\alpha y_\alpha$ contains some edges in $P_\alpha\backslash(f_i \cup f_j \cup f_\beta)$ and in $P_\beta\backslash(f_i \cup f_j \cup f_\alpha)$. First assume that Q_1 does so. Note that either af_ix_α or $x_\alpha f_ic$ contains some edge in $P_i\backslash(f_j \cup f_\alpha \cup f_\beta)$, and either bf_jy_β or $y_\beta f_jd$ contains some edge in $P_j\backslash(f_i \cup f_\alpha \cup f_\beta)$. So there are four possibilities and it is not hard to verify that one can always find a 4-feasible cycle $C(i,j,\alpha,\beta)$ containing Q_1 in each possibility. The proof for the other case (that is, Q_2 contains those edges mentioned above) is analogous, and we can always find a 4-feasible cycle $C(i,j,\alpha,\beta)$ containing Q_2. In any case, we see that both f_α and f_β are contained in $\mathcal{W}_{ij}$, a contradiction. □

By comparing Proposition 15 with Proposition 11, we see that $\mathcal{A}_{ij}, \mathcal{B}_{ij}$ can play the same roles of $\mathcal{M}_{ij}, \mathcal{N}_{ij}$ as in Subsection 4.2. The next result is analogue to Proposition 12.

Proposition 16. *Let $\mathcal{T}$ be any subset of $\mathcal{F}$ with separator $\{x, y\}$ such that $|\mathcal{T}| \geq s - \frac{60\sqrt{n}}{\log n}$ and $x \prec y$. Assume that there do not exist two disjoint subsets $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathcal{T}$ such that $|\mathcal{T}_1| + |\mathcal{T}_2| \geq |\mathcal{T}| - \frac{52\sqrt{n}}{\log^2 n}$, $G(\mathcal{T}_1)$ and $G(\mathcal{T}_2)$ are edge-disjoint, and each $\mathcal{T}_i$ contains at most $\sqrt{n}\log^2 n$ type-II pairs for $i \in [2]$.*

Then there exists $\mathcal{T}' \subseteq \mathcal{T}$ with separator $\{x', y'\}$ such that $|\mathcal{T}'| \geq |\mathcal{T}| - \frac{103\sqrt{n}}{\log^2 n}$, $x \preceq x' \prec y' \preceq y$, and each (x', y')-path in $G(\mathcal{T}')$ can be extended to two distinct (x, y)-paths in $G(\mathcal{T})$.

Proof. The proof will follow the lines as in Proposition 12. So we will omit details when the corresponding argument is the same as before.

First, we may assume that there is a type-II pair $\{f_i, f_j\}$ in $\mathcal{T}$ with $|\mathcal{W}_{ij}| < 51\sqrt{n}/\log^2 n$. Let f_k be the base of $\{f_i, f_j\}$ and let $a \prec b \prec c \prec d$ be vertices in f_k such that $\mathrm{ps}(i,k) = af_ic$ and $\mathrm{ps}(j,k) = bf_jd$. Let $\mathcal{T}_1 = (\mathcal{T} \cap \mathcal{A}_{ij}) \backslash (\mathcal{W}_{ij} \cup \{f_i, f_j, f_k\})$ and $\mathcal{T}_2 = (\mathcal{T} \cap \mathcal{B}_{ij}) \backslash (\mathcal{W}_{ij} \cup \{f_i, f_j, f_k\})$. By Proposition 14,

$$|\mathcal{T}_1| + |\mathcal{T}_2| = |\mathcal{T} \backslash (\mathcal{W}_{ij} \cup \{f_i, f_j, f_k\})| \geq |\mathcal{T}| - 52\sqrt{n}/\log^2 n.$$

Let $\{x_A, y_A\}$ and $\{x_B, y_B\}$ be the separators of $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Then $a \preceq x_A \prec y_A \preceq c$ lie in af_ic, $b \preceq x_B \prec y_B \preceq d$ lie in bf_jd, and by Proposition 15, $G(\mathcal{T}_1)$ and $G(\mathcal{T}_2)$ are disjoint. We claim that

$$\text{for any } \ell \in \{1, 2\} \text{ and any type-II pair } \{f_\alpha, f_\beta\} \text{ in } \mathcal{T}_\ell, \text{ we have } \mathcal{T}_{3-\ell} \subseteq \mathcal{W}_{\alpha\beta}.$$

To see this, note that $f_\alpha \triangle f_\beta \subseteq G(\mathcal{T}_\ell)$. So any path in $\mathcal{T}_{3-\ell}$ does not belong to $\mathcal{A}_{\alpha\beta} \cup \mathcal{B}_{\alpha\beta}$. Then Proposition 14 shows that $\mathcal{T}_{3-\ell} \subseteq \mathcal{W}_{\alpha\beta}$, as claimed.

Suppose $|\mathcal{T}_\ell| \geq 51\sqrt{n}/\log^2 n$ for every $\ell \in [2]$. If there exists some $t \in [2]$ such that $\mathcal{T}_t$ contains more than $\sqrt{n}\log^2 n$ type-II pairs, then the above claim would derive a contradiction to (4); otherwise, each of $\mathcal{T}_1$ and $\mathcal{T}_2$ contains at most $\sqrt{n}\log^2 n$ type-II pairs, a contradiction to our assumption. Therefore, we may assume $|\mathcal{T}_1| < 51\sqrt{n}/\log^2 n$ and thus $|\mathcal{T}_2| \geq |\mathcal{T}| - 103\sqrt{n}/\log^2 n$. Recall the separator $\{x_B, y_B\}$ of $\mathcal{T}_2$ such that $x \preceq a \prec b \preceq x_B \prec y_B \preceq d \preceq y$ lie in f_j. To show that $\mathcal{T}_2$ is the desired $\mathcal{T}'$, it suffices to see that each (x_B, y_B)-path R_0 in $G(\mathcal{T}_2)$ can be extended to two (x, y)-paths $R_1 \cup R_0 \cup y_B f_j y$ and $R_2 \cup R_0 \cup y_B f_j y$ in $G(\mathcal{T})$, where $R_1 = x f_j x_B$ and $R_2 = x f_j a \cup a f_i c \cup c f_k b \cup b f_j x_B$. ☐

We can then prove Lemma 4.3 promptly.

Proof of Lemma 4.3. By replacing Proposition 12 with Proposition 16, this proof is identical to the proof of Lemma 4.2. We left the verification to readers. ☐

Now the proof of the main result Lemma 4.1 in this section is completed.

5 Reordering and partitioning $\mathcal{F}$

The goal of this section is to show that roughly speaking, one can reorder most paths in $\mathcal{F}$ and partition them into a bounded number of intervals such that for every relevant edge e, paths containing e in each interval are listed almost consecutively. The precise statement is as follows.

Lemma 5.1. *There exist disjoint subsets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ in $\mathcal{F}$ and constants β, γ with $\beta \geq n^{1/4}\log n$ and $\gamma = \beta \log n \leq \sqrt{n}$ such that the following hold:*

1). $\sum_{i\in[4]}|\mathcal{P}_i| \geq (1-o(1))\cdot s$*, all* $G(\mathcal{P}_i)$*'s are edge-disjoint, and each* $\mathcal{P}_i$ *contains at most* $2\sqrt{n}\log^2 n$ *pairs of type-I and type-II.*

2). *For any edge* e *in* $G(\mathcal{P}_i)$*, let* $d(e)$ *denote the number of paths in* $\mathcal{P}_i$ *containing* e*. Then there are at most* $9n\log\log n/\log n$ *edges* e *in* $G(\mathcal{P}_i)$ *satisfying that* $\beta \leq d(e) \leq \gamma$*.*

3). *Each* $\mathcal{P}_i$ *has an arrangement* $\{g_j\}_{j\geq 1}$ *and a partition of at most* $3\sqrt{n}/\gamma$ *intervals such that the following holds.*[6] *For any edge* e *in* $G(\mathcal{P}_i)$ *with* $d(e) \geq \gamma$*, one can delete at most* 3β *paths in* $\mathcal{P}_i$ *such that there is at most one interval of* $\mathcal{P}_i$ *which can contain some remaining paths* g_j, g_k *with* $e \in E(g_j)$ *and* $e \notin E(g_k)$ *and moreover, all such paths* g_j, g_k *satisfy* $j < k$*.*

We devote the rest of this section to the proof of Lemma 5.1. We begin by defining the desired subsets $\mathcal{P}_i$ of $\mathcal{F}$. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ be the four disjoint subsets of $\mathcal{F}$ from Lemma 4.1.

Definition 5.2. *For each* $i \in [4]$*, let* $\mathcal{P}_i$ *be obtained from* $\mathcal{F}_i$ *by deleting all paths each of which is contained in at least* $26\sqrt{n}\log n$ *sets* $\mathcal{W}_{jk}$*'s or at least* $n^{1/4}$ *pairs of type-I and type-II in* $\mathcal{F}_i$*.*

From Lemma 4.1 we see that the number of type-I and type-II pairs in each $\mathcal{F}_i$ is at most $2\sqrt{n}\log^2 n$. Together with (4), we have $|\mathcal{P}_i| \geq |\mathcal{F}_i| - n^{1/2}/(2\log n) - 2n^{1/4}\log^2 n$. So by Lemma 4.1 again, we can derive that

$$|\cup_{i\in[4]}\mathcal{P}_i| \geq |\cup_{i\in[4]}\mathcal{F}_i| - 2\sqrt{n}/\log n - 8n^{1/4}\log^2 n = (1-o(1))\cdot s.$$

Since $\mathcal{P}_i \subseteq \mathcal{F}_i$ implies $G(\mathcal{P}_i) \subseteq G(\mathcal{F}_i)$, we see that $G(\mathcal{P}_i)$'s are pairwise edge-disjoint and each $\mathcal{P}_i$ contains at most $2\sqrt{n}\log^2 n$ pairs of type-I and type-II. This proves the first item of Lemma 5.1.

Next we define the constants β and γ. For each edge e contained in $G(\mathcal{P}_i)$ for some $i \in [4]$, we define its *degree* $d(e)$ to be the number of paths in $\mathcal{P}_i$ containing e. So obviously $1 \leq d(e) \leq s+1 \leq 2\sqrt{n}$ for every such e. Let $A = \lceil \frac{\log n}{4\log\log n} \rceil - 2$ and let $\alpha_0, \alpha_1, ..., \alpha_A$ be a geometric sequence of reals such that $\alpha_j = n^{1/4}(\log n)^{j+1}$. By average, there is some $j_0 \in [A]$ such that the number of edges e with $d(e) \in [\alpha_{j_0-1}, \alpha_{j_0}]$ is at most $\frac{2e(G)}{A} \leq \frac{9n\log\log n}{\log n}$. Let $\beta = \alpha_{j_0-1}$ and $\gamma = \alpha_{j_0}$. So $\beta \geq n^{1/4}\log n$ and $\gamma = \beta\log n \leq \sqrt{n}$, as wanted.

It remains to show the third item of Lemma 5.1. For this, in the rest of this section we shall focus on one of $\mathcal{P}_i$'s and express it as $\mathcal{P}$.

We need to collect some properties on the edges in $\mathcal{P}$ first. Let $\{u_0, v_0\}$ be the separator of $\mathcal{P}$ with $u_0 \prec v_0$. Recall the spanning trees L and R from (3). So all paths in $\mathcal{P}$ contain $uLu_0 \cup v_0Rv$.

Proposition 17. *Let* $e \in E(G(\mathcal{P}))$*. If there are* $f_k, f_\ell \in \mathcal{P}$ *with* $e \in E(L_k) \cap E(R_\ell)$*, then* $d(e) \leq 2n^{\frac{1}{4}}$*.*

Proof. Suppose that $d(e) > 2n^{1/4}$. Then there are at least $2n^{1/4}$ paths f_j in $\mathcal{P}\backslash\{f_k, f_\ell\}$ with $e \in E(L_j) \cup E(R_j)$. If $e \in E(R_j)$, then $e \in E(R_j) \cap E(L_k)$ and by Proposition 3, $\{f_j, f_k\}$ must be type-I. Similarly, if $e \in E(L_j)$, then we also can see that $\{f_j, f_\ell\}$ is type-I. Thus, one of f_k and f_ℓ is contained in at least $n^{1/4}$ type-I pairs in $\mathcal{P}$. However this is a contradiction to the definition of $\mathcal{P}$. □

Proposition 18. *All edges* $e \in E(G(\mathcal{P}))$ *with* $d(e) \geq \gamma$ *induce two edge-disjoint trees* $L_{\mathcal{P}}$ *and* $R_{\mathcal{P}}$ *with roots* u_0 *and* v_0*, which are subtrees of* L *and* R*, respectively.*

Proof. Consider such edges e with $d(e) \geq \gamma \geq \alpha_0 > 2n^{1/4}$. By Proposition 17, there are two kinds of such edges e: either (1) all paths f_j in $\mathcal{P}$ containing e satisfy $e \in E(L_j \cup P_j)$, or (2) all paths f_j in

[6] Here, an *interval* of $\mathcal{P}_i$ means a subset of $\mathcal{P}_i$ consisting of paths g_j for all integers j in some interval $[a, b]$.

$\mathcal{P}$ containing e satisfy $e \in E(P_j \cup R_j)$. Let $e = xy$ with $x \prec y$. In the former case (1), we see that all paths in $\mathcal{P}$ containing e also contain the path $L[u_0, y]$, so all edges in $L[u_0, y]$ have degree at least $d(e) \geq \gamma$. This shows that all edges satisfying (1) induce a subtree of L (with root u_0). The analog also holds for the latter case. This finishes the proof. □

Definition 5.3. *Each leaf-edge in the rooted tree $L_\mathcal{P}$ or $R_\mathcal{P}$ is called a transforming edge of $\mathcal{P}$.*

For any paths $f, g \in \mathcal{P}$, let the first and the last splitting vertices in $f \cup g$ (according to the linear ordering $\prec$) be $\mathrm{LS}(f, g)$ and $\mathrm{RS}(f, g)$, respectively.

We now define an arrangement $\{g_j\}_{j\geq 1}$ for $\mathcal{P}$ in the following algorithm.

Algorithm for ordering the paths in $\mathcal{P}$. Initially, set $j = 1$ and $\mathcal{S} = \mathcal{P}$. We iterate the following three steps until $\mathcal{S} = \emptyset$.

(a). If j=1, let $x = u_0, y = v_0$ and $\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S}$. Otherwise, we have $j \geq 2$. Let x be the maximum $\mathrm{LS}(g_{j-1}, f)$ in $\prec$ over all $f \in \mathcal{S}$, and let $\mathcal{S}_1$ be the set of all $f \in \mathcal{S}$ containing the subpath $u_0 L x$. Next, let y be the minimum $\mathrm{RS}(g_{j-1}, f')$ in $\prec$ over all $f' \in \mathcal{S}_1$, and then let $\mathcal{S}_2$ be the set of all $f' \in \mathcal{S}_1$ containing the subpath $y R v_0$.

(b). If there exists some paths in $\mathcal{S}_2$ containing some transforming edge e in $L_\mathcal{P}$ with $x \preceq V(e)$, then let $\mathcal{S}_3$ be the set consisting of all such paths in $\mathcal{S}_2$; otherwise, let $\mathcal{S}_3 = \mathcal{S}_2$. Next, if there exists some paths in $\mathcal{S}_3$ containing some transforming edge e' in $R_\mathcal{P}$ with $V(e') \preceq y$, then let $\mathcal{S}_4$ be the set consisting of all these paths in $\mathcal{S}_3$; otherwise, let $\mathcal{S}_4 = \mathcal{S}_3$.

(c). Pick any path in S_4 and denote it by g_j. Update $j \leftarrow j + 1$ and $\mathcal{S} \leftarrow \mathcal{S} \backslash \{g_j\}$.

We also need some properties on g_j's, which can be collected directly from the above algorithm.

Proposition 19. *The following hold for any $j < k < \ell$:*

(i). If both g_j and g_ℓ contain some subpath $u_0 L w$, then g_k also contains $u_0 L w$.

(ii). $\mathrm{LS}(g_{j-1}, g_j) \succeq \mathrm{LS}(g_{j-1}, g_k)$, and if $\mathrm{LS}(g_{j-1}, g_j) = \mathrm{LS}(g_{j-1}, g_k)$ then $\mathrm{RS}(g_{j-1}, g_j) \preceq \mathrm{RS}(g_{j-1}, g_k)$.

(iii). Suppose that $\mathrm{LS}(g_{j-1}, g_j) = \mathrm{LS}(g_{j-1}, g_k)$ and $\mathrm{RS}(g_{j-1}, g_j) = \mathrm{RS}(g_{j-1}, g_k)$. If g_j contains no transforming edge in $L_\mathcal{P}$, then g_k also contains no transforming edge in $L_\mathcal{P}$.

(iv). Under the same conditions of (iii), if g_k contains a transforming edge in $R_\mathcal{P}$ and g_j does not, then g_j contains a transforming edge in $L_\mathcal{P}$ but g_k does not.

We then partition $\mathcal{P} = \{g_j\}_{j\geq 1}$ into a bounded number of subsets in the next definition.

Definition 5.4. *For any transforming edge e of $\mathcal{P}$, the path $g_j \in \mathcal{P}$ containing e with the minimum j is called a fence of $\mathcal{P}$. Let $\{g_{j_k}\}_{1\leq k<k_\mathcal{P}}$ be the set of all fences of $\mathcal{P}$, where the sequence $\{j_k\}$ is increasing with k. Then the set $\mathcal{I}_k = \{g_j : j_k \leq j < j_{k+1}\}$ for each $0 \leq k < k_\mathcal{P}$ is called an interval of $\mathcal{P}$, where we define $j_0 = 1$ and $j_{k_\mathcal{P}} = |\mathcal{P}| + 1$.*

So $\mathcal{P}$ is partitioned into at most $k_\mathcal{P}$ intervals. By Proposition 18, any path in $\mathcal{P}$ has at most one leaf-edge in $L_\mathcal{P}$ and at most one leaf-edge in $R_\mathcal{P}$. This shows that $k_\mathcal{P} \leq 2|\mathcal{P}|/\gamma \leq 3\sqrt{n}/\gamma$, as desired.

Towards Lemma 5.1, we first prove the following weaker version. Let us recall the definitions of feasible triples and quadruples, stated right before Subsection 4.1.

Proposition 20. *For any edge e in $G(\mathcal{P})$, let g_i be the path in $\mathcal{P}$ containing e with minimum i. Then one can delete at most 2β paths from $\mathcal{P}$ such that for any $k > j > i$, if some remaining path g_j does not contain e, then every remaining path g_k does not contain e.*

Proof. Suppose this fails for some edge xy in $G(\mathcal{P})$ with $x \prec y$. Let g_i be the path in $\mathcal{P}$ containing xy with minimum i. We may assume that $d(xy) > 2\beta$ and there are at least β paths in $\mathcal{P}$ behind g_i which does not contain xy. Let $\mathcal{B}$ be the set of the first β such paths. Note that paths in $\mathcal{B}$ may not be consecutive in $\mathcal{P}$. Let g_{j_0} be the path in $\mathcal{B}$ with maximum j_0. Let $\mathcal{C}$ be the set consisting of all paths in $\mathcal{P}$ behind g_{j_0} and containing xy, and we may also assume that $|\mathcal{C}| \geq \beta$.

Let g_j be the path in $\mathcal{B}$ with minimum j. Then g_j does not contain xy, while g_{j-1} does. As $\beta \geq \alpha_0 \geq n^{1/4} \log n$, we see that there are at least $\beta - 2n^{1/4} \geq \beta/2$ paths g_k in $\mathcal{C}$ such that $\{g_{j-1}, g_k\}$ and $\{g_j, g_k\}$ are normal pairs. From now on, by g_k we mean any one of such paths in $\mathcal{C}$.

Let $a \prec b$ be the two splitting vertices in $g_{j-1} \cup g_k$ (see Figure 4-a). Since $xy \in E(g_{j-1}) \cap E(g_k)$, we have either $y \preceq a$ or $b \preceq x$. If $y \preceq a$, then xy is in the tree L and thus both g_{j-1} and g_k contain the subpath $u_0 L y$, while g_j does not. This contradicts Proposition 19 (i). Hence, $b \preceq x$. Since $xy \notin E(g_j)$, we see that $\mathrm{RS}(g_{j-1}, g_j)$ and $\mathrm{RS}(g_k, g_j)$ are the same vertex, say z, with $b \preceq x \prec y \preceq z$.

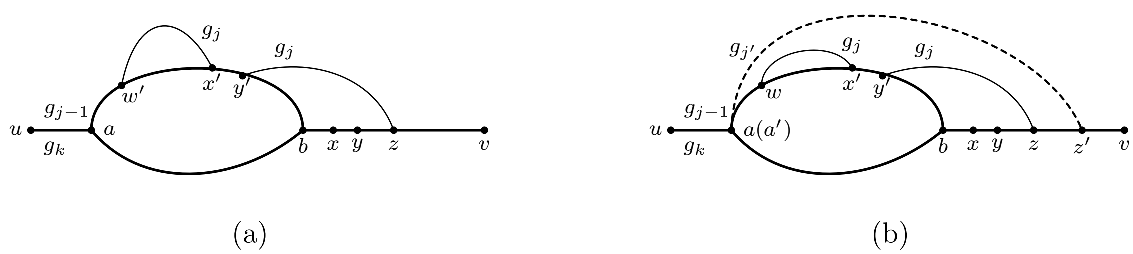


Figure 4

Let $w = \mathrm{LS}(g_{j-1}, g_j)$. We claim that $a \prec w \prec b$. Note that $\{g_j, g_k\}$ is a normal pair and by Proposition 19 (ii), $w \succeq \mathrm{LS}(g_{j-1}, g_k) = a$. If $w \succeq b$, then we have $g_j = u g_{j-1} w \cup w g_j z \cup z g_{j-1} v$, which implies that $\{g_j, g_k\}$ is type-II, a contradiction. So $a \preceq w \prec b$. Now assume that $w = a$. Let $w' = \mathrm{LS}(g_j, g_k)$. So $w' \in V(a g_k b) \backslash \{b\}$ and $g_j = u g_k w' \cup w' g_j z \cup z g_k v$. We see $a = \mathrm{LS}(g_{j-1}, g_j) = \mathrm{LS}(g_{j-1}, g_k)$, but $\mathrm{RS}(g_{j-1}, g_j) = z \succ b = \mathrm{RS}(g_{j-1}, g_k)$, a contradiction to Proposition 19 (ii). This proves the claim that $a \prec w \prec b$. Now a, z are the only two splitting vertices in $g_j \cup g_k$.

If $\{g_{j-1}, g_j\}$ is not type-I, then $g_j = u g_{j-1} w \cup w g_j z \cup z g_{j-1} v$ and further, we see that g_{j-1} is a crossing path of $\{g_j, g_k\}$, contradicting that $\{g_j, g_k\}$ is a normal pair. Thus, $\{g_{j-1}, g_j\}$ is type-I.

Since a, z are the only two splitting vertices in $g_j \cup g_k$, we may assume that the splitting vertices in $g_{j-1} \cup g_j$ are w, x', y', z such that $a \prec w \prec x' \preceq y' \prec b \prec z$ lie in g_{j-1}. We write $f_{\pi(t)} = g_t$ for each $t \in \{j-1, j, k\}$. If $P_{\pi(j)} \subseteq y g_j z$, then $\{g_j, g_k\}$ is type-II with a crossing path g_{j-1}, a contradiction. So by Proposition 6, we can deduce that $P_{\pi(j)} \subseteq w g_j x'$ and $P_{\pi(j-1)} \subseteq y' g_{j-1} z$. In particular, $y' g_{j-1} b$ contains some edge in $P_{\pi(j-1)}$. Let the base of $\{g_{j-1}, g_j\}$ be f_ℓ. Then Proposition 6 also shows that y is an inner vertex of P_ℓ and thus $y' f_\ell z = y' g_j z$ contains some edge in P_ℓ.

Recall the set $\mathcal{B}$. For any chosen $g_k \in \mathcal{C}$ from above, we can find at least $\beta - 3n^{1/4} \geq \beta/2$ paths $g_{j'} \in \mathcal{B}$ such that $g_{j'}$ forms a normal pair with any path in $\{g_{j-1}, g_j, g_k\}$. Notice that we do not require $\{g_{j'}, f_\ell\}$ to be a normal pair as f_ℓ may not be in $\mathcal{P}$. Let $z' = \mathrm{RS}(g_{j'}, g_{j-1})$. As $g_{j'}$ does not contain xy, we see $z' \succeq y$ and thus z' is also the vertex $\mathrm{RS}(g_{j'}, g_k)$. Since $\{g_{j'}, g_{j-1}\}$ and $\{g_{j'}, g_k\}$ both are normal, $g_{j'} \backslash (g_{j-1} \cup g_k)$ forms a path, say $a' g_{j'} z'$, where $a' \in V(u_0 g_{j-1} x \cup a g_k b)$. See Figure 4-b for an illustration.

We claim that $a' = a$. Since $j < j' < k$ and both g_{j-1} and g_k contain $ug_{j-1}a = uLa$, by Proposition 19, $g_{j'}$ also contains $ug_{j-1}a$, which implies that $a' \succeq a$. If $a' \succeq b$, then $g_{j'} = ug_{j-1}a' \cup a'g_{j'}z' \cup z'g_{j-1}v$ and $\{g_{j'}, g_k\}$ forms a type-I pair, a contradiction. If $a' \in V(ag_{j-1}b) \backslash \{a, b\}$, then $\{g_{j'}, g_k\}$ is type-II with a crossing path g_{j-1}, a contradiction. If $a' \in V(ag_kb) \backslash \{a, b\}$, then $\{g_{j'}, g_{j-1}\}$ is type-II with a crossing path g_k, again a contradiction. Therefore we have $a' = a$.

Next we claim that $ag_{j'}z'$ is internally disjoint with $y'g_jz$. Suppose on the contrary that there exists a splitting vertex $w' \in V(ag_{j'}z') \cap V(y'g_jz)$ in $g_{j'} \cup g_j$. Since $\{g_{j'}, g_j\}$ is normal, we have $z' = z$ and $w'g_{j'}z = w'g_jz$. However, as $a \prec y' \prec w' \prec z$ lie in g_j, we see that g_j is a crossing path of $\{g_{j'}, g_{j-1}\}$, a contradiction. This proves the claim.

Recall that $y'g_{j-1}b$ contains some edge in $P_{\pi(j-1)}$ and $y'f_\ell z = yg_jz$ contains some edge in P_ℓ. Thus, $ag_kb \cup bg_{j-1}y' \cup y'f_\ell z \cup zg_kz' \cup z'g_{j'}a$ is a 4-feasible cycle for $\{g_{j-1}, g_{j'}, g_k, f_\ell\}$.

Putting everything together, there are at least $\beta/2$ choices of $g_k \in \mathcal{C}$ and subject to a fixed g_k, there are at least $\beta/2$ choices of $g_{j'} \in \mathcal{B}$ such that $\{g_{j-1}, g_k, g_{j'}\}$ is contained in a feasible quadruple. That is, $g_{j-1} \in \mathcal{P}$ is contained in at least $\beta^2/4 \geq \sqrt{n}\log^2 n/4 > 26\sqrt{n}\log n$ distinct $\mathcal{W}_{pq}$'s. This final contradiction (to the definition of $\mathcal{P}$) completes the proof of Proposition 20. □

Finally, we are ready to complete the proof of Lemma 5.1.

Proof of Lemma 5.1. Putting everything above together, it suffices for us to prove the following statement. For any edge e in $G(\mathcal{P})$ with $d(e) \geq \gamma$, one can delete at most 3β paths in $\mathcal{P}$ such that there is at most one interval of $\mathcal{P}$ which can contain some remaining paths g_j, g_k with $e \in E(g_j)$ and $e \notin E(g_k)$ and moreover, all such paths g_j, g_k satisfy $j < k$.

Suppose this fails for some edge e with $d(e) \geq \gamma$. Let g_i be the path in $\mathcal{P}$ containing e with minimum i. By Proposition 20, one can delete at most 2β paths from $\mathcal{P}$ such that for any $k > j > i$, if some remaining path g_j does not contain e, then every remaining path g_k does not contain e. If g_i is a fence, then only the last interval of $\mathcal{P}$, which has some remaining path containing e, can contain some remaining path g_ℓ with $e \notin E(g_\ell)$, and the conclusion holds. Therefore, we may assume that g_i is not a fence. It will be enough for us to show that by deleting extra β paths, every remaining path in the interval containing g_i contains the edge e.

Let $\mathcal{A}$ be the set of all paths before g_i in this interval. We may assume that $|\mathcal{A}| \geq \beta$. By Proposition 18, e is in either $L_{\mathcal{P}}$ or $R_{\mathcal{P}}$. We consider two cases (see Figure 5 for an illustration).

Case A. e **is in** $L_{\mathcal{P}}$**.**

We obverse that g_i contains no transforming edge which lies below e in $L_{\mathcal{P}}$ (as otherwise, g_i would be the first path in $\mathcal{P}$ containing this transforming edge and thus become a fence). There exists at least one transforming edge e' in $L_{\mathcal{P}}$ such that e lies in the subpath of $L_{\mathcal{P}}$ between u_0 and e'. Let $\mathcal{B}$ be the set of all paths in $\mathcal{P}$ containing e'. Then $|\mathcal{B}| \geq \gamma$ and all paths in $\mathcal{B}$ also contain e. There are at least $\gamma - 2n^{1/4} \geq \beta$ paths $g_k \in \mathcal{B}$ such that $\{g_k, g_{i-1}\}$ and $\{g_k, g_i\}$ are normal. From now on, we fix such a path g_k. Note that g_{i-1} does not contain e, while g_i and g_k contain e.

Let a, b be the two splitting vertices in $g_i \cup g_k$. Clearly $V(e) \preceq a \prec b$. Also we see that $\mathrm{LS}(g_{i-1}, g_i)$ and $\mathrm{LS}(g_{i-1}, g_k)$ are the same vertex, say w, such that $w \preceq V(e)$. We make the following claim.

Claim A. Let $z = \mathrm{RS}(g_{i-1}, g_i)$. Then z is an inner vertex in ag_ib.

Proof of Claim A. Let $z' = \mathrm{RS}(g_{i-1}, g_k)$. If $z' \prec b$, then clearly $z'g_{i-1}v = z'g_kv$, implying that $z' \prec b = z$, a contradiction to Proposition 19 (ii). So $z' \succeq b$. If $z \succeq b$, then we see $z = z'$. In this case, g_k contains some transforming edge (that is e') in $L_{\mathcal{P}}$, while g_i does not, a contradiction to Proposition 19 (iii). If $z \preceq a$, then $wg_{i-1}v = wg_{i-1}z \cup zg_iv$ and thus $\{g_{i-1}, g_k\}$ must be type-I, a contradiction to the choice of g_k. Hence, z is an inner vertex in af_ib (and also $\mathrm{RS}(g_{i-1}, g_k) = b$). □

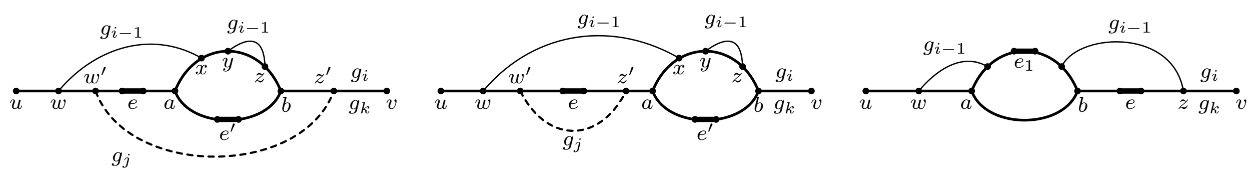
If $\{g_{i-1}, g_i\}$ is not type-I, then $g_{i-1}\backslash(g_i \cup g_k)$ is a subpath $wg_{i-1}z$, which shows that $\{g_{i-1}, g_k\}$ is type-II with a crossing path g_i, a contradiction.

Therefore, $\{g_{i-1}, g_i\}$ is type-I with base f_ℓ and splitting vertices $w \prec x \preceq y \prec z$. Since w, b are the only two splitting vertices in $g_{i-1} \cup g_k$, we see that $w \prec a \prec x \preceq y \prec z \prec b$ lie in g_i. Let $f_{\pi(t)} = g_t$ for every $t \in \{i-1, i, k\}$. If $P_{\pi(i-1)} \subseteq wg_{i-1}x$, then again $\{g_{i-1}, g_k\}$ is type-II with a crossing path g_i. So by Proposition 6, $P_{\pi(i)} \subseteq wg_ix$, $P_{\pi(i-1)} \subseteq yg_{i-1}z$, and x, y are inner vertices of P_ℓ. This shows that

$$ag_ix \text{ contains some edge in } P_{\pi(i)} \text{ and } xg_{i-1}w = xf_\ell w \text{ contains some edge in } P_\ell. \tag{7}$$

There exist at least $\beta - 3n^{1/4} - 1 \geq \beta/2$ paths g_j in $\mathcal{A}\backslash\{f_\ell\}$ such that g_j forms a normal pair with any path in $\{g_{i-1}, g_i, g_k\}$. Any such path g_j does not contain the edge e. So we can set $w' = \mathrm{LS}(g_j, g_i) = \mathrm{LS}(g_j, g_k)$ such that $w' \preceq V(e)$. Then as g_j forms a normal pair with each of g_i and g_k, we can infer that $g_j\backslash(g_i \cup g_k)$ is a path, say $w'g_jz'$ with $z' \in V(y'g_iv \cup ag_kb)$, where we write $e = x'y'$ with $x' \prec y'$. Note that $z' \in V(y'g_ia)$ is impossible, as otherwise we can deduce that $ag_jv = ag_iv = ag_kv$, a contradiction. If $z' \in V(ag_kb)\backslash\{a, b\}$, then $\{g_j, g_i\}$ is type-II with a crossing path g_k, a contradiction. If $z' \in V(ag_ib)\backslash\{a, b\}$, then $\{g_j, g_k\}$ is type-II with a crossing path g_i, a contradiction. So we have $z' \in V(bg_iv)$ and $g_j = ug_iw' \cup w'g_jz' \cup z'g_iv$. Let $f_{\pi(j)} = g_j$. Whenever the two paths $w'g_jz'$ and $wg_{i-1}x = wf_\ell x$ intersect or not (if they do then $w = w'$), $xf_\ell w \cup wg_iw' \cup w'g_jz'$ contains an (x, z')-path Q which contains some edge in P_ℓ and some edge in $P_{\pi(j)}$. By (7), we see that $Q \cup z'g_ka \cup ag_ix$ contains a 4-feasible cycle for $\{g_i, g_j, g_k, f_\ell\}$,

Putting all together, there are at least β choices of $g_k \in \mathcal{B}$ and subject to a fixed g_k, there are at least $\beta/2$ choices of $g_j \in \mathcal{B}$ such that $\{g_i, g_j, g_k\}$ is contained in a feasible quadruple. So g_i in $\mathcal{P}$ is contained in at least $\beta^2/2 > 26\sqrt{n}\log n$ distinct $\mathcal{W}_{jk}$'s. This contradicts the definition of $\mathcal{P}$ and completes the proof of Case A.



Case A: $z' \in V(bg_iv)$ Case A: $z' \in V(y'g_ia)$ Case B: A key step for Claim B

Figure 5

Case B. e is in $R_{\mathcal{P}}$.

The proof of this case is similar to that of Case A. We also see that g_i contains no transforming edge which lies in $R_{\mathcal{P}}$ below e (as otherwise g_i is a fence). So there exists a transforming edge e_0 in $R_{\mathcal{P}}$ such that e lies in the path of $R_{\mathcal{P}}$ between v_0 and e_0. Let $\mathcal{C}$ be the set of all paths in $\mathcal{P}$ containing e_0. Then $|\mathcal{C}| \geq \gamma$ and all paths in $\mathcal{C}$ also contain e. There are at least $\gamma - 2n^{1/4} \geq \beta$ paths $g_k \in \mathcal{C}$ such that $\{g_k, g_{i-1}\}$ and $\{g_k, g_i\}$ are normal. Let a, b be the only two splitting vertices in $g_k \cup g_i$. Note that g_i and g_k contain e, while g_{i-1} does not. So $\mathrm{RS}(g_{i-1}, g_i)$ and $\mathrm{RS}(g_{i-1}, g_k)$ are the same vertex, say z, such that $a \prec b \preceq V(e) \preceq z$.

We need the following claim, which plays the parallel role as Claim A in the previous case.

Claim B. Let $w = \mathrm{LS}(g_{i-1}, g_i)$. Then w is an inner vertex in ag_ib.

Proof of Claim B. First suppose that $b \preceq w$. Since $ug_{i-1}w = ug_iw$ contains ag_ib, we see that $\{g_{i-1}, g_k\}$ must be a type-I pair, a contradiction to the choice of g_k.

To prove Claim B, it suffices to consider when $w \preceq a$ (see Figure 5). Let $w' = \mathrm{LS}(g_{i-1}, g_k)$. If $w' \succ a$, then clearly $ug_{i-1}w' = ug_kw'$, implying that $w = a \prec w'$, a contradiction to Proposition 19 (ii). So $w' \preceq a$. In this case, we see $w = w' \in V(ug_ia)$, that is, g_i and g_k share the same first and last splitting vertices with g_{i-1}. Since g_k contains a transforming edge in $R_{\mathcal{P}}$ while g_i does not, by Proposition 19 (iv), g_i contains a transforming edge (say e_1) in $L_{\mathcal{P}}$ and g_k does not. Clearly, such e_1 is in ag_ib. If g_{i-1} does not contain e_1, then g_i is the first path in $\mathcal{P}$ containing e_1 and thus g_i is a fence, a contradiction. Hence, g_{i-1} also contains $e_1 \in E(L_{\mathcal{P}})$. By Proposition 18, g_{i-1} and g_i contains all edges in the subpath of L from u to e_1. This indicates that $\mathrm{LS}(g_{i-1}, g_i) = w \succ a$, a contradiction. Therefore, w must be an inner vertex in ag_ib. □

The remaining proof is analogous to Case A. In fact, once we are equipped with Claims A and B, these two cases are identical if one revises the linear ordering $\prec$. For simplicity, we omit the detailed verification of the remaining proof of Case B here. We finish the proof of Lemma 5.1. □

6 Proof of the main result

Now we are ready to prove Theorem 1.2, by using a counting strategy motivated by [7]. Let n be sufficiently large and G be an n-vertex 2-connected graph with $n+s$ edges which does not contain two cycles of the same length. Suppose for a contradiction that $s \geq (1+o(1))\sqrt{n}$.

Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ be subsets of $\mathcal{F}$ and β, γ be constants from Lemma 5.1. So $\beta \geq n^{1/4}\log n$, $\gamma = \beta \log n \leq \sqrt{n}$, $s' := \sum_{i\in[4]} |\mathcal{P}_i| \geq (1-o(1))s$, and each $\mathcal{P}_i$ has an arrangement $\{g_j\}_{j\geq 1}$.

Let Φ be the set of all normal pairs $\{g_j, g_k\}$ such that g_j, g_k are from the same interval of $\mathcal{P}_i$ for some $i \in [4]$ and $\beta \leq |j-k| \leq \sqrt{\beta\gamma}$. For $e \in E(G)$ and $\{g_j, g_k\} \in \Phi$, let $\delta(e, g_j, g_k)$ be an index function such that $\delta = 1$ if exactly one of the paths g_j, g_k contains e and $\delta = 0$ otherwise.

In the coming proof, we are estimating the summation Σ of $\delta(e, g_j, g_k)$ over all $e \in E(G)$ and $\{g_j, g_k\} \in \Phi$. First, let us bound the size of Φ. The total number of pairs $\{g_j, g_k\}$ such that g_j, g_k lie in some $\mathcal{P}_i$ and $\beta \leq |j-k| \leq \sqrt{\beta\gamma}$ is at least $\sum_{r=\beta}^{\sqrt{\beta\gamma}}(s'-4r) \geq (1-o(1))s\sqrt{\beta\gamma} = (1-o(1))s\beta\sqrt{\log n}$, where the inequality holds because $s' = \sum_{i\in[4]} |\mathcal{P}_i| \geq (1-o(1))s$ and every $r \leq \sqrt{\beta\gamma} = \gamma/\sqrt{\log n} \leq \sqrt{n/\log n} = o(s)$. By Lemma 5.1, there are at most $3\sqrt{n}/\gamma$ intervals in $\mathcal{P}_i$ for each $i \in [4]$, so there are at most $(12\sqrt{n}/\gamma) \cdot (\sqrt{\beta\gamma})^2 = 12\sqrt{n}\beta$ pairs $\{g_j, g_k\}$ such that g_j, g_k lie in different intervals of some $\mathcal{P}_i$ and $|j-k| \leq \sqrt{\beta\gamma}$. We also know that each $\mathcal{P}_i$ contains at most $2\sqrt{n}\log^2 n$ pairs of type-I and type-II. Putting all together, as $s \geq (1+o(1)\sqrt{n}$ and $\beta \geq n^{1/4}\log n$, we see that

$$|\Phi| \geq (1-o(1))s\beta\sqrt{\log n} - 12\sqrt{n}\beta - 8\sqrt{n}\log^2 n \geq (1-o(1))s\beta\sqrt{\log n}.$$

Since each $\{g_j, g_k\}$ in Φ is a normal pair, the difference $g_j \triangle g_k$ induces a cycle. One can see that all such cycles $g_j \triangle g_k$ are distinct and thus have different lengths. Observe that $\sum_{e\in E(G)} \delta(e, g_j, g_k)$ equals the length of the cycle $g_j \triangle g_k$, denoted by $|g_j \triangle g_k|$. Therefore, the summation

$$\Sigma = \sum_{\{g_j,g_k\}\in\Phi} \left(\sum_{e\in E(G)} \delta(e, g_j, g_k) \right) = \sum_{\{g_j,g_k\}\in\Phi} |g_j \triangle g_k| \geq \sum_{\ell=1}^{|\Phi|} \ell \geq \frac{|\Phi|^2}{2} \geq \left(\frac{1}{2} - o(1)\right) s^2\beta\gamma. \tag{8}$$

Next we fix $e \in E(G)$ and estimate the sum of $\delta(e, g_j, g_k)$ over all $\{g_j, g_k\} \in \Phi$. Note that any paths $g_j, g_k \in \mathcal{P}_i$ share the same edges out of $E(G(\mathcal{P}_i))$, so it suffices for us to consider edges $e \in E(G(\mathcal{P}_i))$ for some $i \in [4]$ (while for other edges e, the above sum always equals zero).

For $e \in E(G(\mathcal{P}_i))$ with $d(e) \geq \gamma$, by Lemma 5.1 one can delete 3β paths in $\mathcal{P}_i$ such that there is at most one interval of $\mathcal{P}_i$ which can contain some remaining paths g_j, g_k with $e \in E(g_j)$ and $e \notin E(g_k)$ and moreover, all such paths g_j, g_k satisfy $j < k$. Let $\Phi(e)$ be the set of pairs in Φ containing at least one of the above 3β paths we delete. So $|\Phi(e)| \leq 3\beta \cdot 2\sqrt{\beta\gamma} = 6\beta\gamma/\sqrt{\log n}$, and for any $\beta \leq r \leq \sqrt{\beta\gamma}$, there are at most r pairs $\{g_j, g_k\}$ in $\Phi\backslash\Phi(e)$ satisfying $\delta(e, g_j, g_k) = 1$. For $e \in E(G(\mathcal{P}_i))$ with $d(e) \leq \beta$, we let $\Phi(e)$ be the set of all pairs in Φ which contains at least one path containing e. Then it is clear that $|\Phi(e)| \leq \beta \cdot 2\sqrt{\beta\gamma} = 2\beta\gamma/\sqrt{\log n}$ and $\delta(e, g_j, g_k) = 0$ for all $\{g_j, g_k\}$ in $\Phi\backslash\Phi(e)$. Hence, for every e with $d(e) \geq \gamma$ or $d(e) \leq \beta$, we have

$$\sum_{\{g_j,g_k\}\in\Phi} \delta(e, g_j, g_k) \leq \left(\sum_{\{g_j,g_k\}\in\Phi\backslash\Phi(e)} \delta(e, g_j, g_k) \right) + |\Phi(e)| \leq \sum_{r=\beta}^{\sqrt{\beta\gamma}} r + 6\beta\gamma/\sqrt{\log n} \leq \left(\frac{1}{2} + o(1)\right)\beta\gamma.$$

By Lemma 5.1, there are at most $9n \log\log n/\log n$ edges e with $\beta \leq d(e) \leq \gamma$. Each such e is contained in at most γ paths g_j, while at most $2\sqrt{\beta\gamma}$ paths g_k can satisfy $\{g_j, g_k\} \in \Phi$. So at most $\gamma \cdot 2\sqrt{\beta\gamma} = 2\beta^2(\log n)^{3/2}$ pairs $\{g_j, g_k\} \in \Phi$ can give $\delta(e, g_j, g_k) = 1$. This implies that

$$\sum_{\beta\leq d(e)\leq\gamma} \sum_{\{g_j,g_k\}\in\Phi} \delta(e, g_j, g_k) \leq \frac{9n \log\log n}{\log n} \cdot 2\beta^2(\log n)^{3/2} = o(n\beta\gamma).$$

Adding all edges $e \in E(G)$ together, we can obtain the following upper bound

$$\Sigma = \sum_{e\in E(G)} \left(\sum_{\{g_j,g_k\}\in\Phi} \delta(e, g_j, g_k) \right) \leq (n + s) \cdot \left(\frac{1}{2} + o(1)\right)\beta\gamma + o(n\beta\gamma) = \left(\frac{1}{2} + o(1)\right)n\beta\gamma. \quad (9)$$

Combining with (8) and (9), we can derive that $s^2 \leq (1 + o(1))n$ and thus $s \leq (1 + o(1))\sqrt{n}$. This finishes the proof of Theorem 1.2. □

7 Concluding remarks

In this paper, we prove that any n-vertex 2-connected graph G with no two cycles of the same length has at most $n + \sqrt{n} + o(\sqrt{n})$ edges. We remark that through a more careful calculation, the present proof can show that G contains at most $n + \sqrt{n} + 20\sqrt{n/\log n}$ edges.

We also would like to point out that all statements in Sections 3–5 can hold only under the assumption (2). Indeed, throughout Sections 3–5, all we need in the proofs is just the upper bound on the total number of cycles, while the stronger assumption that G contains at most one cycle of length i for each $3 \leq i \leq n$ was only used in Section 6. That also says, the structural constraints we develop in Sections 3–5 also apply to 2-connected graphs G with relatively many edges but few cycles. In particular, an analog of Lemma 5.1 still holds for 2-connected graphs G with n vertices, $n + s$ edges and m cycles, assuming that $m \ll s^3$. This may shed some light on an old problem of Entringer from 1973, which asks to determine all graphs G with exactly one cycle of each length between 3 and $|V(G)|$ (see [1], p. 247, Problem 10).

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