

On tight cycles in hypergraphs

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Abstract

A tight k -uniform ℓ -cycle, denoted by TC_ℓ^k , is a k -uniform hypergraph whose vertex set is $v_0, \dots, v_{\ell-1}$, and the edges are all the k -tuples $\{v_i, v_{i+1}, \dots, v_{i+k-1}\}$, with subscripts modulo ℓ . Motivated by a classic result in graph theory that every n -vertex cycle-free graph has at most $n - 1$ edges, Sós and, independently, Verstraëte asked whether for every integer k , a k -uniform n -vertex hypergraph without any tight k -uniform cycles has at most $\binom{n-1}{k-1}$ edges. In this paper, we answer this question in negative.

1 Introduction

A classic result in graph theory says that if an n -vertex graph G contains no cycle, then G has at most $n - 1$ edges. This upper bound is tight by the n -vertex trees. It is natural to ask whether such extremal result can be generalized to k -uniform hypergraphs. There are many different notions of cycles in k -uniform hypergraphs, most notably Berge cycles, loose cycles, and tight cycles (see [22] for the definitions). They all coincide with the usual definitions of cycles when $k = 2$. For the Turán numbers on Berge cycles and loose cycles, there has been extensive research in the literature (see [2, 3, 7, 9, 10, 12, 13, 14, 15]).

In this paper, we consider Turán-type problems of the *tight k -uniform ℓ -cycle*, denoted by TC_ℓ^k (for $\ell \geq k + 1$), which is the k -uniform hypergraph whose vertex set is $v_0, \dots, v_{\ell-1}$, and the edges are all the consecutive k -tuples $\{v_i, v_{i+1}, \dots, v_{i+k-1}\}$, with subscripts modulo ℓ . It seems that the Turán-type problems for tight cycles are more difficult, perhaps because of its interlocking structure. When the desired tight cycle has length linear in n , Allen et al. [1] showed that for any $\delta > 0$, there exists n_0 such that for any $0 \leq \alpha \leq 1$, every k -uniform hypergraph on $n \geq n_0$ vertices with at least $(\alpha + \delta) \binom{n}{k}$ edges contains a tight cycle of length at least αn . For $\alpha = 1$ (i.e. Hamiltonian cycles), Rödl, Ruciński and Szemerédi [20, 21] established an approximate extension of Dirac's Theorem for k -uniform hypergraphs: for all $\gamma > 0$, every sufficiently large k -uniform hypergraph such that every $(k - 1)$ -set of vertices lies in at least $(1/2 + \gamma)n$ edges has a tight Hamilton cycle. Both results uses the regularity lemma which is powerful when the extremal hypergraph is dense. Nevertheless, for shorter tight cycles, in particular cycles of constant lengths divisible by k , the extremal hypergraph is sparse and much less has been known.

Back to our initial question, we call a k -uniform hypergraph *tight-cycle-free* if it does not contain any TC_ℓ^k for $\ell \geq k + 1$. Throughout this paper we denote by $f_k(n)$ the maximum number

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of edges in a k -uniform n -vertex tight-cycle-free hypergraph. In this notation, $f_2(n) = n - 1$. Indeed it is not hard to see that for every integer $k \geq 2$, $f_k(n) \geq \binom{n-1}{k-1}$. A k -uniform hypergraph is called a *full- k -star* if it consists of all the k -subsets of $[n]$ containing a given element called the *center*. When the uniformity is clear from the context, we will simply call it a full-star. One can easily verify that a full- k -star has $\binom{n-1}{k-1}$ edges and contains no tight cycles, since any tight cycle must have length at least $k + 1$ and thus at least one of its edges cannot contain the center. Sós and Verstraëte (see [22]) independently asked whether $f_k(n)$ always equals $\binom{n-1}{k-1}$. This conjecture is still wide open. In particular it is not even clear if $f_k(n) = \Theta(n^{k-1})$. One easy upper bound follows from an observation that TC_{2k}^k is actually a sub-hypergraph of the complete k -partite k -uniform hypergraph $K_{2, \dots, 2}^k$, whose Turán number was shown to be $O(n^{k-(1/2)^{k-1}})$ by Erdős [4]. This gives $f_k(n) = O(n^{k-(1/2)^{k-1}})$. For $k = 3$, currently the best known upper bound is $f_3(n) = O(n^{5/2})$ proved by Verstraëte [23], by showing that the Turán number of TC_{24}^3 is $O(n^{5/2})$.

In this paper, we prove the following theorem, giving a negative answer to the Sós-Verstraëte Conjecture.

Theorem 1.1. *For every integer $k \geq 3$, there exists a constant $c = c(k) > 0$ such that for sufficiently large n ,*

$$f_k(n) \geq (1 + c) \binom{n-1}{k-1}.$$

The rest of the paper is organized as follows. In the next section, we first present an explicit construction showing that $f_3(n)$ is strictly greater than $\binom{n-1}{2}$, but only better by a linear function in n . In Section 3, we generalize this idea to construct denser hypergraphs without tight cycles and prove our main result Theorem 1.1. The final section contains some concluding remarks and open problems. Throughout this paper, we write $[n]$ for the set $\{1, 2, \dots, n\}$.

2 Warm-up

As a warm-up, we construct a 3-uniform n -vertex hypergraph H with $\binom{n-1}{2} + \frac{n-1}{2}$ edges, so that H does not contain any tight cycle. The key idea is that one can slightly modify the full-3-star by removing some edges containing its center, and adding twice as many edges, while preserving the tight-cycle-freeness.

Proposition 2.1. *For odd integers $n \geq 7$,*

$$f_3(n) \geq \binom{n-1}{2} + \frac{n-1}{2}.$$

Proof. Let H be the 3-uniform hypergraph on the vertex set $[n]$ consisting of the following two types of edges. Type-I edges include all the triples containing 1, except for those in the form $\{1, 2i, 2i+1\}$ for $i = 1, \dots, (n-1)/2$. Type-II edges are those in the form $\{2i, 2i+1, 2i+2\}$ or $\{2i, 2i+1, 2i+3\}$, for $i = 1, \dots, \frac{n-3}{2}$, or the two triples $\{n-1, n, 2\}$ and $\{n-1, n, 3\}$. There are $\binom{n-1}{2} - (n-1)/2$ edges of Type I, and $n-1$ edges of Type II. So $e(H) = \binom{n-1}{2} + (n-1)/2$. Notice that the only two edges in H containing some $x \in \{2i, 2i+1\}$ and $y \in \{2i+2, 2i+3\}$ are $\{1, x, y\}$ and $\{2i, 2i+1, y\}$.

Suppose H contains a tight cycle. Such cycle must have length at least 4. In particular, it contains at least one Type-II edge. Without loss of generality, we assume it is the triple $\{2, 3, 4\}$. Modulo symmetry, there are two possibilities of the relative order of these three vertices on the cycle: (i) they are in the order 2, 3, 4; (ii) they are in the order 2, 4, 3.

In the first case, since the only edges containing both 2 and 3 are the Type-II edges $\{2, 3, 4\}$ and $\{2, 3, 5\}$, and the only edges containing both 3 and 4 are $\{2, 3, 4\}$ and a Type-I edge $\{1, 3, 4\}$, it is not hard to see that the cycle must contain consecutive vertices 5, 2, 3, 4, 1. Now the only edge (except $\{2, 3, 5\}$) containing 2 and 5 is $\{1, 2, 5\}$. This shows that the cycle consists of vertices 5, 2, 3, 4, 1. However $\{1, 4, 5\} \notin E(H)$, therefore Case (i) is impossible.

Similarly, for the second case, the cycle must consist of (consecutive) vertices 2, 4, 3, 1 but $\{1, 2, 3\} \notin E(H)$. Therefore both cases are not possible, and H does not contain a tight cycle. This shows that $f_3(n) \geq \binom{n-1}{2} + \frac{n-1}{2}$. \square

We remark that the above construction remains tight-cycle-free, if the range of the index i is changed from $1 \leq i \leq (n-1)/2$ to any integer set of size at least three.

3 Proof of Theorem 1.1

In this section, we prove our main theorem, showing that for every integer $k \geq 3$, there exist tight-cycle-free k -uniform hypergraphs having c -fractional more edges than the full- k -star for some constant $c > 0$.

3.1 The key lemma

The following lemma will be critical for the proof of Theorem 1.1. Loosely speaking, it can be used to reduce Theorem 1.1 to the problem of just finding any tight-cycle-free k -uniform hypergraphs with more edges than the full- k -star.

The t -shadow of a hypergraph H , denoted by $\partial_t(H)$, is defined as the following:

$$\partial_t(H) = \{S : |S| = t, S \subset e \text{ for some } e \in E(H)\}.$$

If H consists of a single edge e , then we will write it as $\partial_t(e)$.

Lemma 3.1. *Let H be a full- k -star with the vertex set $\{0\} \cup [n]$ and the center 0. Let G_1, \dots, G_t be subhypergraphs of H , and F_1, \dots, F_t be k -uniform hypergraphs on $[n]$. Suppose they satisfy the following properties:*

- (i) *The hypergraph $H_i := (H \setminus G_i) \cup F_i$ is tight-cycle-free for all $i = 1, \dots, t$, and*
- (ii) *$\partial_{k-1}(F_i) \cap \partial_{k-1}(F_j) = \emptyset$ for all $1 \leq i < j \leq t$.*

Then

$$H' := \left(H \setminus \bigcup_{i=1}^t G_i \right) \cup \left(\bigcup_{i=1}^t F_i \right)$$

is also tight-cycle-free.

Proof. We prove by contradiction. Suppose that H' contains a tight cycle $C = v_1 v_2 \dots v_\ell$. If $0 \notin V(C)$, then all the edges of C must come from $F_1 \cup \dots \cup F_t$. Note that the $(k-1)$ -shadows of distinct F_i 's are disjoint, therefore all the edges of C must come from the same F_i (as, otherwise, there exist two consecutive edges of C from distinct F_i 's which would share $k-1$ common vertices). However, from (i), each H_i and in particular each F_i is tight-cycle-free, which is impossible.

Now suppose $0 \in V(C)$. Using similar arguments, we can show that the $l-k$ edges of C not containing 0 are from the same F_i . From the definition of H' , we see that the k edges of C containing 0 must be in H_i . Therefore C is completely contained in H_i , again contradicting that H_i has no tight cycle. \square

3.2 The 3-uniform case

We start with the case $k = 3$. We will use the previous lemma to prove the following theorem.

Theorem 3.2. *For sufficiently large n ,*

$$f_3(n) \geq \left(1 + \frac{1}{5}\right) \binom{n-1}{2} + O(n).$$

Proof. It suffices to show that if n is sufficiently large and $n - 7$ is divisible by 30, then $f_3(n) \geq (1 + 1/5)\binom{n-1}{2}$. Consider the full-3-star with the vertex set $\{0, 1, \dots, n-1\}$ and the center 0. Let $F = \{123, 124, 345, 346, 561, 562\}$ and $G = \{012, 034, 056\}$. We call G as the *accompanying set* of F .

By the remark in the end of Section 2, the 3-uniform hypergraph obtained from the full-star by adding edges in F and deleting edges in G is tight-cycle-free. This together with Lemma 3.1 show that if one can find many copies of F (say F_1, \dots, F_t) on the vertex set $\{1, \dots, n-1\}$ whose 2-shadows are pairwise disjoint, then the hypergraph obtained from the full-3-star by adding $F_1 \cup \dots \cup F_t$ and deleting their accompanying sets is also tight-cycle-free. And $e(H) = \binom{n-1}{2} + 3t$.

Let L be the 2-shadow of F , which is a 5-regular graph on 6 vertices, i.e. K_6 . To find such F -copies, it is enough to find many edge-disjoint L -copies in the complete graph K_{n-1} on $\{1, \dots, n-1\}$. By a famous theorem of Wilson [24, 25, 26, 27], if n is sufficiently large and satisfies $6|(n-1)$ and $5|(n-2)$, then one can partition the edge set of K_{n-1} into edge-disjoint copies of L . Since $n - 7$ is divisible by 30, indeed we can apply this result to get

$$t = \frac{1}{e(L)} \binom{n-1}{2} = \frac{1}{15} \binom{n-1}{2}$$

copies of sub-hypergraphs isomorphic to F , whose 2-shadows are pairwise disjoint. Therefore, by Lemma 3.1 we can find a tight-cycle-free 3-graph H with

$$e(H) = \binom{n-1}{2} + 3t = \left(1 + \frac{1}{5}\right) \binom{n-1}{2},$$

completing the proof of Theorem 3.2. □

It is not clear how to generalize the cyclic-type construction in Section 2 to k -uniform hypergraphs for $k \geq 4$. For that, we will show a different construction beating the conjectured $\binom{n-1}{k-1}$ bound for the general case in the next subsection; then applying Lemma 3.1 again would imply a construction with $(1 + c)\binom{n-1}{k-1}$ edges for some $c > 0$.

3.3 The general case: k -uniform

Lemma 3.3. *Let H be a full- k -star with center 0. If we*

- *add new edges $e_1, \dots, e_t \subseteq V(H) \setminus \{0\}$ such that $\partial_{k-1}(e_i) \cap \partial_{k-1}(e_j) = \emptyset$ for all $i \neq j$, and*
- *delete edges $\{0\} \cup f_i$ for all $1 \leq i \leq t$, where f_i is any $(k-1)$ -subset of e_i ,*

then the resulting k -uniform hypergraph H' remains tight-cycle-free and has the same number of edges with the full- k -star.

Proof. Suppose for contradiction that H' contains a tight cycle C . Then C has to contain a new edge, say $e_1 = \{1, \dots, k\}$. Since the $(k-1)$ -shadows of e_i 's are disjoint, it is not hard to see that the only possibility to have a tight cycle C is that $V(C) = \{0, 1, \dots, k\}$. This also means that C is a complete k -uniform hypergraph on $k+1$ vertices. But this is impossible as we have deleted an edge $\{0\} \cup f_1$ for some $(k-1)$ -subset $f_1 \subseteq e_1$. This proves the lemma. □

Lemma 3.4. *For every integer $k \geq 3$, there exists a positive integer $m = m(k)$ such that the following holds for any $n \geq m$. Let H be a full- k -star with the vertex set $\{0\} \cup [n]$ and the center 0. Then there exist two k -uniform hypergraphs F and G such that*

- $V(F)$ is a subset of size at most m in $[n]$,
- G is a subhypergraph of H ,
- $H' := (H \setminus G) \cup F$ is tight-cycle-free with $e(H) + 1$ edges.

Proof. Let $e_0 = [k]$. For all $1 \leq i \leq k$, let $f_i := [k] \setminus \{i\}$ be a $(k-1)$ -subset and $e_i := f_i \cup \{k+i\}$. We will construct $F = \{e_0, e_1, \dots, e_{k+M}\}$ for some positive $M = O(k^3)$ and

$$G = \{\{0\} \cup f_i : f_i \subseteq e_i \text{ and } |f_i| = k-1 \text{ for every } 1 \leq i \leq k+M\}$$

such that $\partial_{k-1}(e_i) \cap \partial_{k-1}(e_j) = \emptyset$ for all $1 \leq i < j \leq k+M$. Let $H' = (H \setminus G) \cup F$. Then $e(H') = e(H) - e(G) + e(F) = e(H) + 1$.

Suppose we have such F and G . By Lemma 3.3, $(H \setminus G) \cup (F \setminus \{e_0\})$ is tight-cycle-free. So all potential tight cycles C in H' must contain the edge e_0 . We will discuss all possibilities for such C as follows. First since the full k -star is tight-cycle-free, C must contain at least one edge from $\{e_0, e_1, \dots, e_{k+M}\}$. If only e_0 appears in C , then $V(C) = \{0, 1, \dots, k\}$, however this is impossible since all the edges $\{0\} \cup f_i$ ($1 \leq i \leq k$) are deleted in H' . Recall that we would choose e_i 's so that their $(k-1)$ -shadows are pairwise disjoint. Therefore C has at least one edge in $\{e_1, \dots, e_k\}$; and it is also not hard to see that the tight cycle C can use at most two edges in $\{e_1, \dots, e_k\}$.

Consider the case that C contains exactly one edge in $\{e_1, \dots, e_k\}$, say e_ℓ for some $\ell \in [k]$. Then the cycle C must have consecutive vertices $\pi(1), \pi(2), \dots, \pi(k), k+\ell$, where $\pi : [k] \rightarrow [k]$ is a permutation, $\pi(1) = \ell$ and $e_\ell = \{\pi(2), \dots, \pi(k), k+\ell\}$. These vertices alone do not form a tight cycle, otherwise it would violate that $\partial_{k-1}(e_i) \cap \partial_{k-1}(e_j) = \emptyset$ for all $1 \leq i < j \leq k+M$. Consider the vertex immediate before $\pi(1)$. For the same reason, this vertex can only be 0, however this is also not possible since for all $(k-1)$ -subset $f_j \subset [k]$, $\{0\} \cup f_j \notin E(H')$.

Now we may assume that C contains two edges in $\{e_1, \dots, e_k\}$, say e_i and e_j for some $1 \leq i < j \leq k$. By similar analysis as above, the only possible tight cycle C that may appear in H' is on the vertex set in the order $k+i, \pi(1), \dots, \pi(k), k+j, 0$ for some permutation π of $[k]$ with $\pi(1) = j$ and $\pi(k) = i$. For all $1 \leq i < j \leq k$ and all sets $\{\pi(4), \dots, \pi(k)\}$ (there are at most $\binom{k}{3}$ such $(k-3)$ -sets), we let

$$f_{k+\alpha} = \{\pi(4), \dots, \pi(k), k+j, k+i\}$$

be all such $(k-1)$ -subsets (say M of them in total), where $M \leq 3\binom{k}{3} = O(k^3)$, and we may label them by $1 \leq \alpha \leq M$.

By removing the edges $\{0\} \cup f_{k+\alpha}$, we destroy all the tight cycle mentioned above. One can find k -subsets e_{k+1}, \dots, e_{k+M} with pairwise distinct $(k-1)$ -shadows such that $f_{k+\alpha} \subset e_{k+\alpha}$ for all $\alpha = 1, \dots, M$. For example, let $e_{k+\alpha} = f_{k+\alpha} \cup \{2k+\alpha\}$. It remains to check that these new $e_{k+\alpha}$'s have distinct $(k-1)$ -shadows from e_1, \dots, e_k . This follows from the observation that $2k+j \notin e_l$ for $1 \leq l \leq k$, and at least one of $\{k+j, k+i\}$ is also not in e_l . We conclude this proof by noting that the value of m can be $O(k^3)$. \square

We are ready to prove our main result – Theorem 1.1. It is a combination of the construction in Lemma 3.4 and some classic hypergraph packing results.

Proof of Theorem 1.1. Let n be sufficiently large and H be the full- k -star with center 0 on $\{0, \dots, n-1\}$. By Lemma 3.4, there exists a k -uniform tight-cycle-free hypergraph $H' = (H \setminus G) \cup F$ with $\binom{n-1}{k-1} + 1$ edges, where F is a k -uniform hypergraph on at most $m = O(k^3)$ vertices with $V(F) \subseteq [n-1]$ and G is a subhypergraph of H . Let us call G the *accompanying* hypergraph of F . By Lemma 3.1, if one can find copies of sub-hypergraphs on $[n-1]$ isomorphic to F , denoted by F_1, \dots, F_t , such that $\partial_{k-1}(F_i)$'s are pairwise disjoint, then

$$H'' = \left(H \setminus \bigcup_{i=1}^t G_i \right) \cup \left(\bigcup_{i=1}^t F_i \right)$$

is also tight-cycle-free, where G_i is the accompanying hypergraph of F_i , and $e(H'') \geq e(H) + t$.

Note that F is contained in a complete k -uniform hypergraph on $m = O(k^3)$ vertices. By the result of Rödl [19] on the Erdős-Hanani Conjecture [6], when n is sufficiently large, one can pack at least

$$t \geq (1 - o(1)) \frac{\binom{n-1}{k-1}}{\binom{m}{k-1}}$$

copies of F in $[n-1]$ so that their $(k-1)$ -shadows are disjoint. Therefore in H'' , the number of edges is equal to

$$e(H'') \geq e(H) + t \geq \binom{n-1}{k-1} + (1 - o(1)) \frac{\binom{n-1}{k-1}}{\binom{m}{k-1}} \geq (1 + c) \binom{n-1}{k-1},$$

for sufficiently large n . Here c is a positive constant which only depends on k (but not n). \square

4 Concluding remarks

In this paper, we construct k -uniform n -uniform hypergraphs that are tight-cycle-free and contains more edges than the full- k -star, disproving a conjecture of Sós and Verstraëte. Below are some observations and related problems.

- To better understand the asymptotic behavior of $f_k(n)$, it naturally leads to the question of determining the Turán number $ex(n, TC_\ell^k)$, i.e., the maximum number of edges in a TC_ℓ^k -free k -uniform n -vertex hypergraph. If ℓ is not divisible by k , TC_ℓ^k is not k -partite. Therefore the complete k -partite k -uniform hypergraph does not contain a copy of TC_ℓ^k and thus $ex(n, TC_\ell^k) = \Theta(n^k)$. If ℓ is divisible by k , it is likely that $ex(n, TC_\ell^k) = \Theta(n^{c_{k,\ell}})$ for some constant $c_{k,\ell} \in (k-1, k)$, and $c_{k,\ell} \rightarrow k-1$ when $\ell \rightarrow \infty$. A problem of Conlon (see [17]) asks for 3-uniform hypergraphs if there is a constant $c > 0$ such that $ex(n, TC_\ell^3) = O(n^{2+c/\ell})$ for all ℓ which are divisible by 3. If this answer is affirmative, then it implies that $f_3(n) = O(n^{2+o(1)})$. One can easily show $ex(n, TC_{ks}^k) = \Omega(n^{k-2} \cdot ex(n, C_{2s}))$. So the C_4 -free projective plane construction immediately gives $ex(n, TC_6^3) = \Omega(n^{5/2})$, while the best upper bound is $ex(n, TC_6^3) = O(n^{11/4})$ by Erdős [4].
- Let H be a 3-uniform hypergraph with vertex set $[n]$. One can define a 3-uniform hypergraph H^* with the vertex set being the 2-shadow $\partial_2(H)$ of H . And every edge $e = \{i, j, k\} \in H$ defines an edge $e^* = \{ij, ik, jk\}$ in H^* . It is easy to see that H^* is a linear 3-graph with $e(H^*) = e(H)$ and $v(H^*) = O(n^2)$. We say a loose cycle $(e_1^*, e_2^*, \dots, e_k^*)$ in H^* is *feasible*, if $|A_i \cap A_{i+1}| = 1$ and for other i, j , $|A_i \cap A_j| = 0$, where $A_i := e_i^* \cap e_{i+1}^*$ is a vertex in H^* and also a 2-subset in $[n]$. It is not hard to show that H is TC_ℓ^3 -free if and only if H^* does not contain a feasible loose cycle of length ℓ . Let G be an n -vertex linear 3-uniform hypergraph. It is known

[3] that if G has no loose cycles of length $2k$, then $e(G) = O(n^{1+1/k})$; and it is also known that if G has no loose cycles of any length, then $e(G) = O(n)$. If these proofs could be adapted for feasible loose cycles as well, then it immediately gives $ex(n, TC_6^3) = O((n^2)^{1+1/3}) = O(n^{8/3})$ confirming a conjecture in [22], and also $f_3(n) = \Theta(n^2)$.

- The following conjecture on tight paths was asked by Kalai, as an attempt to generalize the Erdős-Sós Conjecture [5]:

Conjecture 4.1. *For any integer k and ℓ , suppose H is a k -uniform n -vertex hypergraph not containing a tight k -uniform path of length ℓ , then*

$$e(H) \leq \frac{\ell - 1}{k} \binom{n}{k-1}.$$

Recently, Füredi et al. [8] prove an upper bound upper bound $e(H) \leq \frac{(\ell-1)(k-1)}{k} \binom{n}{k-1}$ for this problem, improving the trivial upper bound $e(H) \leq (\ell - 1) \binom{n}{k-1}$. An earlier result of Patkós [18] gives an upper bound $e(H) \leq \sum_{j=2}^{\ell} \frac{j-1}{k-j+2} \binom{n}{k-1} = O_{\ell}(\frac{1}{k} \binom{n}{k-1})$ for $k \geq 2\ell$, which is better than the previous bound if k is somewhat larger than ℓ . It would be very interesting to further improve bounds for Kalai's Conjecture. As a side note, in [11], Györi et al. proved that any k -uniform n -vertex hypergraph with more than $(\alpha n - k) \binom{n}{k-1}$ edges contains a tight path on αn vertices.

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References

- [1] P. Allen, J. Böttcher, O. Cooley, R. Mycroft, Tight cycles and regular slices in dense hypergraphs, *J. Combin. Theory Ser. A* **149** (2017), 30–100.
- [2] B. Bollobás, E. Györi, Pentagons vs. triangles, *Discrete Math.* **308** (2008), 4332–4336.
- [3] C. Collier-Cartaino, N. Graber, T. Jiang, Linear Turán numbers of r -uniform linear cycles and related Ramsey numbers, *Combin. Probab. Comput.*, to appear.
- [4] P. Erdős, On extremal problems of graphs and generalized graphs, *Israel J. Math.* **2** (1964), 183–190.
- [5] P. Erdős, Extremal problems in graph theory M. Fiedler (Ed.), *Theory of Graphs and its Applications*, Academic Press (1965), 29–36.
- [6] P. Erdős, H. Hanani, On a limit theorem in combinatorial analysis, *Publ. Math. Debrecen* **10** (1963), 10–13.
- [7] Z. Füredi, T. Jiang, Hypergraph Turán numbers of linear cycles, *J. Combin. Theory Ser. A* **123**, no. 1 (2014), 252–270.
- [8] Z. Füredi, T. Jiang, A. Kostochka, D. Mubayi, J. Verstraëte, Tight path and matchings in convex geometric hypergraphs, *arXiv:1709.01173*.
- [9] Z. Füredi, L. Özkahya, On 3-uniform hypergraphs without a cycle of a given length, *arXiv:1412.8083v2*.
- [10] E. Györi, Triangle-free hypergraphs, *Combin. Probab. Comput.* **15** (2006), 185–191.

- [11] E. Gyóri, G. Y. Katona, N. Lemons, Hypergraph extensions of the Erdős-Gallai theorem, *European J. Combin.* **58** (2016), 238–246.
- [12] E. Gyóri, N. Lemons, 3-uniform hypergraphs avoiding a given odd cycle, *Combinatorica* **32** (2012), 187–203
- [13] E. Gyóri, N. Lemons, Hypergraphs with no cycle of a given length, *Combin. Probab. Comput.* **21** (2012), 193–201.
- [14] T. Jiang, J. Ma, Cycles of given lengths in hypergraphs, *arXiv:1609.08212*.
- [15] A. Kostochka, D. Mubayi, J. Verstraete, Turán problems and shadows I: Paths and cycles, *J. Combin. Theory Ser. A* **129** (2015), 57–79.
- [16] F. Lazebnik, V. Ustimenko, A. Woldar, Polarities and $2k$ -cycle-free graphs, *Discrete Mathematics* **197/198** (1999), 503–513.
- [17] D. Mubayi, O. Pikhurko, B. Sudakov, Hypergraph Turán Problem: Some open questions, *manuscript*.
- [18] B. Patkós, A note on traces of set families, *Moscow Journal of Combinatorics and Number Theory*, **2**(1) (2012), 47–55.
- [19] V. Rödl, On a packing and covering problem, *European J. Combin.* **6** (1985), 69–78.
- [20] V. Rödl, A. Ruciński, E. Szemerédi, An approximate Dirac-type theorem for k -uniform hypergraphs, *Combinatorica* **28** (2008), no. 2, 229–260.
- [21] V. Rödl, A. Ruciński, E. Szemerédi, Dirac-type conditions for Hamiltonian paths and cycles in 3-uniform hypergraphs, *Adv. Math.* **227** (2011), no. 3, 1225–1299.
- [22] J. Verstraëte, Extremal problems for cycles in graphs, *Recent Trends in Combinatorics*, Springer, 83–116.
- [23] J. Verstraëte, private communication.
- [24] R. M. Wilson, An existence theory for pairwise balanced designs I. Composition theorems and morphisms, *J. Combin. Theory Ser. A* **13** (1972), 220–245.
- [25] R. M. Wilson, An existence theory for pairwise balanced designs II. The structure of PBD-closed sets and the existence conjectures, *J. Combin. Theory Ser. A* **13** (1972), 246–273.
- [26] R. M. Wilson, An existence theory for pairwise balanced designs III. Proof of the existence conjectures, *J. Combin. Theory Ser. A* **18** (1975), 71–79.
- [27] R. M. Wilson, Decompositions of complete graphs into subgraphs isomorphic to a given graph, In Proc. of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), pages 647–659, Congressus Numerantium, No. XV, Utilitas Math., Winnipeg, Man., 1976.