On the number of triangles in K_4-free graphs

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Abstract

Erdős asked whether for any n-vertex graph G, the parameter $p^*(G) = \min \sum_{i\geq 1}(|V(G_i)| - 1)$ is at most $\lfloor n^2/4 \rfloor$, where the minimum is taken over all edge decompositions of G into edge-disjoint cliques G_i. In a restricted case (also conjectured independently by Erdős), Győri and Keszegh [Combinatorica, 37(6) (2017), 1113–1124] proved that $p^*(G) \leq \lfloor n^2/4 \rfloor$ for all K_4-free graphs G. Motivated by their proof approach, they conjectured that for any n-vertex K_4-free graph G with e edges, and any greedy partition P of G of size r, the number of triangles in G is at least $r(e - r(n - r))$. If true, this would imply a stronger bound on $p^*(G)$. In this paper, we disprove their conjecture by constructing infinitely many counterexamples with arbitrarily large gap. We further establish a corrected tight lower bound on the number of triangles in such graphs, which would recover the conjectured bound once some small counterexamples we identify are excluded.

1 Introduction

The *Turán graph* $T_{n,k-1}$ denotes the complete balanced $(k-1)$-partite graph on n vertices, and let $t_{n,k-1}$ be its number of edges. A graph is K_k*-free* if it contains no copy of the clique K_k as a subgraph. The celebrated Turán's theorem [12], a cornerstone of extremal graph theory, states that the Turán graph $T_{n,k-1}$ is the unique n-vertex K_k-free graph with the maximum number of edges. Exploring various interpretations and extensions of Turán's theorem has long been one of the central themes in extremal graph theory. An early result along this line, due to Erdős, Goodman, and Pósa [5], shows that the edge set of every n-vertex graph can be decomposed into at most $t_{n,2}$ edge-disjoint triangles K_3 and individual edges. Later, this was generalized by Bollobás [2], who proved that for all $k \geq 3$, the edge set of every n-vertex graph can be decomposed into at most $t_{n,k-1}$ edge-disjoint cliques K_k and individual edges. It is clear that this result extends Turán's theorem. Another problem closely related to our study is the determination of the parameter

$$p(G) = \min \sum_{i\geq 1} |V(G_i)|$$

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for any graph G, where the minimum is taken over all edge decompositions of G into edge-disjoint cliques G_i for $i \geq 1$. Chung [3], Győri and Kostochka [8], and Kahn [11] independently proved that $p(G) \leq 2t_{n,2}$, with equality if and only if G is the complete balanced bipartite graph $T_{n,2}$.

Erdős (see [13]) later proposed to study the following enhanced variant of $p(G)$:

$$p^*(G) = \min \sum_{i \geq 1} \big(|V(G_i)| - 1\big)$$

for any graph G, where the minimum is taken over all edge decompositions of G into edge-disjoint cliques G_i for $i \geq 1$. Clearly, $p^*(G) < p(G)$ holds for every graph G. Erdős posed the following challenging problem (see Problem 43 in [13] and Conjecture 3 in [6]): Does every n-vertex graph G satisfy $p^*(G) \leq t_{n,2}$? Recently, the first author, together with Balogh, Krueger, Nguyen, and Wigal [1], proved an asymptotic version of this problem, showing that $p^*(G) \leq (1 + o(1))t_{n,2}$ holds for every n-vertex graph G, where $o(1) \to 0$ as $n \to \infty$. A restricted case of Erdős' problem was considered by Győri [6], who estimated $p^*(G)$ for K_4-free graphs. It turns out that this restricted version is equivalent to a problem of bounding edge-disjoint triangles in K_4-free graphs, which was independently conjectured by Erdős (see [10]) and later resolved by Győri and Keszegh [7] in the following theorem.

Theorem 1.1 (Győri and Keszegh [7]). *Every K_4-free graph with n vertices and $t_{n,2} + m$ edges contains at least m edge-disjoint triangles.*

A crucial concept in their proof [7] is the following special partition of vertex-disjoint cliques. We call it *greedy* because it can be obtained by iteratively applying the following greedy procedure: at each step, select a largest clique in the remaining graph and then delete the vertices of this clique.

Definition 1.2. *A greedy partition P of a graph G is a partition of $V(G)$ into disjoint cliques T_i for $i \geq 1$ such that $|T_i| \geq |T_{i+1}|$ for each $i \geq 1$ and, for each $\ell \geq 1$, the union of cliques with size at most ℓ induces a $K_{\ell+1}$-free subgraph. The size $r(P)$ of P denotes the number of cliques in this partition.*

Throughout, let $t(G)$ denote the number of triangles in a graph G, and let $t_e(G)$ denote the maximum number of edge-disjoint triangles in G.[1] A useful lemma of Huang and Shi [9] relates these parameters via greedy partitions: for any K_4-free graph G and any greedy partition P of G, we have

$$t_e(G) \geq t(G)/r(P). \tag{1}$$

Győri and Keszegh [7] employed an approach based on greedy partitions to show

$$t(G) \geq r(P)\big(e(G) - t_{n,2}\big) \tag{2}$$

for any n-vertex graph G, without requiring K_4-freeness. Combined with (1), this immediately implies Theorem 1.1. To explain the proof of (2) in more detail, for any greedy partition P in an n-vertex graph G with e edges, we define $r := r(P)$, $t := t(G)$, and

$$g(G, P) := r(e - r(n - r)) - t.$$

[1] When the graph G is clear from context, we simply write t and t_e.

Using symmetrization arguments, they [7] showed that it suffices to verify (2) for complete multipartite graphs G. Since $g(G, P) \leq 0$ for any complete multipartite graph G (see [7, Lemma 8]), it follows that $\frac{t}{r} \geq e - r(n-r) \geq e - t_{n,2}$ for such graphs, which establishes (2) in full.

Motivated by this approach, Győri and Keszegh [7] proposed the following stronger conjecture.

Conjecture 1.3 (Győri and Keszegh [7], Conjecture 2)**.** *Let G be an n-vertex K_4-free graph with e edges, and let P be any greedy partition of G of size $r := r(P)$. Then*

$$t(G) \geq r\big(e - r(n-r)\big), \quad \textit{or equivalently,} \quad g(G, P) \leq 0,$$

and consequently, $t_e(G) \geq e - r(n-r)$.

In this paper, we first disprove Conjecture 1.3 by constructing infinitely many counterexamples in a strong sense: for some K_4-free graphs G and greedy partitions P, the quantity $g(G, P)$ is postive and can be arbitrarily large.

Theorem 1.4. *For any positive integer λ, there exists an n-vertex K_4-free graph G with e edges and a greedy partition P of size r such that $t(G) \leq r\big(e - r(n-r)\big) - \lambda$.*

Our proof of Theorem 1.4 begins by constructing four special graphs F_1, F_2, F_3, and F_4 (all defined in Section 2, each formed from at most three cliques). The first two are minimal counterexamples to Conjecture 1.3, while the "3-blow-up" of F_3 and F_4 yield additional counterexamples. We then perform certain operations on these graphs to generate infinitely many larger, non-isomorphic counterexamples.

Our second contribution is to establish a corrected lower bound on the number of triangles in K_4-free graphs, using a new approach that is distinct from the method of Győri and Keszegh [7]. To state the result, we first introduce some notation. Let $P = \{T_1, T_2, \ldots, T_r\}$ be any greedy partition of G of size r. For indices $1 \leq i < j < k \leq r$, we say that a triple (i, j, k) is *P-bad* if the induced subgraph $G[T_i \cup T_j \cup T_k]$ is isomorphic to one of F_1, F_2, F_3, or F_4. We then define $\omega(P)$ to be the total number of P-bad triples. The following result shows that the lower bound on the number of triangles in Conjecture 1.3 can be corrected for all K_4-free graphs by subtracting $\omega(P)$.

Theorem 1.5. *Let G be an n-vertex K_4-free graph with e edges, and let P be any greedy partition of G of size r. Then*

$$t(G) \geq r\big(e - r(n-r)\big) - \omega(P).$$

In particular, if G contains none of the induced subgraphs F_1, F_2, F_3, or F_4, the conclusion of Conjecture 1.3 holds.

The rest of the paper is organized as follows. We prove Theorem 1.4 in Section 2. The proof of Theorem 1.5 will be presented in Section 3. In Section 4, we give some concluding remarks.

2 Proof of Theorem 1.4: Counterexamples to Conjecture 1.3

In this section, we prove Theorem 1.4 by constructing infinitely many counterexamples to Conjecture 1.3. The constructions are divided into two types, presented in the following two subsections.

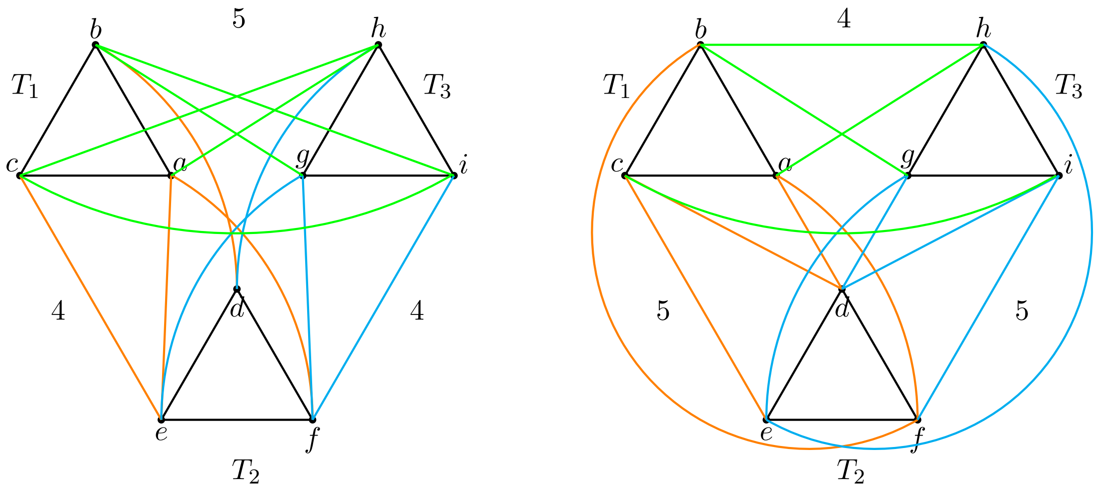

Figure 1: Counterexamples F_1 (left) and F_2 (right) to Conjecture 1.3.

2.1 Counterexamples of Type I

We define two graphs F_1 and F_2 as follows. Let $T_1 = \{a, b, c\}$, $T_2 = \{d, e, f\}$, and $T_3 = \{g, h, i\}$ be three disjoint triangles.

- **The graph** F_1: Let $V(F_1) = T_1 \cup T_2 \cup T_3$ and $E(F_1)$ consists of 4 edges between T_1 and T_2, 4 edges between T_2 and T_3, and 5 edges between T_1 and T_3 (see Figure 1, on left).

We see $P_1 = \{T_1, T_2, T_3\}$ defines a greedy partition of F_1. It is also easy to see that $v(F_1) = 9, e(F_1) = 22, r(P_1) = 3$. Thus this yields

$$12 = r(P_1)(e(F_1) - r(P_1)(v(F_1) - r(P_1))) > t(F_1) = 11.^2$$

Since any 4 vertices induce at most 2 triangles, F_1 is K_4-free.

- **The graph** F_2: Let $V(F_2) = T_1 \cup T_2 \cup T_3$ and $E(F_2)$ consists of 5 edges between T_1 and T_2, 5 edges between T_2 and T_3, and 4 edges between T_1 and T_3 (see Figure 1, on right).

We see $P_2 = \{T_1, T_2, T_3\}$ defines a greedy partition of F_2. It is also easy to see that $v(F_2) = 9, e(F_2) = 23, r(P_2) = 3$. Thus this yields

$$15 = r(P_2)(e(F_2) - r(P_2)(v(F_2) - r(P_2))) > t(F_2) = 14.^3$$

Since any 4 vertices induce at most 2 triangles, F_2 is also K_4-free.

Proof of Theorem 1.4 (Type I). We will construct infinitely many counterexamples by blowing up F_1 and F_2. For a positive integer vector $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}_+^3$, for $i \in \{1, 2\}$ and $j \in \{1, 2, 3\}$, the $\mathbf{k}$-blow-up of F_i, denoted by $F_i^{\mathbf{k}}$, is the graph obtained by replacing every vertex v of T_j with k_j different vertices where a copy of u is adjacent to a copy of v in $F_i^{\mathbf{k}}$ if and only if u is adjacent to v in F_i (see Figure 2 for $F_1^{\mathbf{k}}$). Note that the blow-up graph $F_i^{\mathbf{k}}$ is also K_4-free. We denote the k_j copies of T_j by $T_j^{(1)}, \ldots, T_j^{(k_j)}$ for $j \in \{1, 2, 3\}$. We see that $P_i^{\mathbf{k}} = \{T_1^{(1)}, \ldots, T_1^{(k_1)}, T_2^{(1)}, \ldots, T_2^{(k_2)}, T_3^{(1)}, \ldots, T_3^{(k_3)}\}$ is

[2] There are 11 triangles in F_1: abc, ace, ach, aef, bci, bgi, chi, def, efg, fgi, ghi.
[3] There are 14 triangles in F_2: abc, abf, abh, acd, adf, bgh, cde, cdi, def, deg, dfi, dgi, egh, ghi.

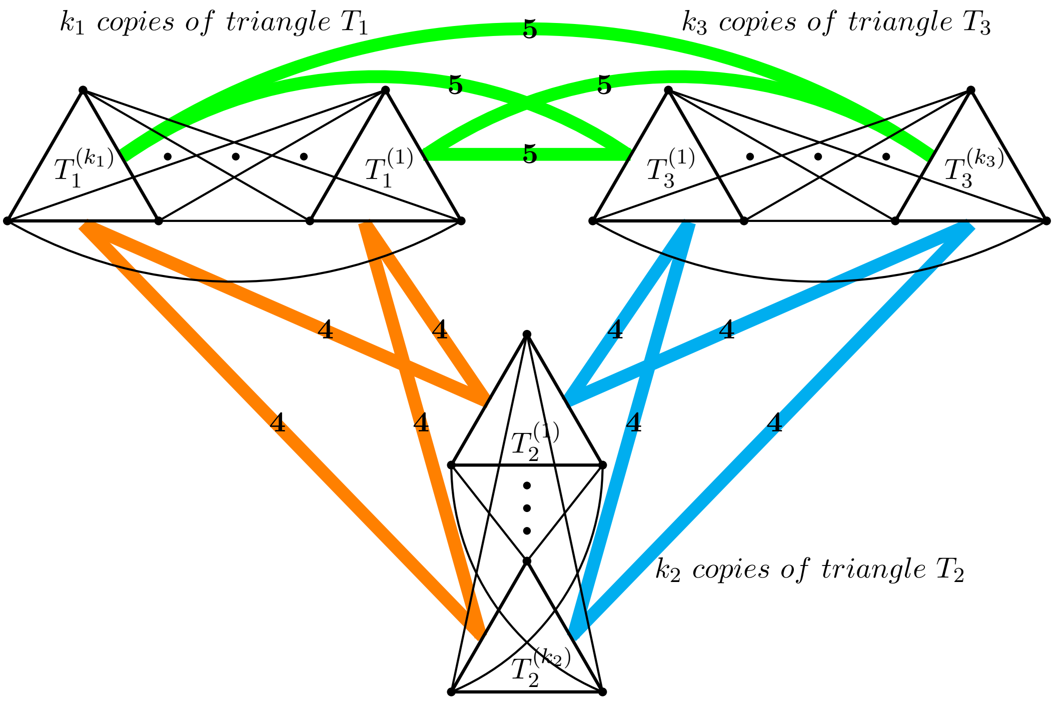

Figure 2: Graph $F_1^{\mathbf{k}}$ with the greedy partition $P_1^{\mathbf{k}}$.

a greedy partition of $F_i^{\mathbf{k}}$ for $i \in \{1,2\}$. It is not hard to see that $v(F_1^{\mathbf{k}}) = 3(k_1+k_2+k_3), e(F_1^{\mathbf{k}}) = 3(k_1^2+k_2^2+k_3^2)+4k_1k_2+5k_1k_3+4k_2k_3, r(P_1^{\mathbf{k}}) = k_1+k_2+k_3$ and $t(F_1^{\mathbf{k}}) = k_1^3+k_2^3+k_3^3+k_1^2k_2+2k_1^2k_3+k_2^2k_1+k_2^2k_3+2k_3^2k_1+k_3^2k_2$. Thus this yields

$$t(F_1^{\mathbf{k}}) - r(P_1^{\mathbf{k}})(e(F_1^{\mathbf{k}}) - r(P_1^{\mathbf{k}})(v(F_1^{\mathbf{k}}) - r(P_1^{\mathbf{k}}))) = -k_1k_2k_3 < 0.$$

Similarly, it is also not hard to see that $v(F_2^{\mathbf{k}}) = 3(k_1+k_2+k_3), e(F_2^{\mathbf{k}}) = 3(k_1^2+k_2^2+k_3^2)+5k_1k_2+4k_1k_3+5k_2k_3, r(P_2^{\mathbf{k}}) = k_1+k_2+k_3$ and $t(F_2^{\mathbf{k}}) = k_1^3+k_2^3+k_3^3+2k_1^2k_2+k_1^2k_3+2k_2^2k_1+2k_2^2k_3+k_3^2k_1+2k_3^2k_2+k_1k_2k_3$. Thus this yields

$$t(F_2^{\mathbf{k}}) - r(P_2^{\mathbf{k}})(e(F_2^{\mathbf{k}}) - r(P_2^{\mathbf{k}})(v(F_2^{\mathbf{k}}) - r(P_2^{\mathbf{k}}))) = -k_1k_2k_3 < 0.$$

Therefore the graphs $F_1^{\mathbf{k}}$ with greedy partition $P_1^{\mathbf{k}}$ and $F_2^{\mathbf{k}}$ with greedy partition $P_2^{\mathbf{k}}$ are counterexamples to Conjecture 1.3. Moreover, as $n = 3(k_1+k_2+k_3)$, the discrepancy $r(e-r(n-r))-t = k_1k_2k_3$ approaches infinity as $n \to \infty$, which completes the proof of Theorem 1.4. □

2.2 Counterexamples of Type II

We define two graphs F_3 and F_4 as follows. Let $T_1 = \{a,b,c\}$, $T_2 = \{d,e,f\}$, and $T_3 = \{g,h,i\}$ be three disjoint triangles.

- **The graph** F_3: Let $V(F_3) = T_1 \cup T_2 \cup T_3$ and $E(F_3)$ consists of 5 edges between T_1 and T_2, 5 edges between T_2 and T_3, and 3 edges between T_1 and T_3 (see Figure 3, on left).

It is easy to see that $v(F_3) = 9$, $e(F_3) = 22$ and $t(F_3) = 13$.[4] Since any 4 vertices induce at most 2

[4]There are 13 triangles in F_3: abc, abf, abh, acd, adf, bgh, cde, def, deg, dfi, dgi, egh, ghi.

triangles, F_3 is K_4-free.

Let $T'_1 = \{a, b, c\}$ and $T'_3 = \{f, g, h\}$ be two disjoint triangles and $T'_2 = \{d, e\}$ be an edge.

- **The graph** F_4: Let $V(F_4) = T'_1 \cup T'_2 \cup T'_3$ and $E(F_4)$ consists of 3 edges between T'_1 and T'_2, 3 edges between T'_2 and T'_3, and 5 edges between T'_1 and T'_3 (see Figure 3, on right).

It is easy to see that $v(F_4) = 8$, $e(F_4) = 18$ and $t(F_4) = 10$.[5] Since any 4 vertices induce at most 2 triangles, F_4 is also K_4-free.

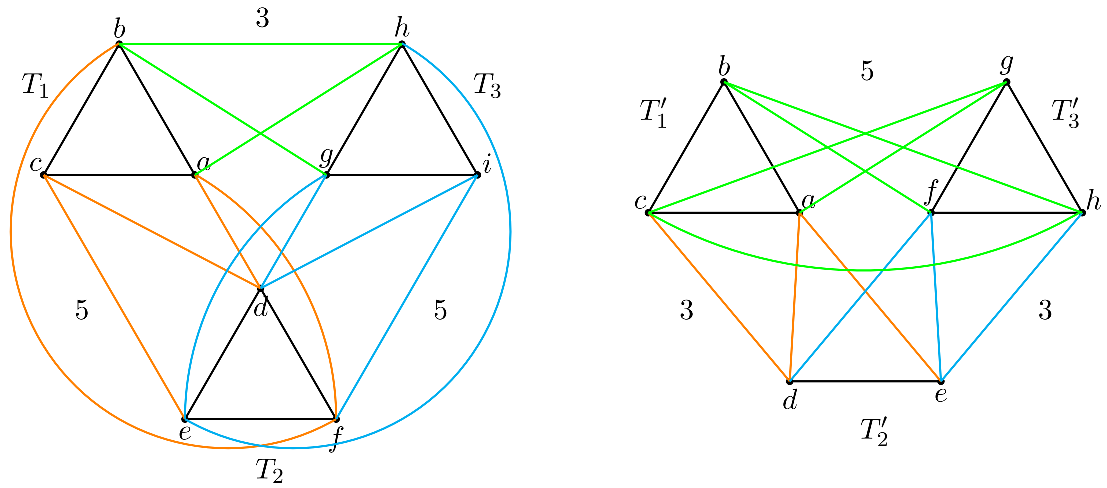

Figure 3: Graphs F_3 (left) and F_4 (right).

Proof of Theorem 1.4 (Type II). While F_3 and F_4 are not counterexamples to Conjecture 1.3, some blow-ups of these graphs are. Indeed, by a similar proof of Theorem 1.4 (Type I), for a positive integer vector $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3_+$, for $j \in \{1, 2, 3\}$, we define $F_3^{\mathbf{k}}$ (respectively, $F_4^{\mathbf{k}}$) to be the graph obtained by replacing every vertex v of T_j (respectively, T'_j) with k_j different vertices. We denote the k_j copies of T_j by $T_j^{(1)}, \ldots, T_j^{(k_j)}$ for $j \in \{1, 2, 3\}$. We see that $P_3^{\mathbf{k}} = \{T_1^{(1)}, \ldots, T_1^{(k_1)}, T_2^{(1)}, \ldots, T_2^{(k_2)}, T_3^{(1)}, \ldots, T_3^{(k_3)}\}$ is a greedy partition of $F_3^{\mathbf{k}}$. It is not hard to see that $v(F_3^{\mathbf{k}}) = 3(k_1 + k_2 + k_3)$, $e(F_3^{\mathbf{k}}) = 3(k_1^2 + k_2^2 + k_3^2) + 5k_1k_2 + 3k_1k_3 + 5k_2k_3$, $r(P_3^{\mathbf{k}}) = k_1 + k_2 + k_3$, and $t(F_3^{\mathbf{k}}) = k_1^3 + k_2^3 + k_3^3 + 2k_1^2k_2 + k_1^2k_3 + 2k_2^2k_1 + 2k_2^2k_3 + k_3^2k_1 + 2k_3^2k_2$. Thus this yields

$$t(F_3^{\mathbf{k}}) - r(P_3^{\mathbf{k}})(e(F_3^{\mathbf{k}}) - r(P_3^{\mathbf{k}})(v(F_3^{\mathbf{k}}) - r(P_3^{\mathbf{k}}))) = k_1k_3(k_1 - k_2 + k_3).$$

Thus for infinitely many (k_1, k_2, k_3) satisfying the inequality $k_1 + k_3 < k_2$, the graph $F_3^{\mathbf{k}}$ with greedy partition $P_3^{\mathbf{k}}$ is a counterexample to Conjecture 1.3. Moreover, as $n = 3(k_1 + k_2 + k_3)$, the discrepancy $r(e - r(n - r)) - t = k_1k_3(k_2 - k_1 - k_3)$ may approach infinity as $n \to \infty$.

Similarly, we denote the k_j copies of T'_j by $T_j^{'(1)}, \ldots, T_j^{'(k_j)}$ for $j \in \{1, 2, 3\}$. We see that $P_4^{\mathbf{k}} = \{T_1^{'(1)}, \ldots, T_1^{'(k_1)}, T_2^{'(1)}, \ldots, T_2^{'(k_2)}, T_3^{'(1)}, \ldots, T_3^{'(k_3)}\}$ defines a greedy partition of $F_4^{\mathbf{k}}$. It is not hard to see that $v(F_4^{\mathbf{k}}) = 3k_1 + 2k_2 + 3k_3$, $e(F_4^{\mathbf{k}}) = 3k_1^2 + k_2^2 + 3k_3^2 + 3k_1k_2 + 5k_1k_3 + 3k_2k_3$, $r(P_4^{\mathbf{k}}) = k_1 + k_2 + k_3$, and $t(F_4^{\mathbf{k}}) = k_1^3 + k_3^3 + k_1^2k_2 + 2k_1^2k_3 + k_2^2k_1 + k_2^2k_3 + 2k_3^2k_1 + k_3^2k_2$. Thus this yields

$$t(F_4^{\mathbf{k}}) - r(P_4^{\mathbf{k}})(e(F_4^{\mathbf{k}}) - r(P_4^{\mathbf{k}})(v(F_4^{\mathbf{k}}) - r(P_4^{\mathbf{k}}))) = k_2(k_1k_2 - k_1k_3 + k_2k_3).$$

[5]There are 10 triangles in F_4: abc, acd, acg, ade, bch, bfh, cgh, def, efh, fgh.

Thus for infinitely many (k_1, k_2, k_3) satisfying the inequality $k_1k_2 + k_2k_3 < k_1k_3$, the graph $F_4^{\mathbf{k}}$ with greedy partition $P_4^{\mathbf{k}}$ is a counterexample to Conjecture 1.3. Moreover, as $n = 3k_1 + 2k_2 + 3k_3$, the discrepancy $r(e - r(n - r)) - t = k_2(k_1k_3 - k_1k_2 - k_2k_3)$ may approach infinity as $n \to \infty$, which completes the proof of Theorem 1.4. □

3 Proof of Theorem 1.5

In this section, we first reduce the proof of Theorem 1.5 to a key lemma (Lemma 3.2) in Subsection 3.1, and then prove Lemma 3.2 in Subsection 3.2. A proof outline of Theorem 1.5 is also provided in Subsection 3.1.

Throughout the rest of this section, let G be an n-vertex K_4-free graph with e edges, and let $P = \{T_1, ..., T_r\}$ be a greedy partition of G of size r. Since G is K_4-free, each T_i has size $3, 2$ or 1. Let a, b, c be the number of cliques of size $3, 2$ and 1, respectively. Thus $r = a + b + c$, and $n = 3a + 2b + c$.

- For $1 \le i < j \le r$, we define

$$e_{ij} = e(G[T_i \cup T_j]) \quad \text{and} \quad t_{ij} = \text{the number of triangles in } G[T_i \cup T_j].$$

- For $1 \le i < j < k \le r$, we define

$$e_{ijk} = e(G[T_i \cup T_j \cup T_k]) \quad \text{and} \quad t_{ijk} = \text{the number of triangles in } G[T_i \cup T_j \cup T_k].$$

3.1 Completing the Proof, Assuming Lemma 3.2

Our proof strategy of Theorem 1.5 employs double counting technique to analyze the contribution of triangles. Specifically, by double counting the number of triangles that contribute to t_{ijk}, we can write t as a sum of these t_{ijk} terms and a term related to t_{ij} (see equation (3)). Next, in Lemma 3.2, we analyze how t_{ij} affects t_{ijk} locally, then apply the induction to extend the effect to the whole graph, and thus obtain the desired lower bound on the number of triangles.

For $i \in [3]$, let $M_i(G; P)$ be the number of triangles with three vertices lying in exactly i different T_j's. Clearly, we have $M_1(G; P) = a$, and $t(G) = M_1(G; P) + M_2(G; P) + M_3(G; P)$. By double counting the number of triangles, we have

$$\begin{aligned}\sum_{1\le i<j<k\le r} t_{ijk} &= M_1(G;P)\cdot\binom{r-1}{2} + M_2(G;P)\cdot\binom{r-2}{1} + M_3(G;P)\\ &= t(G) + (r-3)M_2(G;P) + \frac{a}{2}(r-1)(r-2) - a,\end{aligned}$$

which implies that

$$t(G) = \sum_{1\le i<j<k\le r} t_{ijk} - (r-3)M_2(G;P) - \frac{a}{2}(r-1)(r-2) + a. \tag{3}$$

To lower bound the right-hand side of (3), we first estimate $M_2(G;P)$. We define the pair deficiency

$$M_0(G;P) := \sum_{1\le i<j\le r} 2(e_{ij} - 2(|T_i| + |T_j| - 2)) - a(r-1). \tag{4}$$

The following lemma states that $M_2(G;P)$ is bounded below by the pair deficiency $M_0(G;P)$.

Lemma 3.1. *Let G be a K_4-free graph with greedy partition P. Then we have $M_2(G;P) \ge M_0(G;P)$.*

Proof. We first claim that for any $1 \le i < j \le r$, we have $t_{ij} \ge 2(e_{ij} - 2(|T_i| + |T_j| - 2))$. Indeed, this follows from case analysis of all possible sizes for T_i and T_j (which can only be 1, 2, or 3 vertices due to the K_4-free condition). We omit the detail.

Summing up all the t_{ij}'s and by (4), we obtain that

$$\sum_{1\le i<j\le r} t_{ij} \ge \sum_{1\le i<j\le r} 2\left(e_{ij} - 2(|T_i| + |T_j| - 2)\right) = M_0(G;P) + a(r-1).$$

By the definition of the greedy partition, we know that $G[T_{a+1}, ..., T_r]$ is triangle-free. By double counting the number of triangles in $G[T_i \cup T_j]$, we have

$$\sum_{1\le i<j\le r} t_{ij} = M_2(G;P) + a(r-1),$$

and this implies that $M_2(G;P) \ge M_0(G;P)$, which completes the proof of Lemma 3.1. □

Next, we give a lower bound on the term $\sum_{1\le i<j<k\le r} t_{ijk}$ in (3). We define the triple deficiency motivated by the bound suggested in Conjecture 1.3

$$F_0(G;P) := \sum_{1\le i<j<k\le r} 3(e_{ijk} - 3(|T_i| + |T_j| + |T_k| - 3)). \tag{5}$$

We present our key lemma as follows and postpone the proof to Subsection 3.2.

Lemma 3.2. *Let G be a K_4-free graph and P be any greedy partition of G of size r. Let $M_2(G;P) = M_0(G;P) + C$ for some $C \ge 0$, then we have $\sum\limits_{1\le i<j<k\le r} t_{ijk} \ge F_0(G;P) + (r-2)\cdot C - \omega(P)$.*

Now we are ready to finish the proof of Theorem 1.5.

Proof of Theorem 1.5. By Lemma 3.1, we may assume that $M_2(G;P) = M_0(G;P) + C$ for some $C \ge 0$. By Lemma 3.2, it follows that

$$\sum_{1\le i<j<k\le r} t_{ijk} \ge F_0(G;P) + (r-2)\cdot C - \omega(P). \tag{6}$$

Combining (3) with (6), we have

$$\begin{aligned} t(G) \ge& F_0(G;P) + (r-2)\cdot C - \omega(P) - (r-3)[M_0(G;P) + C] - \frac{a}{2}(r-1)(r-2) + a \\ =& F_0(G;P) - (r-3)M_0(G;P) - \frac{a}{2}(r-1)(r-2) + a - \omega(P) + C \\ =& r(e - r(n-r)) - \omega(P) + C \ge r(e - r(n-r)) - \omega(P), \end{aligned} \tag{7}$$

where the last equality holds by the definitions of $M_0(G; P)$ and $F_0(G; P)$ (see its justification in Appendix A). This completes the proof of Theorem 1.5. ■

3.2 Proof of Lemma 3.2

In this subsection, we finish the proof of Lemma 3.2 by induction on r, the size of the greedy partition P of G. First, we prove the base case for $r \leq 3$ in the following claim.

Claim 3.3. *Let G be a K_4-free graph and P be any greedy partition of G of size r for some $r \leq 3$. Let $M_2(G; P) = M_0(G; P) + C$ for some $C \geq 0$, then we have*

$$\sum_{1 \leq i<j<k \leq r} t_{ijk} \geq F_0(G; P) + (r-2) \cdot C - \omega(P).$$

Proof. When $r = 1$ or 2, by definition, we have $\sum_{1 \leq i<j<k \leq r} t_{ijk} = F_0(G; P) = \omega(P) = 0$, and we are done. The verification of $r = 3$ involves detailed case analysis and computer assistance. We defer the proof to Appendix B. □

Now we are ready to finish the proof of Lemma 3.2.

Completing the proof of Lemma 3.2. By Claim 3.3, we may assume that the statement is true for any K_4-free graph with any greedy partition of size at most $r-1$ for some $r \geq 4$. We now consider a K_4-free graph G with greedy partition $P = \{T_1, ..., T_r\}$ of size r.

Let a be the number of T_i's in P of size 3. For $\ell \in [r]$, let $G_\ell = G[V(G) \setminus T_\ell]$, $P_\ell = P \setminus \{T_\ell\}$ (a greedy partition of G_ℓ of size $r-1$), and let

$$M_2(G_\ell, T_\ell) := M_2(G; P) - M_2(G_\ell; P_\ell).$$

By Lemma 3.1, suppose that $M_2(G; P) = M_0(G; P) + C$ for some $C \geq 0$. Then for $\ell \in [r]$,

$$\begin{aligned} M_2(G_\ell; P_\ell) &= M_2(G; P) - M_2(G_\ell, T_\ell) = M_0(G; P) + C - M_2(G_\ell, T_\ell) \\ &= M_0(G_\ell; P_\ell) + [M_0(G; P) + C - M_2(G_\ell, T_\ell) - M_0(G_\ell; P_\ell)]. \end{aligned}$$

Let $C_\ell := M_0(G; P) + C - M_2(G_\ell, T_\ell) - M_0(G_\ell; P_\ell)$. By Lemma 3.1 again, we have $C_\ell \geq 0$ for $\ell \in [r]$.

Let I_ℓ be the set of all triples (i, j, k) such that $1 \leq i < j < k \leq r$ and $i, j, k \in [r] \setminus \{\ell\}$. By the inductive hypothesis, we have

$$\sum_{(i,j,k) \in I_\ell} t_{ijk} \geq F_0(G_\ell; P_\ell) + (r-3) \cdot C_\ell - \omega(P_\ell). \tag{8}$$

Summing up all $\ell \in [r]$, by (8) and the definition of C_ℓ, we have

$$\begin{aligned} \sum_{\ell \in [r]} \sum_{(i,j,k) \in I_\ell} t_{ijk} &\geq \sum_{\ell \in [r]} F_0(G_\ell; P_\ell) + (r-3) \cdot \sum_{\ell \in [r]} C_\ell - \sum_{\ell \in [r]} \omega(P_\ell) \\ &= X + (r-3) \left[r(M_0(G; P) + C) - \sum_{\ell \in [r]} M_2(G_\ell, T_\ell) - Y \right] - \sum_{\ell \in [r]} \omega(P_\ell), \end{aligned} \tag{9}$$

where $X = \sum_{\ell \in [r]} F_0(G_\ell; P_\ell)$, and $Y = \sum_{\ell \in [r]} M_0(G_\ell; P_\ell)$.

For the left-hand side of (9), by double counting the contribution of t_{ijk}, we have

$$\sum_{\ell \in [r]} \sum_{(i,j,k) \in I_\ell} t_{ijk} = (r-3) \cdot \sum_{1 \le i < j < k \le r} t_{ijk}. \tag{10}$$

Recall that $F_0(G; P) = \sum_{1 \le i < j < k \le r} 3(e_{ijk} - 3(|T_i| + |T_j| + |T_k| - 3))$ (see (5)), for any triple $1 \le i < j < k \le r$, the term $3(e_{ijk} - 3(|T_i| + |T_j| + |T_k| - 3))$ is counted $r-3$ times in X. Consequently,

$$X = (r-3) \cdot F_0(G; P). \tag{11}$$

Similarly, by the definition of $M_0(G; P) = \sum_{1 \le i < j \le r} 2(e_{ij} - 2(|T_i| + |T_j| - 2)) - a(r-1)$ (see (4)), each term $2(e_{ij} - 2(|T_i| + |T_j| - 2))$ is counted $r-2$ times in Y. Moreover, the term $a(r-2)$ appears $r-a$ times in Y (for $\ell \in [r] \setminus [a]$), while the term $(a-1)(r-2)$ appears a times in Y (for $\ell \in [a]$). Thus,

$$\begin{aligned} Y &= (r-2) \sum_{1 \le i < j \le r} 2(e_{ij} - 2(|T_i| + |T_j| - 2)) - a(r-2)(r-a) - (a-1)(r-2)a \\ &= (r-2) \sum_{1 \le i < j \le r} 2(e_{ij} - 2(|T_i| + |T_j| - 2)) - a(r-1)(r-2) = (r-2) \cdot M_0(G; P). \end{aligned} \tag{12}$$

Finally, by double counting the number of triangles and induced subgraphs isomorphic to F_1, F_2, F_3 or F_4 contributing to $M_2(G; P)$ and $\omega(P)$, respectively, we have

$$\sum_{\ell \in [r]} M_2(G_\ell, T_\ell) = 2M_2(G; P), \text{ and } \sum_{\ell \in [r]} \omega(P_\ell) = (r-3) \cdot \omega(P). \tag{13}$$

Combining equations (10), (11), (12), and (13) with inequality (9), we obtain that

$$\begin{aligned} (r-3) \sum_{1 \le i < j < k \le r} t_{ijk} \ge &(r-3) \cdot F_0(G; P) + (r-3) \cdot [r(M_0(G; P) + C) - 2M_2(G; P) - \\ &(r-2) \cdot M_0(G; P)] - (r-3) \cdot \omega(P). \end{aligned}$$

Since $M_2(G; P) = M_0(G; P) + C$, we conclude that

$$\begin{aligned} \sum_{1 \le i < j < k \le r} t_{ijk} &\ge F_0(G; P) + 2M_0(G; P) - 2M_2(G; P) + rC - \omega(P) \\ &= F_0(G; P) + (r-2) \cdot C - \omega(P), \end{aligned}$$

which completes the proof of Lemma 3.2. ■

4 Concluding Remarks

In this paper, we first disprove the assertion in Conjecture 1.3, proposed by Győri and Keszegh [7], concerning the number $t(G)$ of triangles in K_4-free graphs G under size and greedy partition constraints. Second, we provide a corrected lower bound, showing that $t(G) \ge r(e - r(n-r)) - \omega(P)$ for any n-vertex K_4-free graph G with e edges and any greedy partition P of size r. It would be interesting to further improve this bound, particularly the term involving $\omega(P)$.

We remark that the bound given by Theorem 1.5 can be tight for infinitely many K_4-free graphs. Let G be any complete 3-partite graph with parts X, Y and Z. Suppose that the sizes of X, Y and Z are x, y and z, respectively and $x \geq y \geq z$. It is easy to see that the greedy partition P of G is unique, which consists of z triangles, $y-z$ edges and $x-y$ isolated vertices. Thus we have $n = x+y+z$, $e = xy+xz+yz$, $r = r(P) = x$, and moreover it is easy to see that $\omega(P) = 0$. By Theorem 1.5, we have $xyz = t \geq r(e - r(n-r)) = xyz$, thus our theorem is tight for complete 3-partite graph. For any positive integer vector $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}_+^3$ and $i \in \{1, 2\}$, Theorem 1.5 is also tight for graphs $F_i^{\mathbf{k}}$ that we constructed in Subsection 2.1, since the value $\omega(P_i^{\mathbf{k}}) = k_1 k_2 k_3$ exactly matches the gap between t and $r(e - r(n-r))$.

Our second remark concerns the maximum of the quantity $g(G, P)$ as a function of n. Formally, let $g(n) = \max_{(G,P)} g(G, P)$, where the maximum is taken over all n-vertex K_4-free graphs G and all greedy partitions P of G. We claim that $g(n) = \Theta(n^3)$. Indeed, for any greedy partition P of G of size r, $\omega(P) \leq \binom{r}{3} \leq \binom{n}{3}$. By Theorem 1.5, we have $t \geq r(e - r(n-r)) - \binom{n}{3}$, which yields that $g(n) \leq \binom{n}{3} = O(n^3)$. On the other hand, when $9 \mid n$, let $\mathbf{k} = (\frac{n}{9}, \frac{n}{9}, \frac{n}{9})$. Consider the graph $F_1^{\mathbf{k}}$ and its greedy partition $P_1^{\mathbf{k}}$ defined in Subsection 2.1 (when $9 \nmid n$, just consider a suitable n-vertex subgraph of $F_1^{\mathbf{k}}$ for $\mathbf{k} = (\lceil \frac{n}{9} \rceil, \lceil \frac{n}{9} \rceil, \lceil \frac{n}{9} \rceil)$). It follows that $r(P_1^{\mathbf{k}})(e(F_1^{\mathbf{k}}) - r(P_1^{\mathbf{k}})(n - r(P_1^{\mathbf{k}}))) - t(P_1^{\mathbf{k}}) = k_1 k_2 k_3$, which implies that $g(n) = \Omega(n^3)$, as desired.

Finally, it is worth noting that, although the counterexamples in Section 2 disprove the assertion of Conjecture 1.3 on $t(G)$, they do not violate the desired inequality $t_e(G) \geq e - r(n-r)$. As a consequence of Theorem 1.5, when $\omega(P) < r$, applying the lemma of Huang and Shi [9] that $t_e(G) \geq t(G)/r(P)$, we see that the lower bound on t_e in Conjecture 1.3 holds. It remains an interesting open question whether $t_e(G) \geq e - r(n-r)$ holds in general.

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Appendix A: Justification of the inequality (7)

Starting from inequality (7), it suffices to show that

$$F_0(G;P) - (r-3)M_0(G;P) - \frac{a}{2}(r-1)(r-2) + a = r(e - r(n-r)).$$

Since $n = 3a + 2b + c$ and $r = a + b + c$, we have $3a + b = n - r + a$. From the definition of $M_0(G;P)$ in (4), we obtain that

$$\begin{aligned} M_0(G;P) =& 2\sum_{1\le i<j\le r} e_{ij} - 4n(r-1) + 8\binom{r}{2} - a(r-1) = 2e + 2(3a+b)(r-2) - (r-1)(4n-4r+a) \\ =& 2e + 2(n-r+a)(r-2) - (r-1)(4n-4r+a) = 2e - 2(n-r)r + a(r-3). \end{aligned} \tag{14}$$

Similarly, by the definition of $F_0(G;P)$ in (5), we have

$$\begin{aligned} F_0(G;P) =& \sum_{1\le i<j<k\le r} 3(e_{ijk} - 3(|T_i| + |T_j| + |T_k| - 3)) = 3\sum_{1\le i<j<k\le r} e_{ijk} - 9n\binom{r-1}{2} + 27\binom{r}{3} \\ =& 3(e - 3a - b)(r-2) + 3(3a+b)\binom{r-1}{2} - \frac{9}{2}(n-r)(r-1)(r-2) \\ =& 3e(r-2) - 3(n-r+a)(r-2) + \frac{3}{2}(n-r+a)(r-1)(r-2) - \frac{9}{2}(n-r)(r-1)(r-2) \\ =& 3e(r-2) - 3(n-r)r(r-2) + \frac{3}{2}a(r-2)(r-3). \end{aligned} \tag{15}$$

Combining (14) and (15), we obtain that

$$\begin{aligned} & F_0(G;P) - (r-3)M_0(G;P) - \frac{a}{2}(r-1)(r-2) + a \\ =& 3e(r-2) - 3(n-r)r(r-2) + \frac{3}{2}a(r-2)(r-3) - (r-3)[2e - 2(n-r)r + a(r-3)] - \frac{a}{2}r(r-3) \\ =& er - (n-r)r^2 + \frac{a}{2}(r-3)[3(r-2) - 2(r-3) - r] = r(e - r(n-r)), \end{aligned}$$

which completes the proof of (7). ■

Appendix B: Proof of Claim 3.3

Note that, for $r = 3$, we have $F_0(G;P) = 3(e(G) - 3(v(G) - 3))$. We point out that it is always the case that $\omega(P) \in \{0, 1\}$. The claim reduces to proving

$$t(G) \geq F_0(G;P) + M_2(G;P) - M_0(G;P) - \omega(P). \tag{16}$$

Let $P = \{T_1, T_2, T_3\}$ and let a, b, c be the number of cliques of size $3, 2$ and 1, respectively.

Suppose $|T_1| \leq 2$, i.e., $a = 0$. By the definition of greedy partition, we have $t(G) = M_2(G;P) = \omega(P) = 0$, and $M_0(G;P) = \sum_{1 \leq i < j \in [3]} 2(e_{ij} - 2(|T_i| + |T_j| - 2)) - 2a = 2(e + v(G) - 3) - 8v(G) + 24$. Thus inequality (16) is equivalent to $e(G) - 3v(G) + 9 \leq 0$, which holds by straightforward case analysis.

We verify the remaining subcases for $r = 3$ with $|T_1| = 3$ (i.e., $a \geq 1$) via computer assistance (see the program below for details). For every K_4-free graph with parameters (a, b, c) where $a \geq 1$, our program verifies whether or not the input graph satisfies the inequality

$$t(G) \geq F_0(G;P) + M_2(G;P) - M_0(G;P). \tag{17}$$

If it satisfies, then evidently it satisfies (16); otherwise, the program identifies graphs that do not satisfy this inequality (called *Counterexamples to* (17)), along with the parameters $\{t(G), M_2(G;P), e(G)\}$. These are summarized in Table 1. Note that $\omega(P) = 1$ holds for all Counterexamples to (17). After thorough calculations, inequality (16) holds for all Counterexamples to (17), thus Claim 3.3 holds and we are done. ■

(a,b,c)	$M_0(G;P)$	$F_0(G;P)$	Precise expression of Inequality (17)	Counterexamples to (17) and parameters $\{t(G), M_2(G;P), e(G)\}$
$(3,0,0)$	$2e-36$	$3e-54$	$t(G) \geq M_2(G;P) + e(G) - 18$	$F_1 : \{11, 8, 22\}$ $F_2 : \{14, 10, 23\}$ $F_3 : \{13, 10, 22\}$
$(2,1,0)$	$2e-30$	$3e-45$	$t(G) \geq M_2(G;P) + e(G) - 15$	$F_4 : \{10, 8, 18\}$
$(2,0,1)$	$2e-24$	$3e-36$	$t(G) \geq M_2(G;P) + e(G) - 12$	$\emptyset$
$(1,2,0)$	$2e-24$	$3e-36$	$t(G) \geq M_2(G;P) + e(G) - 12$	$\emptyset$
$(1,1,1)$	$2e-18$	$3e-27$	$t(G) \geq M_2(G;P) + e(G) - 9$	$\emptyset$
$(1,0,2)$	$2e-12$	$3e-18$	$t(G) \geq M_2(G;P) + e(G) - 6$	$\emptyset$

Table 1: All the subcases for $r = 3$ with $|T_1| = 3$ in Claim 3.3.

The program for $r = 3$ with $|T_1| = 3$ in Claim 3.3.

```python
import networkx as nx
import itertools
import os
import matplotlib.pyplot as plt

class GraphSolution:
    # Initialize the graphs: we first fix the structure between two of {T_1,T_2,T_3} (self.isomorphism_list) and then traverse all the remaining edges (self.possible_add_edge1/2).
    # For different types, the difference between the inequalities is stored in self.constant, i.e., if t(G)<m_2(G)+e(G)-self.constant.
    def __init__(self):
        self.graph = None
        self.isomorphism_list = []
        self.possible_add_edge1 = []
        self.possible_add_edge2 = []
        self.constant = 0
        self.ori_tri_list = []          # The number of triangles in {G[T_1],G[T_2]}, {G[T_1],G[T_3]}, and {G[T_2],G[T_3]}.
        self.subgraphs = []             # The induced subgraphs on vertex sets T_1UT_2, T_1UT_3, and T_2UT_3.
        self.save_root = './graph_result'

    # Verify if the graph G is k_4-free by checking the number of edges of subgraphs induced on any 4 vertices of G.
    def is_k4_free(self, G=None):
        if G is None:
            G = self.graph
        for nodes in itertools.combinations(G.nodes, 4):
            subgraph = G.subgraph(nodes)
            if subgraph.number_of_edges() == 6:
                return False
        return True

    # Calculate m_2(G) by adding up m_2(G[T_1UT_2]), m_2(G[T_1UT_3]), and m_2(G[T_2UT_3]).
    def triangle_nums_bt_tri(self, G=None):
        if G is None:
            G = self.graph
        count = 0
        for i, selected_nodes in enumerate(self.subgraphs):
            subgraph = G.subgraph(selected_nodes)
            triangle_count = sum(nx.triangles(subgraph).values()) // 3 - self.ori_tri_list[i]
            count += triangle_count
        return count

    # Determine whether the graph is a new graph up to isomorphism.
    def isomorphic_list(self, graph_list, new_graph):
        i = 0
        for _,graph in enumerate(graph_list):
            GM = nx.isomorphism.GraphMatcher(graph,new_graph)
            if GM.is_isomorphic():
                break
            else:
                i += 1
        if i == len(graph_list):
            graph_list.append(new_graph)
        else:
            new_graph = None
        return graph_list, new_graph

    # Initialize graphs according to different types.
    def initialize_graph(self, graph_type):
        initializers = {
            1: self._initial_graph_1,
            2: self._initial_graph_2,
            3: self._initial_graph_3,
            4: self._initial_graph_4,
            5: self._initial_graph_5,
            6: self._initial_graph_6,
        }
        # Read graphs, and the structure between the two given cliques.
        # Traverse all the remaining edges, and select different inequalities.
        if graph_type in initializers:
            self.graph, self.isomorphism_list, self.possible_add_edge1, self.possible_add_edge2, self.constant, self.ori_tri_list, self.subgraphs = initializers[graph_type]()
        else:
            raise ValueError("Invalid graph type")

    # Add edges and verify if the inequalities hold.
    def forward(self):
        os.makedirs(self.save_root, exist_ok=True)
        graph_list = []

        count = 0
```

```python
        # Read subgraphs induced on the vertex set of the two fixed cliques (G_1).
        for i, isomorphism in enumerate(self.isomorphism_list):
            print(f'Starting to add {i+1} isomorphism')
            # Traverse all the subgraphs induced on the vertex set of one of the undetermined structure between two cliques (G_2).
            for j in range(1, len(self.possible_add_edge1) + 1):
                combinations1 = list(itertools.combinations(self.possible_add_edge1, j))
                # Traverse all the subgraphs induced on the vertex set of the other undetermined structure between two cliques (G_3).
                for k in range(1, len(self.possible_add_edge2) + 1):
                    combinations2 = list(itertools.combinations(self.possible_add_edge2, k))
                    for combo1 in combinations1:
                        for combo2 in combinations2:
                            G_new = self.graph.copy()
                            G_new.add_edges_from(isomorphism) # Add edges for subgraph G_1.
                            G_new.add_edges_from(combo1)      # Add edges for subgraph G_2.
                            G_new.add_edges_from(combo2)      # Add edges for subgraph G_3.
                            # Verify if the graph G is k_4-free
                            if self.is_k4_free(G_new):
                                num_edge = G_new.number_of_edges()                         # Calculate the number of edges of G.
                                triangle_count = sum(nx.triangles(G_new).values()) // 3   # Calculate the number of triangles of G.
                                m2_num = self.triangle_nums_bt_tri(G_new)                 # Calculate m_2(G).
                                # Verify if t(G)<m_2(G)+e(G)-self.constant
                                if triangle_count < m2_num + num_edge - self.constant:
                                    graph_list, new_graph = self.isomorphic_list(graph_list, G_new) # Determine whether the graph is a new graph.
                                    if new_graph is not None:
                                        plt.figure(figsize=(8, 6))
                                        pos = nx.spring_layout(G_new) # Select a layout algorithm
                                        nx.draw(G_new, with_labels=True, node_color='lightblue', edge_color='gray')
                                        filename = os.path.join(self.save_root, f'graph_edge={num_edge}_triangle={triangle_count}_m2={m2_num}_{count}.png')
                                        count += 1
                                        plt.savefig(filename)  # Save the graph.
                                        plt.close()

    # Type I (a=3, and T_1={1,2,3}, T_2={4,5,6}, T_3={7,8,9}).
    def _initial_graph_1(self):
        G = nx.Graph()
        G.add_nodes_from([1, 2, 3, 4, 5, 6, 7, 8, 9])
        G.add_edges_from([(1, 2), (1, 3), (2, 3), (4, 5), (4, 6), (5, 6), (7, 8), (7, 9), (8, 9)])
        # Different K_4-free graphs on vertex set T_1UT_2.
        isomorphism_list = [
            [],
            [(1,4)], [(1,4),(1,5)], [(1,4),(2,5)], [(1,4),(1,5),(3,5)], [(1,4),(1,5),(2,6)],
            [(1,4),(2,5),(3,6)], [(1,4),(1,5),(2,5),(2,6)], [(1,4),(1,5),(3,5),(2,6)],
            [(1,4),(1,5),(3,4),(2,6)],
            [(1,4),(1,5),(3,4),(2,5),(2,6)],
            [(1,4),(1,5),(3,4),(2,5),(2,6),(3,6)]
        ]
        # Add possible edges between T_1 and T_3.
        possible_add_edge1 = list(set(itertools.combinations([1, 2, 3, 7, 8, 9], 2)) - set(G.edges))
        # Add possible edges between T_2 and T_3.
        possible_add_edge2 = list(set(itertools.combinations([4, 5, 6, 7, 8, 9], 2)) - set(G.edges))
        # self_constant=18, i.e., check if t(G)<m_2(G)+e(G)-18
        return G, isomorphism_list, possible_add_edge1, possible_add_edge2, 18, [2,2,2], [[1, 2, 3, 4, 5, 6], [1, 2, 3, 7, 8, 9], [4, 5, 6, 7, 8, 9]]

    # Type II (a=2, b=1, and T_1={1,2,3}, T_2={4,5,6}, T_3={7,8}).
    def _initial_graph_2(self):
        G = nx.Graph()
        G.add_nodes_from([1, 2, 3, 4, 5, 6, 7, 8])
        G.add_edges_from([(1, 2), (1, 3), (2, 3), (4, 5), (4, 6), (5, 6), (7, 8)])
        isomorphism_list = [
            [],
            [(1,4)], [(1,4),(1,5)], [(1,4),(2,5)], [(1,4),(1,5),(3,5)], [(1,4),(1,5),(2,6)],
            [(1,4),(2,5),(3,6)], [(1,4),(1,5),(2,5),(2,6)], [(1,4),(1,5),(3,5),(2,6)],
            [(1,4),(1,5),(3,4),(2,6)],
            [(1,4),(1,5),(3,4),(2,5),(2,6)],
            [(1,4),(1,5),(3,4),(2,5),(2,6),(3,6)]
        ]
        possible_add_edge1 = list(set(itertools.combinations([1, 2, 3, 7, 8], 2)) - set(G.edges))
        possible_add_edge2 = list(set(itertools.combinations([4, 5, 6, 7, 8], 2)) - set(G.edges))
        # self_constant=15, i.e., check if t(G)<m_2(G)+e(G)-15
        return G, isomorphism_list, possible_add_edge1, possible_add_edge2, 15, [2,1,1], [[1, 2, 3, 4, 5, 6], [1, 2, 3, 7, 8], [4, 5, 6, 7, 8]]

    # Type III (a=2, c=1, and T_1={1,2,3}, T_2={4,5,6}, T_3={7}).
    def _initial_graph_3(self):
        G = nx.Graph()
        G.add_nodes_from([1, 2, 3, 4, 5, 6, 7])
        G.add_edges_from([(1, 2), (1, 3), (2, 3), (4, 5), (4, 6), (5, 6)])
        isomorphism_list = [
            [],
            [(1,4)], [(1,4),(1,5)], [(1,4),(2,5)], [(1,4),(1,5),(3,5)], [(1,4),(1,5),(2,6)],
            [(1,4),(2,5),(3,6)], [(1,4),(1,5),(2,5),(2,6)], [(1,4),(1,5),(3,5),(2,6)],
            [(1,4),(1,5),(3,4),(2,6)],
```

```
            [(1,4),(1,5),(3,4),(2,5),(2,6)],
            [(1,4),(1,5),(3,4),(2,5),(2,6),(3,6)]
        ]
        possible_add_edge1 = list(set(itertools.combinations([1, 2, 3, 7], 2)) - set(G.edges))
        possible_add_edge2 = list(set(itertools.combinations([4, 5, 6, 7], 2)) - set(G.edges))
        # self_constant=12, i.e., check if t(G)<m_2(G)+e(G)-12.
        return G, isomorphism_list, possible_add_edge1, possible_add_edge2, 12, [2,1,1], [[1, 2, 3, 4, 5, 6], [1, 2, 3, 7], [4, 5, 6, 7]]

    # Type IV (a=1, b=2, and T_1={1,2,3}, T_2={4,5}, T_3={6,7}).
    def _initial_graph_4(self):
        G = nx.Graph()
        G.add_nodes_from([1, 2, 3, 4, 5, 6, 7])
        G.add_edges_from([(1, 2), (1, 3), (2, 3), (4, 5), (6, 7)])
        # Different K_3-free graphs on vertex set T_2UT_3.
        isomorphism_list = [[], [(4,6)], [(4,6),(5,7)]]
        # Add possible edges between T_1 and T_2.
        possible_add_edge1 = list(set(itertools.combinations([1, 2, 3, 4, 5], 2)) - set(G.edges))
        # Add possible edges between T_1 and T_3.
        possible_add_edge2 = list(set(itertools.combinations([1, 2, 3, 6, 7], 2)) - set(G.edges))
        # self_constant=12, i.e., check if t(G)<m_2(G)+e(G)-12.
        return G, isomorphism_list, possible_add_edge1, possible_add_edge2, 12, [1,1,0], [[1, 2, 3, 4, 5], [1, 2, 3, 6, 7], [4, 5, 6, 7]]

    # Type V (a=1, b=1, c=1, and T_1={1,2,3}, T_2={4,5}, T_3={6}).
    def _initial_graph_5(self):
        G = nx.Graph()
        G.add_nodes_from([1, 2, 3, 4, 5, 6])
        G.add_edges_from([(1, 2), (1, 3), (2, 3), (4, 5)])
        # Different K_3-free graphs on vertex set T_2UT_3.
        isomorphism_list = [[], [(4,6)]]
        possible_add_edge1 = list(set(itertools.combinations([1, 2, 3, 4, 5], 2)) - set(G.edges))
        possible_add_edge2 = list(set(itertools.combinations([1, 2, 3, 6], 2)) - set(G.edges))
        # self_constant=9, i.e., check if t(G)<m_2(G)+e(G)-9.
        return G, isomorphism_list, possible_add_edge1, possible_add_edge2, 9, [1,1,0], [[1, 2, 3, 4, 5], [1, 2, 3, 6], [4, 5, 6]]

    # Type VI (a=1, c=2, and T_1={1,2,3}, T_2={4}, T_3={5}).
    def _initial_graph_6(self):
        G = nx.Graph()
        G.add_nodes_from([1, 2, 3, 4, 5])
        G.add_edges_from([(1, 2), (1, 3), (2, 3)])
        # Different K_2-free graphs on vertex set T_2UT_3.
        isomorphism_list = [[]]
        possible_add_edge1 = list(set(itertools.combinations([1, 2, 3, 4], 2)) - set(G.edges))
        possible_add_edge2 = list(set(itertools.combinations([1, 2, 3, 5], 2)) - set(G.edges))
        # self_constant=6, i.e., check if t(G)<m_2(G)+e(G)-6.
        return G, isomorphism_list, possible_add_edge1, possible_add_edge2, 6, [1,1,0], [[1, 2, 3, 4], [1, 2, 3, 5], [4, 5]]

# Run all the types and output the counterexamples
graph_solution = GraphSolution()
for i in range(6,0,-1):
    print(i)
    graph_solution.initialize_graph(i)
    graph_solution.forward()
```