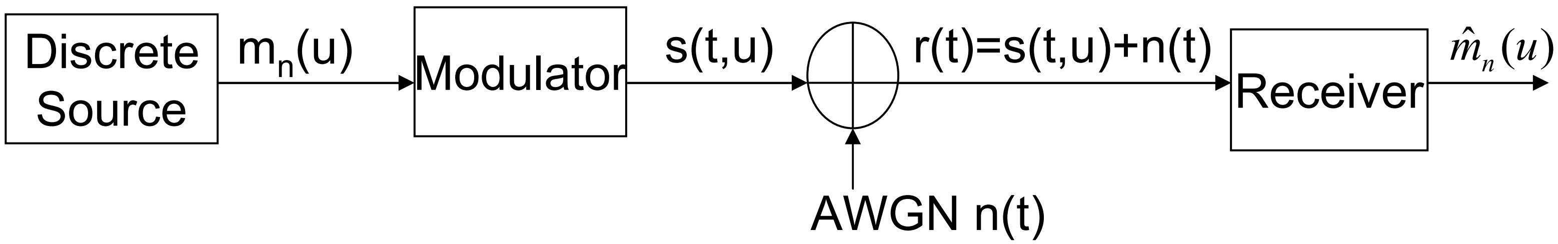
Digital Modulation Methods



- Modulator maps a sequence of messages into analog waveforms that matches the characteristics of the channel
- $m_n(u)$ represents the *n*th random message which takes values in a message set $\{m_0, m_1, \cdots, m_{M-1}\}$
- To transmit $m_n(u)$, the modulator then selects a signal $s_{i_n}(t)$ from a signal set $\{s_0(t), s_1(t), \cdots, s_{\hat{M}-1}(t)\}$ for transmission over the channel during the nth interval [(n-1)T,nT), where $\hat{M} \geq M$
- When the signal set is given, any mapping

$$\{m_n(u)\}_{n=-\infty}^{+\infty} \rightarrow \{i_n\}_{n=-\infty}^{+\infty}$$

defines a modulator.

Digital Modulation Methods (cntd)

- If the mapping is memoryless, that is, the *n*th integer in depends only on $m_n(u)$, the modulator is called memoryless. Otherwise, the modulator is said to have memory.
- For memoryless modulator, $\hat{M} = M$, and we consider only the interval [0,T].
- For modulators having memory, $\hat{M}$ may be strictly greater than M to allow the freedom of selecting different signals during different intervals even when the corresponding message are the same.
- We first look at memoryless modulation methods.

Pulse Amplitude Modulation (PAM)

$S_i(t) = R_e\{ A_i\, g(t)\, e^{j2\pi f_c t} \} = A_i\, g(t) \cos 2\pi f_c t \quad 0 \leqslant t \leqslant T,\ 0 \leqslant i \leqslant M\text{-}1$

- $\{ A_i \}$ denotes the set of M possible amplitudes,

  $A_i = (2i+1-M)d,\ i = 0, \cdots, M-1$

- g(t) represents the generic pulse whose shape influences the spectrum of the transmitted signal.

$$g(t) = \begin{cases} 1 & 0 \le \mathrm{t} \le \mathrm{T} \\ 0 & \text{oterwise} \end{cases}$$

- f_c is the carrier frequency and often chosen equal to n_c/T for some integer n_c.
- Digital PAM is also called amplitude-shifting-keying (ASK)

Signal Space of PAM Signals

Let $\Phi_1(t)$ be the normalized version of $g(t)\cos 2\pi f_c t$. Then $\Phi_1(t)$ is an orthonormal basis for the PAM signals $S_i(t)$. When

$$g(t) = \begin{cases} 1 & 0 \le \mathrm{t} \le \mathrm{T} \\ 0 & \text{oterwise} \end{cases}$$

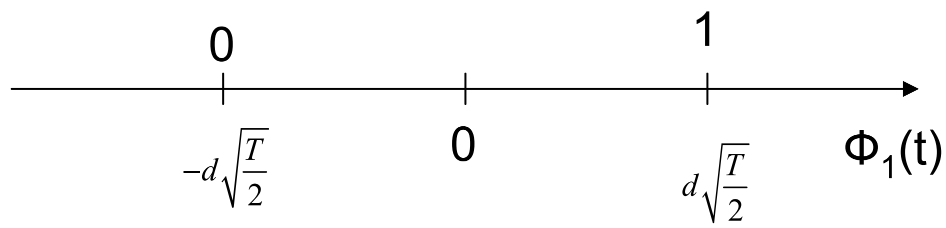
$$\phi_1(t) = \frac{g(t)\cos 2\pi f_c t}{\sqrt{\int_0^T g^2(t)\cos^2 2\pi f_c t}} = \sqrt{\frac{T}{2}}\cos 2\pi f_c t, \ \ 0 \le t \le T$$

$$\Rightarrow \ s_i(t) = A_i \sqrt{\frac{T}{2}}\phi_1(t)$$

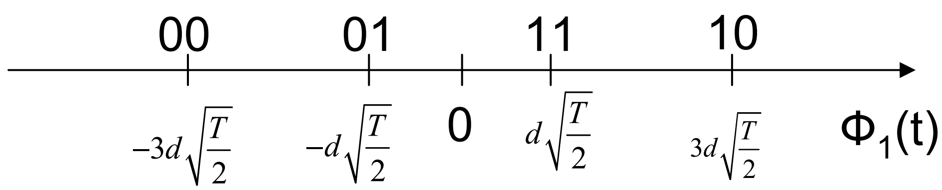
$$0 \le i \le M-1$$

Special Cases

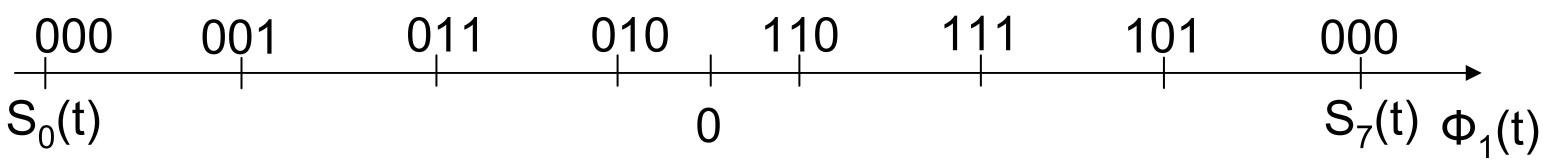
1) M=2, BPSK or antipodal signaling



2) M=4



3) M=8



Gray Encoding

- A Gray mapping (or code) is a mapping C from (0,1, $\cdots 2^{k-1}$) to $(0,1)^k$ such that for any i, C(i) and C(i+1) differ in only one location.
- When $M=2^K$ for some K, the mapping from the message set denoted by {0,1, $\cdots$, 2^{K-1}} to the PAM signal set {$s_i(t)$: i=0,1, $\cdots 2^{k-1}$} is often as Gray encoding.

Digital Phase Modulation

$S_i(t) = \text{Re}\{ g(t)\, e^{j(2\pi fct+2\pi i/M)} \} = g(t) \cos(2\pi f_c t+2\pi i/M)$

$0 \leqslant t \leqslant T,\ 0 \leqslant i \leqslant M\text{-}1$

- The transmitted information is carried by the M possible phases of the carrier.
- Digital phase modulation is often called phase-shift keying (PSK). PSK with M signals is called M-PSK.
- Signal space

Let g(t) be the unit rectangular pulse.

$$s_i(t) = \cos\frac{2\pi i}{M}\cos 2\pi f_c t - \sin\frac{2\pi i}{M}\sin 2\pi f_c t$$

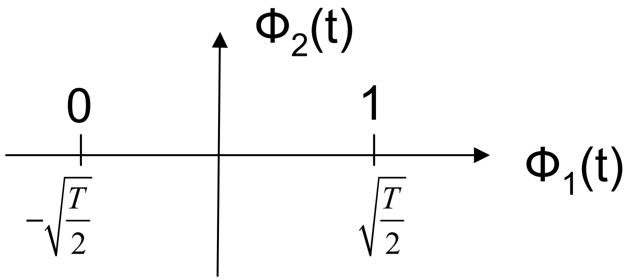
$$\text{Let } \phi_1(\text{t})=\sqrt{\frac{2}{\text{T}}}\cos 2\pi f_c t, \phi_2(\text{t})=\sqrt{\frac{2}{\text{T}}}\sin 2\pi f_c t$$

$\{\Phi_1(t), \Phi_2(t)\}$ is an orthonormal basis for the signal set $\{s_i(t)\}$

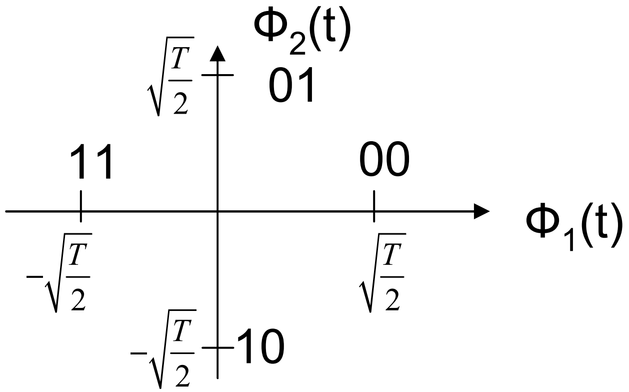
$$s_i(t) \leftrightarrow \underline{s_i} = (\sqrt{\frac{\text{T}}{2}}\cos\frac{2\pi i}{M}, \sqrt{\frac{\text{T}}{2}}\sin\frac{2\pi i}{M})\ 0 \le \text{i} \le \text{M-1}$$

Special Cases

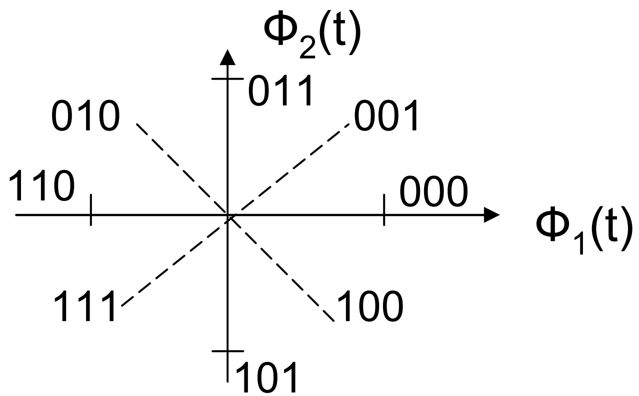
1) M=2, BPSK



2) M=4



3) M=8



When $M=2^K$, the mapping from the message set to the set of M possible phases is often chosen as Gray mapping so as to reduce the corresponding bit error prob.

Quadrature Amplitude Modulation (QAM)

Combining amplitude and phase modulation, we get

$S_i(t) = \mathrm{Re}\{ (A_{ic}+jA_{is})g(t)\, e^{j2\pi fct} \} = A_{ic}g(t) \cos 2\pi f_c t - A_{is}g(t)\sin 2\pi f_c t$

$= V_i\, g(t)\cos(2\pi f_c t+\theta_i)$

$$V_i = \sqrt{A_{ic}^2 + A_{is}^2}, Q_i = \tan^{-1}(A_{is} / A_{ic})$$

- The term $\cos 2\pi f_c t$ is typically referred to as the inphase carrier and the term $\sin 2\pi f_c t$ as the quadrature carrier.
- In QAM, the transmitted information is carried by both the amplitudes of the inphase carrier and the amplitudes of the quadrature carrier.
- If V_i is allowed to take one of M_1 possible values, θ_i to take one of M_2 possible values, then the corresponding QAM is an $M=M_1M_2$ combined PAM-PSM signal constellation

Signal Space

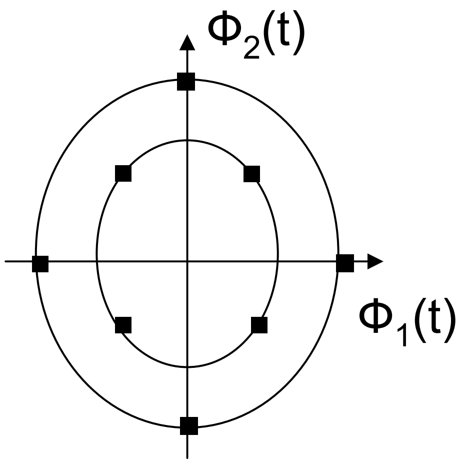
$$\text{Let } g(t) = \begin{cases} 1 & 0 \le \mathrm{t} \le \mathrm{T} \\ 0 & \text{oterwise} \end{cases}$$

$$\phi_1(\mathrm{t}) = \sqrt{\frac{2}{\mathrm{T}}} \cos 2\pi f_c t, \;\; \phi_2(\mathrm{t}) = \sqrt{\frac{2}{\mathrm{T}}} \sin 2\pi f_c t$$

are an orthonormal basis for $\{\mathrm{s_i(t)}\}$

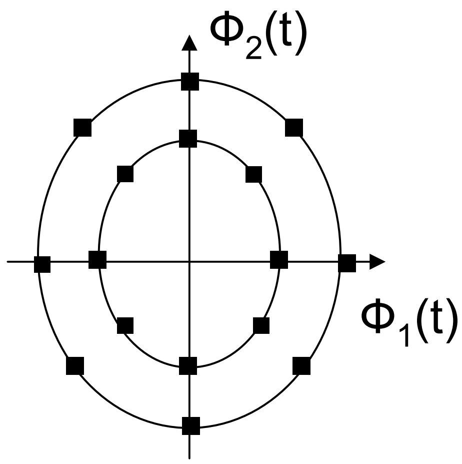
$$\mathrm{s_i(t)} \leftrightarrow \underline{s_i} = (A_{ic}, A_{is})$$

Special Cases



Comobined PAM-PSK

M=8, M_1=2 and M_2=4



16-star QAM

M=16, M_1=4 and M_2=4

Other cases: see pages 152 and 153

Multidimensional Signals

- In PAM, we have a one dimensional signal space. In QAM, we have a two dimensional signal space.
- To get a high dimensional space, we can
  - Partition the time interval [0,T] into small time intervals of length $T_1=T/N$
  - or partition the corresponding frequency band into N frequency slots of equal length
  - Due to the bandwidth limit, you can not partition the time interval into any number of sub-intervals.

Example:

- If PAM is used in each time (or frequency) slot, then the N slots can transmit N-dimensional signal vectors.
- If QAM is used in each time (or frequency) slot, then the N slots can transmit 2N-dimensional signal vectors.

Orthogonal Signals

$$S_i(t) = \sqrt{\frac{2\varepsilon}{T}} \cos(2\pi f_c t + 2\pi i \Delta f t)$$

$$0 \le t \le T, \quad 0 \le \mathrm{i} \le \text{M-1}$$

To ensure that $s_i(t)$ and $s_j(t)$ are orthogonal for any $i \neq j$, Δf must be $n/(2T)$ for some integer n.

Δf is the frequency separation.

$$\int_0^T s_i(t) s_j(t) dt = \frac{2\varepsilon}{T} \int_0^T [\cos(2\pi f_c t + 2\pi i \Delta f t) \times \cos(2\pi f_c t + 2\pi j \Delta f t)] dt$$

$$= \frac{\varepsilon}{T} \int_0^T [\cos(4\pi f_c t + 2\pi(i+j)\Delta f t) + \cos(2\pi(i-j)\Delta f t)] dt$$

$$= \frac{\varepsilon}{T} [\frac{\sin 2\pi(i+j)\Delta f T}{4\pi f_c + 2\pi(i+j)\Delta f} + \int_0^T \cos 2\pi(i-j)\Delta f t dt]$$

$$= \begin{cases} 0 & \text{if i} \neq \text{j} \\ \varepsilon & \text{i=j} \end{cases} \quad \text{under the condition that } \Delta f = n/2T$$

The transmitted information is carried by different frequencies. This type of modulation is called frequency-shift-keying (FSK) or M-FSK

Signal Space

$$\phi_j(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t + 2\pi j \Delta f t), \;\; 1 \le \mathrm{j} \le \mathrm{M}$$

$$s_0(t) \leftrightarrow \underline{s}_0 = [\sqrt{\varepsilon}, 0, 0 \cdots, 0]$$

$$s_1(t) \leftrightarrow \underline{s}_1 = [0, \sqrt{\varepsilon}, 0 \cdots, 0]$$

$$\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots$$

$$s_{M-1}(t) \leftrightarrow \underline{s}_{M-1} = [0, 0 \cdots, 0, \sqrt{\varepsilon}]$$

Biorthorgonal Signals

M is even.

$s_0(t), \cdots, s_{M/2-1}(t)$ are orthogonal and have the same energy ε.

$$s_{\frac{M}{2}}(t) = -s_0(t)$$

$$s_{\frac{M}{2}+1}(t) = -s_1(t)$$

$$\vdots \qquad \qquad \vdots$$

$$s_{M-1}(t) = -s_{\frac{M}{2}-1}(t)$$

$$s_0(t) \leftrightarrow \underline{s}_0 = [\sqrt{\varepsilon}, 0, 0 \cdots, 0]$$

$$s_{\frac{M}{2}}(t) \leftrightarrow \underline{s}_{\frac{M}{2}} = [-\sqrt{\varepsilon}, 0, 0 \cdots, 0]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$s_{\frac{M}{2}-1}(t) \leftrightarrow \underline{s}_{\frac{M}{2}-1} = [0, 0 \cdots, 0, \sqrt{\varepsilon}]$$

$$s_{M-1}(t) \leftrightarrow \underline{s}_{M-1} = [0, 0 \cdots, 0, -\sqrt{\varepsilon}]$$

Simplex Signals

$s_0(t), \cdots, s_{M-1}(t)$ are orthogonal and have the same energy ε. The mean of this signal set is

$$\overline{s}(t) = \frac{1}{M}\sum_{i=0}^{M-1} s_i(t)$$

We construct a new set of M signals by substracting the mean $\overline{s}(t)$ from each of the M orthogonal signals:

$$s_i^{'}(t) = s_i(t) - \overline{s}(t) \quad 0 \le \mathrm{i} \le \mathrm{M\text{-}1}$$

The resulting signals $s_i^{'}(t)$ are called simplex (or transorthogonal) signals.

Signal Space

$$\phi_j(t) = \sqrt{\frac{2}{T}}\cos(2\pi f_c t + 2\pi j\Delta f t),\ 1 \leq \mathrm{j} \leq \mathrm{M}$$

$$s_0(t) \quad \leftrightarrow \underline{s}_0 \quad = [\sqrt{\varepsilon}, 0, 0 \cdots, 0]$$

$$s_1(t) \quad \leftrightarrow \underline{s}_1 \quad = [0, \sqrt{\varepsilon}, 0 \cdots, 0]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$s_{M-1}(t) \leftrightarrow \underline{s}_{M-1} = [0, 0 \cdots, 0, \sqrt{\varepsilon}]$$

$$\overline{s}(t) \quad \leftrightarrow \quad \underline{\overline{s}} = \frac{1}{M}[\sqrt{\varepsilon}, \sqrt{\varepsilon} \cdots, \sqrt{\varepsilon}]$$

$$s_0'(t) \quad \leftrightarrow \underline{s}_0' \quad = [(1 - \frac{1}{M})\sqrt{\varepsilon}, -\frac{\sqrt{\varepsilon}}{M}, \cdots, -\frac{\sqrt{\varepsilon}}{M}]$$

$$s_1'(t) \quad \leftrightarrow \underline{s}_1' \quad = [-\frac{\sqrt{\varepsilon}}{M}, (1 - \frac{1}{M})\sqrt{\varepsilon}, -\frac{\sqrt{\varepsilon}}{M} \cdots, -\frac{\sqrt{\varepsilon}}{M}]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$s_{M-1}'(t) \leftrightarrow \underline{s}_{M-1}' = [-\frac{\sqrt{\varepsilon}}{M} \cdots, -\frac{\sqrt{\varepsilon}}{M}, (1 - \frac{1}{M})\sqrt{\varepsilon}]$$

Signal Space

$$\| s_i'(t) \|^2 = \| \underline{s}_i' \|^2 = \frac{\varepsilon}{M^2}(M-1) + \frac{(M-1)^2}{M^2}\varepsilon$$

$$\Rightarrow s_i'(t), 0 \le i \le M-1, \text{ have the same energy } (1-\frac{1}{M})\varepsilon$$

Furthermore,

$$\frac{< s_i'(t), s_j'(t) >}{\| s_i'(t) \| \| s_j'(t) \|} = \frac{\underline{s}_i' \cdot \underline{s}_j'}{\| \underline{s}_i' \| \| \underline{s}_i' \|} = \frac{\frac{M-2}{M}\varepsilon - 2\frac{M-1}{M^2}\varepsilon}{(1-\frac{1}{M})\varepsilon}$$

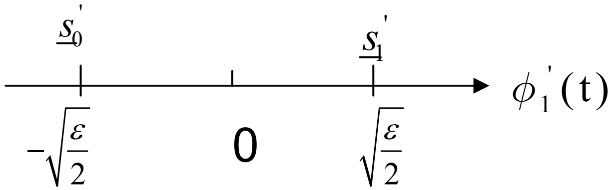
$$= -\frac{1}{M-1} \text{ for any } i \ne j$$

$\Rightarrow$ Simplex signals are equally correlated and require less energy than the original signals.

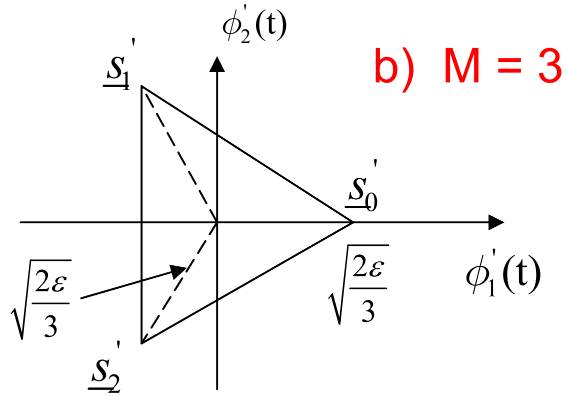
Examples

Note that simplex signals $s_i'(t)$ lies in an M-1 dimensional space

a) M = 2



b) M = 3



c) M=4. By selecting $\phi_1'(t)$, $\phi_2'(t)$ and $\phi_3'(t)$ appropriately

$$s_0' = \frac{\sqrt{\varepsilon}}{2}(1,1,1)$$

$$s_1' = \frac{\sqrt{\varepsilon}}{2}(1,-1,-1)$$

$$s_2' = \frac{\sqrt{\varepsilon}}{2}(-1,1,-1)$$

$$s_3' = \frac{\sqrt{\varepsilon}}{2}(-1,-1,1)$$

Signals Waveforms from Binary Codes

- Assume that we have a binary code consisting of M binary codeword $C_m = (C_{m1}, C_{m2}, \cdots, C_{mN})$, $0 \leqslant m \leqslant M-1$, where C_{mj} is 0 or 1.
- To generate signal waveform corresponding to the binary code, we first partition the time interval [0, T] into *N* non-overlapping time slots $[(j-1)T_c, jT_c]$, $1 \leqslant j \leqslant N$ where $T_c = T/N$.
- Assume that PAM is used for generating signal waveforms. We next generate a set of M waveform signals $s_i(t)$ from the binary code. The correspondence between C_i and $s_i(t)$

$$C_{ij} = 1 \Rightarrow s_i(t) = \sqrt{\frac{2\varepsilon_c}{T_c}} \cos 2\pi f_c t, t \in [(j-1)Tc, jTc]$$

$$C_{ij} = 0 \Rightarrow s_i(t) = -\sqrt{\frac{2\varepsilon_c}{T_c}} \cos 2\pi f_c t, t \in [(j-1)Tc, jTc]$$

$$0 \leq i \leq M-1, 1 \leq j \leq N$$

Signals Waveforms from Binary Codes

$$\phi_j(t) = \begin{cases} \sqrt{\dfrac{2}{T_c}} \cos 2\pi f_c t, & t \in [(j-1)T_c, jT_c] \\ 0, & \text{otherwise} \end{cases}$$

$$1 \le j \le N$$

In the space spanned by $\{\phi_j(t)\}_{j=1}^{N}$, we have

$$s_i(t) \leftrightarrow \underline{s_i} = (s_{i1}, s_{i2}, \cdots s_{iN}), s_{ij} = (2cij - 1)\sqrt{\varepsilon_c}$$

$$0 \le i \le M-1,\ 1 \le j \le N$$

$$| s_i(t) - s_k(t) \| = \| s_i - s_k \| = 2\sqrt{\varepsilon_c d(c_i, c_k)}$$

$d(c_i,c_k)$= the Hamming distance between C_i and C_k

= the # of locations in which c_{ij} and c_{kj} differ

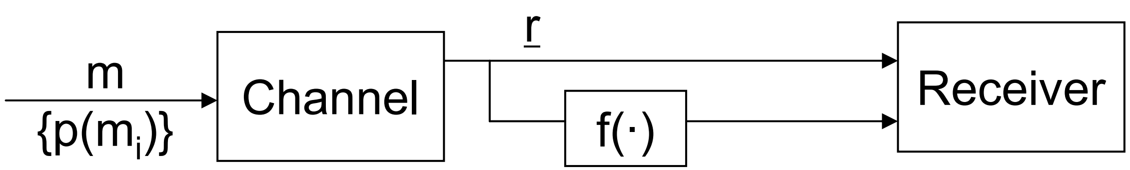
Signals Waveforms from Binary Codes

- With respect to the time slot [0, T_c], signal waveforms generated from binary codes can be regarded as coded waveforms and the corresponding modulation schemes are called coded modulation schemes.
- The are 2^N possible waveforms. That is, $\hat{M} = 2^N$ Usually, M is much smaller than $\hat{M}$. Then one has many choices of selecting a subset of M signals from 2^N possible waveforms.
- The problem is to find a subset of M signals such that the distance between different signals is as large as possible
- Modulation methods with memory include differential PSK (DPSK), MSK (minimum shift keying), etc.

General Rules for Signal Design

- Theorem of reversibility

Let f be any one to one mapping. Form $\underline{r}$, we can compute f($\underline{r}$); conversely, from f($\underline{r}$), we can compute $\underline{r}$. Let us form an additional observation f($\underline{r}$)



m—>f($\underline{r}$) —>$\underline{r}$ forms a Markov chain. f($\underline{r}$) is a sufficient statistic with respect to ($\underline{r}$, f($\underline{r}$)). Therefore, only the observation f($\underline{r}$) is needed in the design of optimal receivers.

General Rules for Signal Design

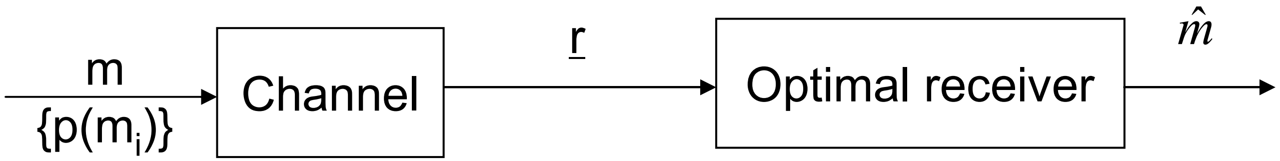


Fig 3.5.1

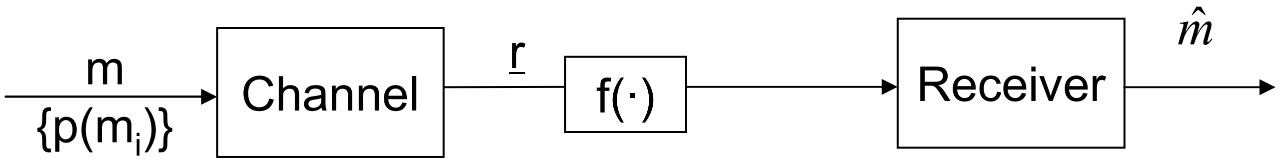


Fig 3.5.2

Theorem 3.5.1: The decision maker can make equally well decisions when observing $\underline{r}$ and $f(\underline{r})$ provided that $f(\cdot)$ is invertible. In other words, any reversible operation of data does not affect the performance of any optimal receiver.

Applications of the Theorem to Translations

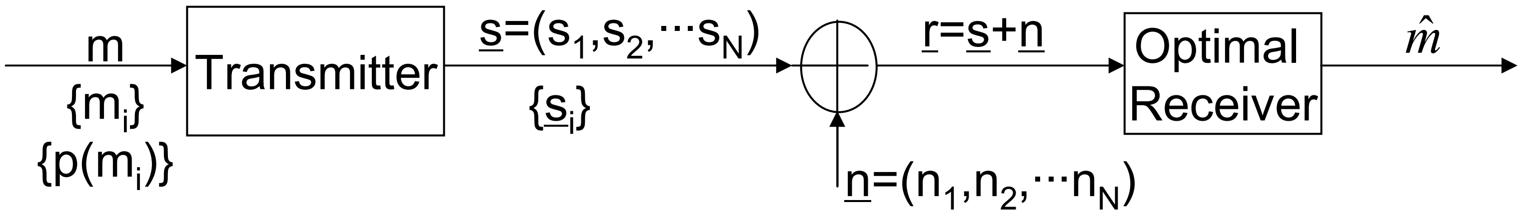


Figure a

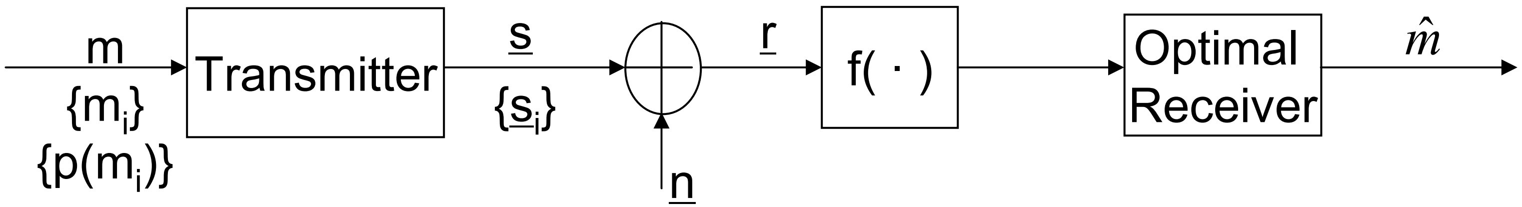


Figure b

Assume the channel is additive. f(·) is a translation. The the optimal receivers in figures a and b have the same structure. In this case, the signal set $\{\underline{s}_i\}$ is transmitted into a new signal set $\{\underline{s}_i'\}$ with $\underline{n}$ unaffected:

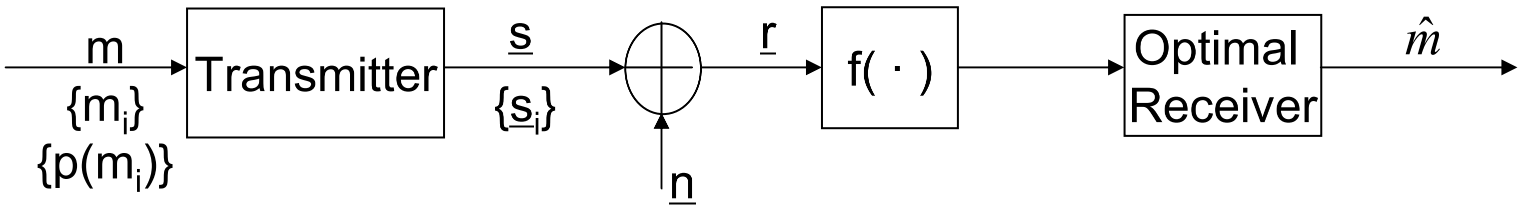
$$\underline{s}_i' = \underline{s}_i + \underline{a} \quad 0 \leq \mathrm{i} \leq \mathrm{M}\text{-}1$$

$\underline{a}$ is an translation vector

Signal Design Rule 1

Translations of a signal set do not change performance when the channel is an additive interference channel.

Applications of the Theorem to Rotations



Assume the noise vector $\underline{n}$ be Gaussian and i.i.d with zero mean and variance $\sigma^2 I_{NxN}$. Let the mapping f(·) be a rotation

$$f(\underline{r}) = A\,\underline{r}$$

where A is an orthogonal matrix, i.e.,

$$A^T = A^{-1} \text{ (or } A^T A = I \text{ identity matrix)}$$

Since $\underline{r} = \underline{s} + \underline{n}$, it follows that

$$f(\underline{r}) = A\underline{s} + A\underline{n}$$

The signal set $\{s_i\}$ is now roated into a new signal vector $\{\underline{s}_i^{'}\}$

$$\underline{s}_i^{'} = A\underline{s}_i \quad 0 \le \mathrm{i} \le \mathrm{M\text{-}1}$$

The noise vector $\underline{n}$ is rotated into a new noise vector $\underline{n}' = A\underline{n}$.

Lemma: Assume $\underline{n}$ is Gaussian and i.i.d with zero mean and variance $\sigma^2 I_{NxN}$, the transformed sequence $\underline{n}'$ = A $\underline{n}$ has the same joint pdf as n if and only if $AA^T = I$

Proof:

Since $E(\underline{n}) = \underline{0}$, we get $E(\underline{n}') = \underline{0}$.

The covariance matrix $R_{n'}$ of the vector $\underline{n}'$ is

$$R_{n'} = E[\underline{n}'(\underline{n}')^T] = E[A\,\underline{n}\,\underline{n}^T A^T] = A R_n A^T$$

Since $R_n = \sigma^2 I_{NxN}$, we get $R_{n'} = \sigma^2 A A^T$

Therefore, the sufficient and necessary condition for $\underline{n}' = A\underline{n}$ to have the same distribution as $\underline{n}$ is

$$AA^T = I$$

Signal Design Rule 2

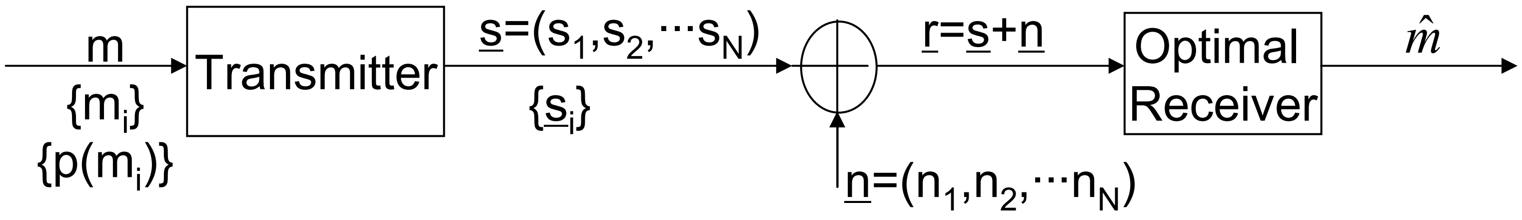


Figure a

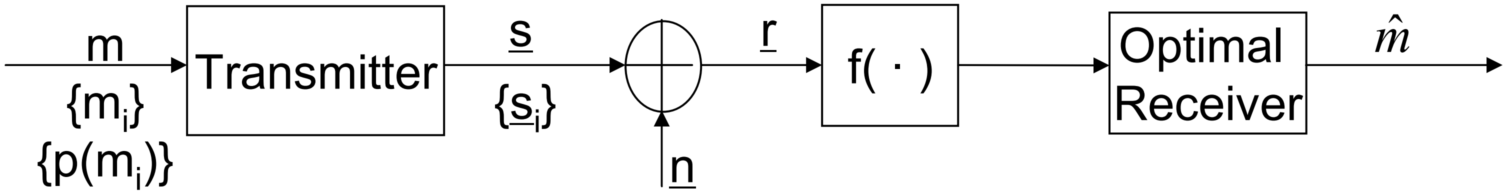


Figure b

Rotations of a signal set do not change performance when the channel is an AWGN channel

Applications of the Theorem to Rotations & Translations

Example from page 170

Ask students to do it

Translations Minimizing the Average Energy

Translations affect the energy required to transmit each signal. Given a signal set $\{\underline{s}_i\}_{i=0}^{M-1}$, let $\underline{a}$ be the mean vector of the set $\{\underline{s}_i\}_{i=0}^{M-1}$

$$\underline{a} = \sum_{i=0}^{M-1} p(m_i)\underline{s}_i$$

For any vector $\underline{b}$, let

$$\underline{s}_i^{'} = \underline{s}_i - \underline{b}, \quad 0 \leq \mathrm{i} \leq \mathrm{M\text{-}1}$$

Then the average energy of the translated signal set $\{\underline{s}_i^{'}\}$ is

$$\sum_{i=0}^{M-1} p(m_i) \| \underline{s}_i^{'} \|^2 = \sum_{i=0}^{M-1} p(m_i) \| \underline{s}_i - \underline{b} \|^2$$

$$= \sum_{i=0}^{M-1} p(m_i) \| \underline{s}_i - \underline{a} + (\underline{a} - \underline{b}) \|^2$$

$$= \sum_{i=0}^{M-1} p(m_i) \| \underline{s}_i - \underline{a} \|^2 + \sum_{i=0}^{M-1} p(m_i) \| \underline{a} - \underline{b} \|^2$$

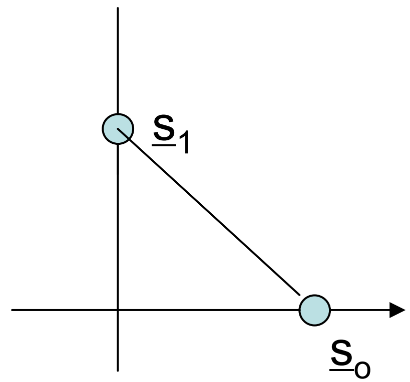
$$= \| \underline{a} - \underline{b}) \|^2 + \sum_{i=0}^{M-1} p(m_i) \| \underline{s}_i - \underline{a} \|^2$$

Translations Minimizing the Average Energy

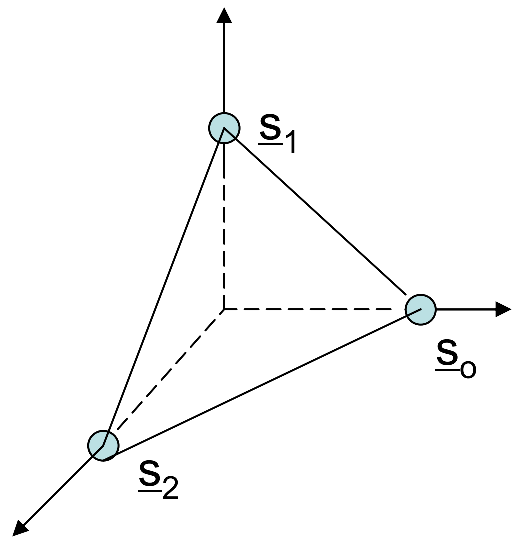
The average energy in a translated signal set is minimized when the mean vector of the original signal set is moved to the origin (i.e., $\underline{b} = \underline{a}$)

Translations Maintaining the Equal Energy Character

Hyperplane



1 dimensional hyperplane
within 2 dimensional space



2 dimensional hyperplane
within 3 dimensional signal space

M signals $\underline{s}_0$, $\underline{s}_1$, ···, $\underline{s}_{M-1}$ can be considered an *(M-1)*-dimensional hyperplane within n dimensional space.

We can use M-1 vectors $\underline{s}_i - \underline{s}_0$ $(1 \leqslant i \leqslant M-1)$ to represent the *(M-1)*-dimensional hyperplane.

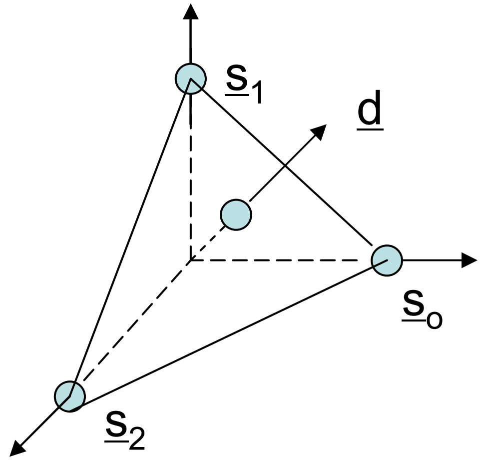
Translations Maintaining the Equal Energy Character

Assume that we have M signals $\underline{s}_0, \underline{s}_1, \cdots, \underline{s}_{M-1}$ which are linearly independent and have the same energy. Let $\underline{d}$ be a unit vector normal to the hyperplane, i.e., $\underline{d} \cdot (\underline{s}_i - \underline{s}_0) = 0, \quad 1 \leqslant i \leqslant M-1$

Then the distance form the hyperplane to the origin is

$$\rho = \underline{d} \cdot \underline{s}_0 \quad (\text{Assume } \rho \geq 0)$$

If we move the hyperplane toward to the origin, then we get a new equal energy set of smaller length.



$$\underline{s}_i^{'} = \underline{s}_i - \rho \underline{d}, \quad 0 \leq i \leq \text{M-1}$$

$$\text{Then } \|\underline{s}_i^{'}\|^2 = \|\underline{s}_i\|^2 - \rho^2, \ 0 \leq i \leq \text{M-1}$$

$$\Rightarrow \underline{s}_i^{'}, 0 \leq i \leq \text{M-1 have the same energy and } \|\underline{s}_i^{'}\| < \|\underline{s}_i\|$$

Special cases: simplex signals

Whitening Transformation

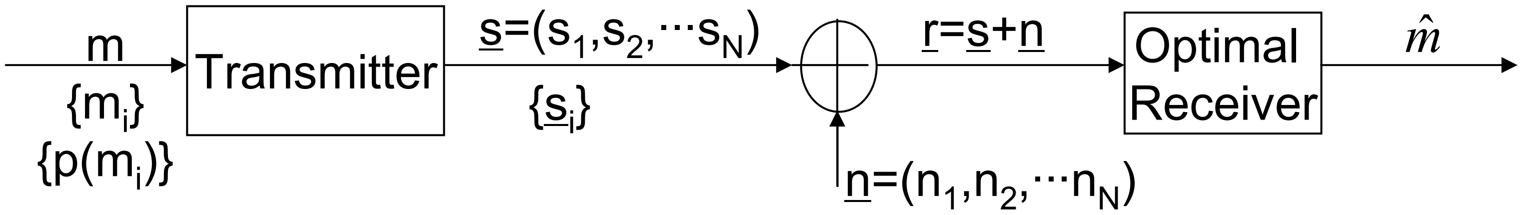


Figure a

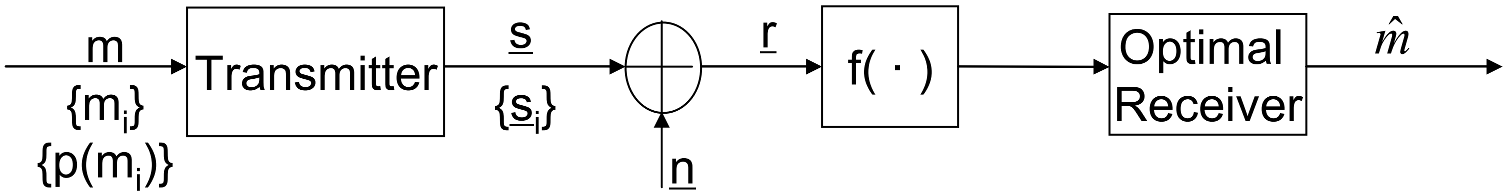


Figure b

The systems shown in Figures a and b have the same performance as long as f(·) is reversible, but their receivers have different structures in general.

If the noise vector $\underline{n}$ is Gaussian but colored, then one can select a special transformation f(·) to simplify the structure of the corresponding receiver.

Whitening Transformation (cntd)

Assume that $\underline{n}$ is a Gaussian vector with zero mean vector and positive covariance matrix R_n. Since R_n is positive, one can decompose R_n as follows:

$$R_n = A^T \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix} A$$

where $A^T A = I_{NxN}$. Let f be a rotation f($\underline{r}$) = A $\underline{r}$. Then the new vector $\underline{n}'$ =A$\underline{n}$ is a Gaussian vector with zero mean vector and covariance matrix

$$R_{n'} = E[A\underline{n}\underline{n}^T A^T] = AR_n A^T$$

$$= AA^T \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} AA^T = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

Noncorrelated components $n_1' n_2' \cdots n_N'$ of $\underline{n}$' are now independent. The above transformation given by A is referred to as whitening transformation

Example

(Colored Gaussian noise) One of two messages m_0 and m_1 is to be communicated over a two dimensional vector channel which adds to the transmitted vector a Gaussian vector $\underline{n} = (n_1, n_2)$ with zero mean vector and covariance matrix

$$\begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}$$

where $-1 < \rho < 1$. Assume that the transmitter uses signal vectors $\underline{s}_0 = (-2,2)$ and $\underline{s}_1 = (2,-2)$ with the correspondence $m_i \leftrightarrow s_i$. The message m_0 and m_1 are not necessarily equally likely.

(a) Use the MAP rule to design an optimal receiver

(b) Compute the probability of error in terms of the Q function

Union Bound

For two equally likely messages m_0 and m_1, no matter what the locations of the corresponding signal vector $\underline{s}_0$ and $\underline{s}_1$ are, one always have

$$P_e = P(\hat{m} \neq m) = Q(\frac{\| \underline{s}_0 - \underline{s}_1 \|}{\sqrt{2N_0}})$$

$$P(\hat{m} \neq m \mid m = m_0) = P(\hat{m} \neq m \mid m = m_1) = Q(\frac{\| \underline{s}_0 - \underline{s}_1 \|}{\sqrt{2N_0}})$$

In general, however, when M and N are large, it it impossible to find a closed form formula for P_e. We shall instead derive a simple upper bound to Pe.

Assumption:

- All messages m_0, m_1, ⋯, m_{M-1} are equally likely
- The channel is AWGN channel

The MAP rule reduces to the minimum distance decision rule.

Union Bound (cntd)

$$D_j = \left\{ \underline{r} \in R^N : \| \underline{r} - \underline{s}_j \| = \min_i [\| \underline{r} - \underline{s}_i \|] \right\}$$

$$\Rightarrow D_j^c = \bigcup_{\substack{i=0 \\ i \neq j}}^{M-1} \left\{ \underline{r} \in R^n : \| \underline{r} - \underline{s}_i \| < \| \underline{r} - \underline{s}_j \| \right\}$$

$$\Rightarrow P(\hat{m} \neq m \mid m = m_j) = P(\hat{m} \neq m_j \mid m = m_j)$$

$$= P(\underline{r} \in D_j^c \mid m = m_j)$$

$$\leq \sum_{\substack{i=0 \\ i \neq j}}^{M-1} P(\| r - \underline{s}_i \| < \| \underline{r} - \underline{s}_j \| \mid m = m_j)$$

$$= \sum_{\substack{i=0 \\ i \neq j}}^{M-1} Q\left(\frac{\| \underline{s}_i - \underline{s}_j \|}{\sqrt{2N_0}} \right)$$

$$P_e = \frac{1}{M} \sum_{j=0}^{M-1} P(\hat{m} \neq m \mid m = m_j)$$

$$\leq \frac{1}{M} \sum_{j=0}^{M-1} \sum_{\substack{i=0 \\ i \neq j}}^{M-1} Q\left(\frac{\| \underline{s}_i - \underline{s}_j \|}{\sqrt{2N_0}} \right)$$

Union bound

Union Bound (cntd)

$$P(\hat{m} \neq m \mid m = m_j) = P(\underline{r} \in D_j^c \mid m = m_j)$$

$$\geq P(\| r - \underline{s}_i \| < \| \underline{r} - \underline{s}_j \| \mid m = m_j)$$

$$= Q\left(\frac{\| \underline{s}_i - \underline{s}_j \|}{\sqrt{2N_0}}\right) \quad \text{for any } i \neq j$$

$$\Rightarrow P(\hat{m} \neq m \mid m_j) \geq \max_{i \neq j} Q\left(\frac{\| \underline{s}_i - \underline{s}_j \|}{\sqrt{2N_0}}\right)$$

Thus

$$\max_{i \neq j} Q\left(\frac{\| \underline{s}_i - \underline{s}_j \|}{\sqrt{2N_0}}\right) = Q\left(\frac{\min_{i \neq j} \| \underline{s}_i - \underline{s}_j \|}{\sqrt{2N_0}}\right)$$

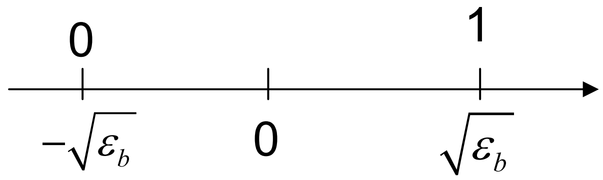
$$\leq P(\hat{m} \neq m \mid m_j) \leq \sum_{\substack{i=0 \\ i \neq j}}^{M-1} Q\left(\frac{\| \underline{s}_i - \underline{s}_j \|}{\sqrt{2N_0}}\right)$$

Thus the above upper and lower bounds become tight as the signal-to-noise ratio (SNR) increases

Performance of the Optimal Receiver for Memoryless Modulation

Binary Modulation

1) Binary PAM signals
$s_0(t) = -s_1(t)$ equally likely



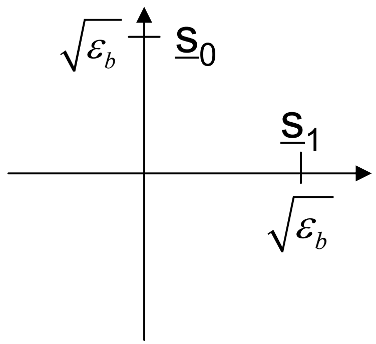
$$P_e = P(\hat{m} \neq m) = P(\hat{m} \neq m \mid m = m_0)$$

$$= P(\hat{m} \neq m \mid m = m_1) = Q\left(\frac{2\sqrt{\varepsilon_b}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{2\varepsilon_b}{N_0}}\right)$$

The signal ε_b/N0 is referred to as the signal-to-noise ratio per bit (or bit SNR). Since M=2, the symbol error prob. P_e is the same as the bit error probability P_b in this case.

Binary Modulation

2) Binary orthogonal signals



$s_0(t)$ and $s_1(t)$ are orthogonal and have the same ε_b

The two signals are equally likely

$$P_b = P_e = P(\hat{m} \neq m)$$

$$= Q\left(\frac{\| \underline{s}_0 - \underline{s}_1 \|}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{\varepsilon_b}{N_0}}\right)$$

To maintain the same bit error probability, binary orthogonal signals require $10\log_{10}2$ = 3 dB more bit SNR

M-ary Orthogonal Signals

$s_0(t), \cdots, s_{M-1}(t)$ are orthogonal and have the same energy ε_s. All messages are equally likely. MAP rule reduces to

$$\text{choose } \hat{\text{m}}=\text{m}_\text{j} \text{ iff } \underline{r}\cdot\underline{s}_j = \max_i \underline{r}\cdot\underline{s}_i$$

$$\Leftrightarrow \sqrt{\varepsilon_s}\, r_{j+1} = \max_i \sqrt{\varepsilon_s}\, r_{i+1} \Leftrightarrow r_{j+1} = \max_i r_{i+1}$$

Where s_i, $0 \leqslant i \leqslant M-1$, are the vector representations of $s_i(t)$:

$$\begin{aligned}
s_0(t) &\leftrightarrow \underline{s}_0 = [\sqrt{\varepsilon_s}, 0, 0 \cdots, 0] \\
s_1(t) &\leftrightarrow \underline{s}_1 = [0, \sqrt{\varepsilon_s}, 0 \cdots, 0] \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots \\
s_{M-1}(t) &\leftrightarrow \underline{s}_{M-1} = [0, 0 \cdots, 0, \sqrt{\varepsilon_s}]
\end{aligned}$$

M-ary Orthogonal Signals (cntd)

$$P(\hat{m}=m \mid m=m_0)=P(r_1=\max_i r_{i+1} \mid m=m_0)=P(r_2\le r_1, r_3\le r_1,\cdots,r_M\le r_1 \mid m=m_0)$$

$$=P(n_2\le\sqrt{\varepsilon_s}+n_1,\cdots,n_M\le\sqrt{\varepsilon_s}+n_1)$$

Since $n_1, n_2\cdots n_N$ are i.i.d and $\underline{n}\sim \mathrm{N}(0,\ \frac{\mathrm{N}_0}{2}I_{N\times N})$, we have

$$P(\hat{m}=m \mid m=m_0)=P(\frac{n_2}{\sqrt{N_0/2}}\le\sqrt{\frac{2\varepsilon_s}{N_0}}+\frac{n_1}{\sqrt{N_0/2}},\cdots\frac{n_M}{\sqrt{N_0/2}}\le\sqrt{\frac{2\varepsilon_s}{N_0}}+\frac{n_1}{\sqrt{N_0/2}})$$

$$=\int_{-\infty}^{+\infty}[\int_{-\infty}^{\sqrt{\frac{2\varepsilon_s}{N_0}}+x}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt]^{M-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$=\int_{-\infty}^{+\infty}[\int_{-\infty}^{y}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt]^{M-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-\sqrt{2\varepsilon_s/N_0})^2}{2}}dy$$

$$=\int_{-\infty}^{0}[Q(|y|)]^{M-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-\sqrt{2\varepsilon_s/N_0})^2}{2}}dy$$

$$+\int_{0}^{+\infty}[1-Q(y)]^{M-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{(y-\sqrt{2\varepsilon_s/N_0})^2}{2}}dy$$

M-ary Orthogonal Signals (cntd)

By symmetry, $\forall$i

$$P(\hat{m} = m \mid m = m_i) = \int_{-\infty}^{0} [Q(|\,y\,|)]^{M-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sqrt{2\varepsilon_s/N_0})^2}{2}} dy$$

$$+ \int_{0}^{+\infty} [1 - Q(y)]^{M-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sqrt{2\varepsilon_s/N_0})^2}{2}} dy$$

$$P_c = P(\hat{m} = m \mid m = m_i)$$

$$P_e = 1 - P_c \text{(Symbol error probability)}$$

Since a symbol now represents logM bits, it follows that

$$\varepsilon_b = \frac{\varepsilon_s}{\log M} \text{ and } \frac{\varepsilon_s}{N_0} = \frac{\varepsilon_b}{N_0} \log M$$

Bit Error Probability

Suppose $M=2^K$. Then each message m_i, $0 \leqslant i \leqslant M-1$, represents a distinct binary sequence of length K:

$$m_i \leftrightarrow C(i)$$

In the case of M-ary orthogonal signals, it is not necessary to choose C as a Gray mapping. The bit error probability P_b is defined as

$$\begin{aligned}
p_b &= \frac{\text{Average \# of erroneous bits}}{\text{K}} \\
&= \frac{1}{\text{K}} \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} p(m = m_i, \hat{m} = m_j) \times d(c(i), c(j)) \\
&= \sum_{\text{i=0}}^{\text{M-1}} p(m_i) [\frac{1}{K} \sum_{j \neq i} p(\hat{m} = m_j \mid m_i) d(c(i), c(j))]
\end{aligned}$$

where $d(c(i), c(j))$ = the Hamming distance between $C(i)$ and $C(j)$

we need to compute $\sum_{j \neq i} p(\hat{m} = m_j \mid m_i) d(c(i), c(j)$

Bit Error Probability

Since $P_e = P(\hat{m} \neq m \mid m_i)$

The conditional symbol errors given m=m_i are equi-probable due to the fact that M-ary orthogonal signals are equally separated, we have

$$P(\hat{m} = m_j \mid m_i) = \frac{P_e}{M-1}, \forall j \neq i$$

$$\Rightarrow \sum_{j\neq i} P(\hat{m} = m_j \mid m_i) d(C(i), C(j)) = \frac{P_e}{2^K - 1} \sum_{j \neq i} d(C(i), C(j))$$

$$= \frac{P_e}{2^K - 1} \sum_{l=1}^{K} l \binom{k}{l} = \frac{P_e}{2^K - 1} K 2^{K-1}$$

$$\Rightarrow P_b = \frac{P_e}{2^K - 1} 2^{K-1} \approx \frac{P_e}{2} \text{ for large K}$$

When K increases, one can reduce the bit SNR required to achieve a given bit error prob. To achieve a $p_b = 10^{-5}$, the required bit SNR is a little more than 12 dB for M=2 (or K=1). But if M is increased to 64 (K=6), the required bit SNR is approximately 6dB.

M-ary Biorthogonal Signals

Let M be even. $s_0(t)$, $s_2(t)$, $s_4(t)$, ⋯, $s_{M-2}(t)$ are orthogonal and have the same energy ε_s. For any $0 \leqslant i \leqslant M/2-1$. $s_{2i+1}(t) = -s_{2i}(t)$. Assume that all messages are equally likely

$$\underline{s}_0 = [\sqrt{\varepsilon_s}, 0, 0 \cdots, 0]$$

$$\underline{s}_1 = [-\sqrt{\varepsilon_s}, 0, 0 \cdots, 0]$$

$$\underline{s}_2 = [0, \sqrt{\varepsilon_s}, 0 \cdots, 0]$$

$$\underline{s}_3 = [0, -\sqrt{\varepsilon_s}, 0 \cdots, 0]$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\underline{s}_{M-2} = [0, 0 \cdots, 0, \sqrt{\varepsilon_s}]$$

$$\underline{s}_{M-1} = [0, 0 \cdots, 0, -\sqrt{\varepsilon_s}]$$

M-ary Biorthogonal Signals (cntd)

$$P(\hat{m}=m \mid m=m_0)=P(r\cdot \underline{s}_0=\max_i r\cdot \underline{s}_i \mid m=m_0)$$

$$=P(r\cdot \frac{\underline{s}_0}{\sqrt{\varepsilon_s}}=\max_i r\cdot \frac{\underline{s}_i}{\sqrt{\varepsilon_s}} \mid m=m_0)$$

$$=P(r_1\geq 0,|r_2|\leq r_1,\cdots,|r_{\frac{M}{2}}|\leq r_1 \mid m=m_0)$$

$$=P(n_1+\sqrt{\varepsilon_s}\geq 0,|n_2|\leq \sqrt{\varepsilon_s}+n_1,\cdots,|n_M|\leq \sqrt{\varepsilon_s}+n_1)$$

$$=P(\frac{n_1}{\sqrt{N_0/2}}\geq -\sqrt{\frac{2\varepsilon_s}{N_0}},|\frac{n_2}{\sqrt{N_0/2}}|\leq \sqrt{\frac{2\varepsilon_s}{N_0}}+\frac{n_1}{\sqrt{N_0/2}},\cdots|\frac{n_M}{\sqrt{N_0/2}}|\leq \sqrt{\frac{2\varepsilon_s}{N_0}}+\frac{n_1}{\sqrt{N_0/2}})$$

$$=\int_0^{+\infty}[1-2Q(x)]^{\frac{M}{2}-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sqrt{2\varepsilon_s/N_0})^2}{2}}dx$$

By symmetry, $\forall$i

$$P(\hat{m}=m \mid m=m_i)=\int_0^{+\infty}[1-2Q(x)]^{\frac{M}{2}-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sqrt{2\varepsilon_s/N_0})^2}{2}}dx$$

$$\Rightarrow P_c=\int_0^{+\infty}[1-2Q(x)]^{\frac{M}{2}-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sqrt{2\varepsilon_s/N_0})^2}{2}}dx$$

$$P_e=1-P_c$$

$$\varepsilon_b=\frac{\varepsilon_s}{\log M},\frac{\varepsilon_b}{N_0}=\frac{\varepsilon_s}{N_0\log M}$$

Bit Error Probability

$$p_b = \frac{\text{Average \# of erroneous bits}}{\text{K}}$$

$$= \frac{1}{\text{K}} \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} p(m = m_i, \hat{m} = m_j) \times d(c(i), c(j))$$

$$= \sum_{\text{i=0}}^{\text{M-1}} p(m_i)[\frac{1}{K} \sum_{j \neq i} p(\hat{m} = m_j \mid m_i) d(c(i), c(j))]$$

where $d(c(i), c(j))$ = the Hamming distance between $C(i)$ and $C(j)$

We need to compute $P(\hat{m} = m_j \mid m_i)$. By symmetry, we select i = 0.

Given m=m_0, there are two types of symbol errors:

1) The negative signal $-s_0(t)$ is selected, i.e., $\hat{m} = m_1$
2) Any of the remaining M-2 signals is selected, i.e.,

$$\hat{m} = m_j, j \geq 2$$

Bit Error Probability (cntd)

$$P(\hat{m}=m_1 \mid m_0)=P(r_1 \le 0, |r_2| \le |r_1|, \cdots, |r_{\frac{M}{2}}| \le |r_1| \mid m=m_0)$$

$$=P(n_1+\sqrt{\varepsilon_s} \le 0, |n_2| \le |\sqrt{\varepsilon_s}+n_1|, \cdots, |n_M| \le |\sqrt{\varepsilon_s}+n_1|)$$

$$=P(\frac{n_1}{\sqrt{N_0/2}} \le -\sqrt{\frac{2\varepsilon_s}{N_0}}, |\frac{n_2}{\sqrt{N_0/2}}| \le |\sqrt{\frac{2\varepsilon_s}{N_0}}+\frac{n_1}{\sqrt{N_0/2}}|, \cdots |\frac{n_M}{\sqrt{N_0/2}}| \le |\sqrt{\frac{2\varepsilon_s}{N_0}}+\frac{n_1}{\sqrt{N_0/2}}|)$$

$$=\int_{-\infty}^{0}[1-2Q(|x|)]^{\frac{M}{2}-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sqrt{2\varepsilon_s/N_0})^2}{2}}dx$$

$$P(\hat{m}=m_j \mid m_0)=\frac{P_e-P(\hat{m}=m_1 \mid m_0)}{M-2}$$

$$=\frac{1}{M-2}[1-\int_{-\infty}^{0}[1-2Q(|x|)]^{\frac{M}{2}-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sqrt{2\varepsilon_s/N_0})^2}{2}}dx]\ \forall \mathrm{j} \ge 2$$

$$\Rightarrow \sum_{j\ne 0} p(\hat{m}=m_j \mid m_0)d(C(0),C(j))$$

$$=p(\hat{m}=m_1 \mid m_0)d(C(0),C(1))+\sum_{j=2}^{M-1} p(\hat{m}=m_j \mid m_0)d(C(0),C(j))$$

- The conditional symbol error probability of selecting the negative signal $–s_i(t)$ of the transmitted signal $s_i(t)$ is greater than the conditional symbol error probability of the other type.
- To minimize p_b, we need to select the mapping C so that the binary sequence corresponding to $s_i(t)$ and $–s_i(t)$ are complementary. That is, C should be selected so that for any $0 \leqslant i \leqslant M/2-1$

$$C(2i) = C(2i+1) \oplus (1,1,\cdots,1)$$

$$\text{Thus } d(C(0),C(1)) = \frac{1}{K}\sum_{j=2}^{M-1} d(C(0),C(j)) = k2^{k-1} - k$$

$$\Rightarrow \frac{1}{K}\sum_{j\neq 0} p(\hat{m} = m_j \mid m_0) d(C(0),C(j))$$

$$= P(\hat{m} = m_1 \mid m_0) + \frac{P_e - P(\hat{m} = m_1 \mid m_0)}{2^k - 2}(2^{k-1} - 1)$$

$$= \frac{Pe}{2} + \frac{1}{2}P(\hat{m} = m_1 \mid m_0)$$

$$= \frac{Pe}{2} + \frac{1}{2}\int_{-\infty}^{0}[1 - 2Q(|x|)]^{\frac{M}{2}-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sqrt{2\varepsilon_s/N_0})^2}{2}}dx$$

By symmetry

$$P_b = \frac{Pe}{2} + \frac{1}{2}\int_{-\infty}^{0}[1 - 2Q(|x|)]^{\frac{M}{2}-1}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sqrt{2\varepsilon_s/N_0})^2}{2}}dx$$

Simplex Signals

- The calculation of the symbol error prob. and bit error prob. is the same as those of orthogonal signals except for the adjustment in energy
- Let ε_s be the energy of the simplex signals. The corresponding orhtogonal signals have the energy $\varepsilon_s M/(M-1)$.

By symmetry, $\forall$i

$$P_c = \int_{-\infty}^{0} [Q(|y|)]^{M-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sqrt{2\varepsilon_s M/(M-1)N_0})^2}{2}} dy$$

$$+ \int_{-\infty}^{0} [1-Q(y)]^{M-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sqrt{2\varepsilon_s M/(M-1)N_0})^2}{2}} dy$$

$$P_e = 1 - P_c \quad \text{(Symbol error probability)}$$

$$P_b = \frac{P_e}{2^K - 1} 2^{K-1} \approx \frac{P_e}{2} \text{ for large K}$$

M-ary Binary-coded Signals

- A binary codebook *B* consisting of M binary codewords $C_i=(C_{i1},C_{i2},\cdots C_{iN})$, $0\leqslant i\leqslant M-1$, $C_{ij}=1$ or 0
- The corresponding coded signals $s_i(t)$ are given by

$$s_i(t) = (2c_{ij}-1)\sqrt{\frac{2\varepsilon_c}{T_b}}\cos(2\pi f_c t)$$

$$1 \le j \le N, Tc = T/N, t \in [(j-1)T_c, jT_c]$$

- Assume that all messages are equally likely. Then the MAP rule reduces to the minimum distance decision rule:

$$\text{choose } \hat{\mathrm{m}}=\mathrm{m}_{\mathrm{j}} \text{ iff } \|r-\underline{s}_j\| = \min_i \|\underline{r}-\underline{s}_i\|$$

$$\text{where } \underline{s}_i = \sqrt{\varepsilon_c}[(2c_{i1}-1),\cdots(2c_{iN}-1)], 0 \le i \le M-1$$

are the vector representations of $s_i(t)$.

In general, it is difficult to find closed formulas for p_b and p_e. Nevertheless, we can apply the union bound to get an upper bound on P_e.

M-ary Binary-coded Signals (cntd)

$$P_e \le \frac{1}{M}\sum_{j=0}^{M-1}\sum_{\substack{i=0\\ i\neq j}}^{M-1} Q\left(\frac{\| \underline{s}_i - \underline{s}_j \|}{\sqrt{2N_0}}\right)$$

$$\le (M-1)Q\left(\frac{\min_{i\neq j} \| \underline{s}_i - \underline{s}_j \|}{\sqrt{2N_0}}\right)$$

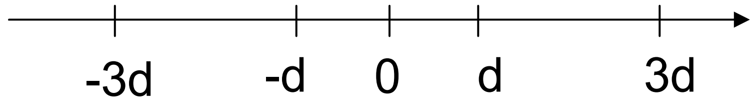
$$= (M-1)Q\left(\frac{2\sqrt{\varepsilon_c d_H(B)}}{\sqrt{2N_0}}\right) = (M-1)Q\left(\sqrt{\frac{2\varepsilon_c d_H(B)}{N_0}}\right)$$

where $d_H(B) = \min_{i\neq j} d(C_i, C_j)$

M-ary PAM Signal

$S_i(t) = = A_i\,(2/T)^{1/2}\cos 2\pi f_c t \quad 0 \leqslant t \leqslant T$

$A_i = (2i+1-M)d,\ i = 0, \cdots, M-1$



$||s_i(t)||^2=(2i+1-M)^2d^2$

Assume that the average energy $E_{AV/s}$ per symbol is

$$\mathrm{E}_{\mathrm{AV/s}} = \frac{1}{\mathrm{M}}\sum_{i=0}^{M-1} \| s_i(t) \|^2 = \frac{1}{3}(M^2-1)d^2$$

Apply the minimum distance decision rule

$$P_c = \frac{M-2}{M}[1-2Q\left(\sqrt{\frac{2d^2}{N_0}}\right)] + \frac{2}{M}\left[1-Q\left(\sqrt{\frac{2d^2}{N_0}}\right)\right]$$

$$= 1 - \frac{2(\mathrm{M}-2)+2}{M} Q\left(\sqrt{\frac{2d^2}{N_0}}\right)$$

$$\Rightarrow P_e = 2(1-\frac{1}{M})Q(\sqrt{\frac{2d^2}{N_0}}) = 2(1-\frac{1}{M})Q(\sqrt{\frac{6E_{AV/s}}{(M^2-1)N_0}})$$

$$= 2(1-\frac{1}{M})Q(\sqrt{\frac{(6\log M)E_b}{(M^2-1)N_0}})$$

where $\mathrm{E_b}$=bit energy= $\mathrm{E_{AV/s}} / \log M$

M-ary PAM Signal (cntd)

- Given any symbol to bit mapping, the bit error probability can be computed exactly in a closed form as in the computing for orthogonal signals.
- For any Gray mappings and large bit SNR values, P_b is approximately

$$\mathrm{P}_{\mathrm{b}} \approx \frac{P_e}{\log M} = \frac{P_e}{K}$$

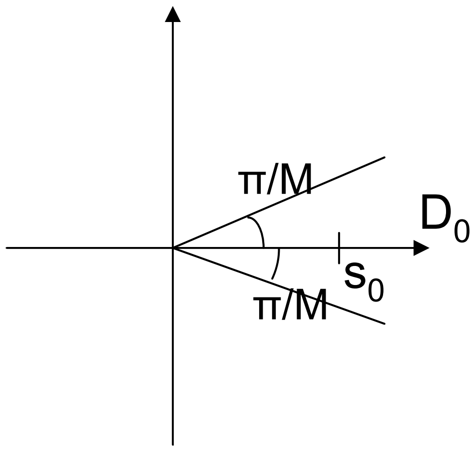
Coherent M-ary PSK Signals

$$s_i(t) = \sqrt{\frac{2Es}{T}} \cos(2\pi f_c t + \frac{2i\pi}{M})$$

$$0 \leq \mathrm{i} \leq \mathrm{M\text{-}1},\ 0 \leq \mathrm{t} \leq \mathrm{T}$$

$$\|s_i(t)\|^2 = E_s \text{ (Energy per symbol)}$$

$$s_i(t) \leftrightarrow \underline{s_i} = (\sqrt{E_s}\cos\frac{2\pi i}{M}, \sqrt{E_s}\sin\frac{2\pi i}{M})$$



Assume that all messages are equally likely. MAP rule reduces to the minimum distance decision rule. By symmetry,

$$P_c = \frac{1}{M}\sum_{i=0}^{M-1} P(\hat{m} = m_i \mid m = m_i) = P(\hat{m} = m_0 \mid m = m_0)$$

$$= \int_{D_0} \frac{1}{\pi N_0} \exp\{-\frac{(r_1 - Es)^2 + r_2^2}{N_0}\} dr_1 dr_2$$

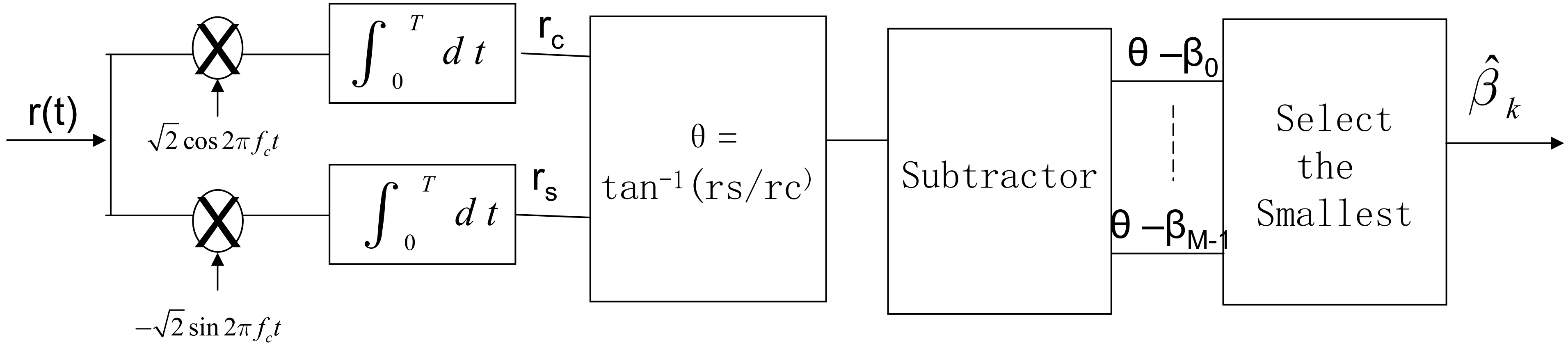
$$= 1 - \frac{1}{\pi}\int_{\frac{\pi}{M}}^{\pi} \exp\{-\frac{1}{N_0}\left(\frac{\sqrt{Es}\sin\frac{\pi}{M}}{\sin(\theta - \frac{\pi}{M})}\right)^2\} d\theta$$

$$P_e = 1 - P_c$$

Using the same approach as before, we can get a closed form formula of bit error probability. For Gray mappings and large bit SNR values, P_b is approximately equal to

$$P_b = \frac{P_e}{\log M} = \frac{Pe}{K}$$

Coherent Receivers



Coherent QAM Signals

Assume that all messages are equally likely. Then the average energy is

$$E_{AV/s} = \frac{1}{M}\sum_{i=0}^{M-1}(A_{ic}^2 + A_{is}^2)$$

The symbol error P_e and bit error prob. P_b depend on specific signal points.

$$S_i(t) == A_{ic}\sqrt{\frac{2}{T}}\cos 2\pi f_c t \ \text{-}\ A_{is}\sqrt{\frac{2}{T}}\sin 2\pi f_c t$$

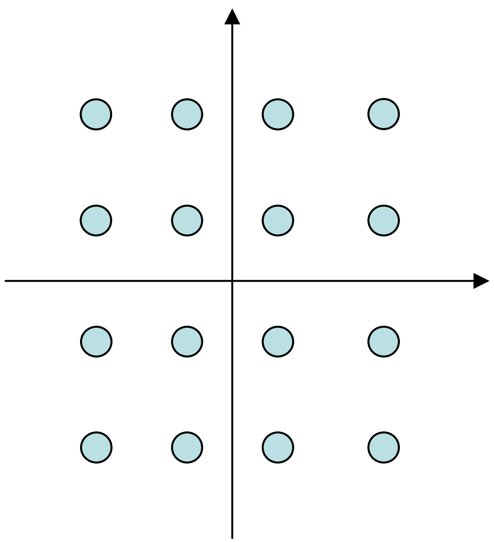
$$0 \le t \le T, 0 \le i \le M\text{-}1$$

Rectangular QAM signal constellations

A_{ic} and A_{is} take any values in

$$\left\{(1-\sqrt{M})d, (3-\sqrt{M})d \cdots (\sqrt{M-1})d\right\}$$

where $\sqrt{M}$ is an integer.



A rectangular QAM signal is generated from
Two PAM signals respectively on the inphase
And the quadrature carriers.

Coherent QAM Signals

The probability of correct decision P_c is

$$P_c = (1 - P_{e,\sqrt{M}})^2$$

where $P_{e,\sqrt{M}}$ is the symbol error prob. of a $\sqrt{M}$-ary PAM

$$\Rightarrow \mathrm{P_e} = 2P_{e,\sqrt{M}} - (P_{e,\sqrt{M}})^2$$

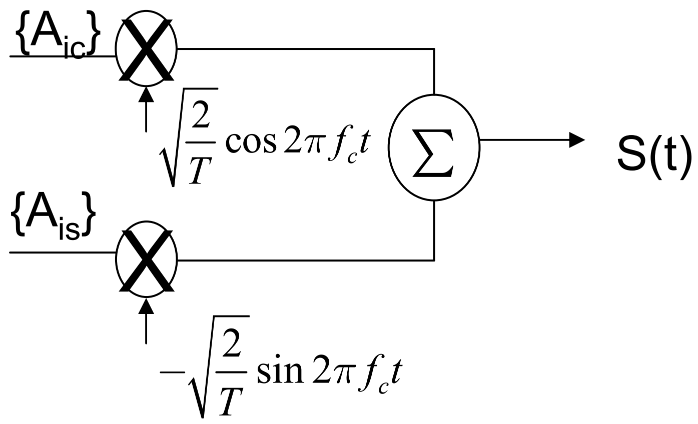
$$\Rightarrow Pe = 4(1 - \frac{1}{\sqrt{M}})Q(\sqrt{\frac{2E_{AV/s}}{(M-1)N_0}}) - 4(1 - \frac{1}{\sqrt{M}})^2 Q^2(\sqrt{\frac{2E_{AV/s}}{(M-1)N_0}})$$

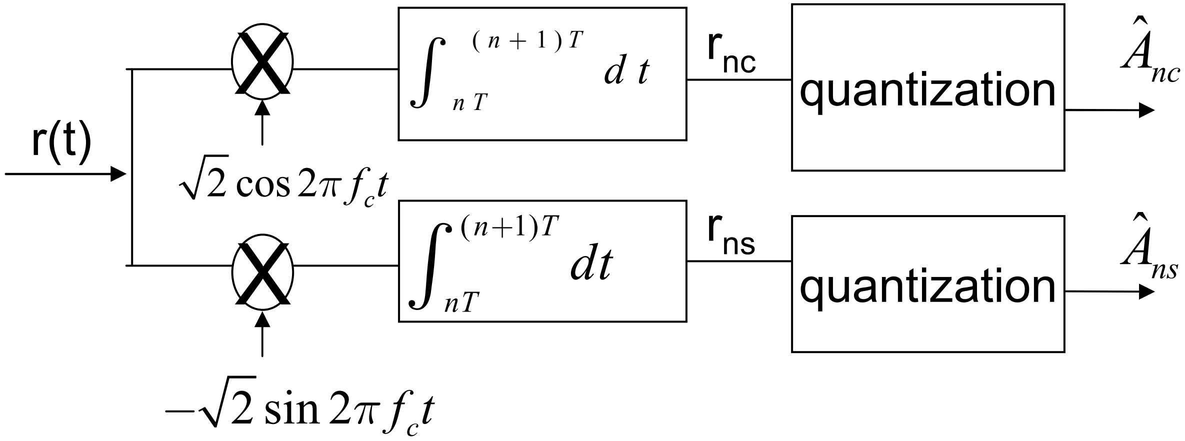
In terms of $\mathrm{E_b/N_0}$

$$Pe = 4(1 - \frac{1}{\sqrt{M}})Q(\sqrt{\frac{3\log M}{(M-1)}\frac{E_b}{N_0}}) - 4(1 - \frac{1}{\sqrt{M}})^2 Q^2(\sqrt{\frac{3\log M}{(M-1)}\frac{E_b}{N_0}})$$

The bit error prob. Of an M rectangular QAM is the same as that of a $\sqrt{M}$-ary PAM

Modulators and Demodulators of Rectangular QAMs





QAM (cntd)

4 rectangular QAM is equivalent to QPSK, that is, 4 rectangular QAM, QPSK and BPSK all have the same error prob, P_b

SNR advantage of M-ary QAM over M-PSK

M	E_b/N_0(QAM)	E_b/N_0(PSK)
4	9.6 dB	9.6 dB
16	13.4 dB	17.4 dB
64	17.8 dB	27.5 dB

$P_b = 10^{-5}$