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Dynamics and stability of transverse vibrations of nonlocal nanobeams with a variable axial load

C Li[1,2,3], **C W Lim**[2,3] **and J L Yu**[1,2]

[1] Department of Modern Mechanics, University of Science and Technology of China, Hefei 230026, People's Republic of China
[2] USTC-CityU Joint Advanced Research Centre, Suzhou 215123, People's Republic of China
[3] Department of Building and Construction, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong SAR, People's Republic of China

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Abstract
This paper investigates the natural frequency, steady-state resonance and stability for the transverse vibrations of a nanobeam subjected to a variable initial axial force, including axial tension and axial compression, based on nonlocal elasticity theory. It is reported that the nonlocal nanoscale has significant effects on vibration behavior, which results in a new effective nonlocal bending moment different to but dependent on the corresponding nonlocal bending moment. The effects of nonlocal nanoscale and the variation of initial axial force on the natural frequency as well as the instability regions are analyzed by the perturbation method. It concludes that both the nonlocal nanoscale and the initial tension, including static and dynamic tensions, cause an increase in natural frequency, while an initial compression causes the natural frequency to decrease. Instability regions are also greatly influenced by the nonlocal nanoscale and initial tension and they become smaller with stronger nonlocal effects or larger initial tension.

1. Introduction

In recent years, dynamics, stability and its control in mechanical vibration has become a basic and inseparable branch of study in applied mechanics and engineering due to the rapid progress in engineering technology and increasingly stringent design requirements. Although generally considered as a harmful source to design components which should be controlled and suppressed, mechanical vibration in many circumstances is a useful phenomenon that can be utilized to better serve human life.

With the advent of nanoengineering and nanotechnology, nanotubes and nanobeams are now potential design candidates which are likely to play key roles in many engineering devices or components at the nanometer scale, such as micro- or nano-electromechanical systems (MEMS or NEMS). Currently, although classical mechanics theories for linear and nonlinear vibrations of beam, plate or shell-like structures at macroscales are well established, the vibration behavior and stability of size-dependent structures at the nanoscale are far from being understood. This is particularly true when the structures are subjected to some variable axial forces.

Because of the scarcity of research in vibrations of nanobeams subjected to a variable axial tension or compression, an account of research work at the macroscale and nanoscale is described here. Wicket and Mote [1] summarized the previous studies on vibration and stability of axially moving materials and later they made a further analysis of the classical vibration of axially moving continua [2]. Subsequently, Öz *et al* [3] presented the nonlinear dynamics and stability of an axially moving classical beam with a time-dependent velocity. Transverse nonlinear parametric vibration of an axially accelerating viscoelastic string was introduced by Chen *et al* [4] and the method of multiple scales was applied directly to the nonlinear partial differential equation to investigate the principal resonance. Chen and Yang [5] studied the steady-state response of an axially moving viscoelastic beam with pulsating velocity by comparing two nonlinear models. Recently, nonlinear parametric vibration and stability of an axially moving Timoshenko beam were considered for two dynamic models by Ghayesh and Balar [6]. For a micro/nanoscale structure, Ukita *et al* [7] presented the relation between photothermal deflection of a microbeam and an antireflection-coated bimorph structure via an analytical approach. Guo and Zhao [8] proposed a theoretical model to investigate the size-dependent bending elastic properties with the influence of surface effects. Chen *et al* [9] investigated the tip trajectories of a smart micro-cantilever beam consisting of an atomic force microscope probe with an additional segment of piezoelectric material on top of the probe.

The nonlocal nanobeam models have received increasing interest in the past few years. Nonlocal continuum theories regard the stress state at a point as a function of the strain states of all points in the body while the classical continuum mechanics assumes the stress state at a given point to be dependent uniquely on the strain state at that same point. The static deformation of nanobeams based on a simplified nonlocal model was obtained firstly by Eringen [10]. This nonlocal model received relatively limited attention because the nonlocal stress was expressed in an integral relation throughout the domain of concern and an analytical constitutive relation was difficult to be solved. Using an alternative differential nonlocal constitutive relation by Eringen [11], the application of nonlocal continuum theories to nanobeams was revived recently by Peddieson *et al* [12] who focused on cantilever nanobeams which were often used as actuators in nanoscale systems. The nonlocal theory of elasticity was used to study applications in nanomechanics including lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, fracture mechanics, surface tension fluids, etc [13–26]. In a recent work, Lim *et al* [26] established analytical solutions for the transverse vibrations of a simply supported nanobeam with a constant axial force based on nonlocal continuum theories. They concluded, through the variational principle, some new results for the natural frequency where the exact nonlocal effects were taken into account. In particular, they derived an effective nonlocal bending moment in an infinite series of nonlocal bending moments [21, 23]. This new definition, instead of the nonlocal bending moment, should be used to replace the classical bending moment in the equilibrium equation or the equation of motion of the classical model for a beam.

In this paper, the nonlocal nanobeam with variable initial axial force effects is investigated. Based on the work by Lim *et al* [26], we consider an initial axial tension which varies with respect to time. Subsequently, the perturbation method is developed to determine the stability for transverse vibrations. The effects of variable initial axial force and nonlocal stress on the vibration behavior of nanobeams are discussed in detail.

2. Equations of motion based on exact nonlocal elasticity

Considering transverse linear vibrations with small deformation for a nanobeam with an initial axial tension N at the ends, the non-dimensional governing equation of motion obtained based on the D'Alembert principle and a new exact nonlocal stress model [26] can be expressed as

$$\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} - \bar{N}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \tau^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}^2} + (\tau^2 \bar{N} + 1)\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} - 2\tau^2 \frac{\partial^6 \bar{w}}{\partial \bar{x}^6} = 0 \quad (1)$$

where

$$\bar{x} = \frac{x}{L}, \qquad \bar{w} = \frac{w}{L}, \qquad \bar{t} = t\sqrt{\frac{EI}{\rho A L^4}}, \qquad \tau = \frac{e_0 a}{L}, \qquad \bar{N} = \frac{NL^2}{EI}. \quad (2)$$

In equations (1) and (2), x is the axial longitudinal coordinate, t time, w transverse displacement, ρ mass density, A cross-sectional area, L length of nanobeam and EI flexural stiffness in which E is the Young's modulus and I the area moment of inertia. Two other quantities e_0 and a which represent the nonlocal effects are, respectively, a constant dependent on material and an internal characteristic length, e.g. for lattice parameter, carbon–carbon or C–C single bond length.

Most of the previous studies assumed the initial axial force to be absent or a constant [12, 15, 16, 20, 22, 26]. In fact, the axial force may vary with respect to time. In this work, the initial axial tension is assumed to vary as

$$N = N_0 + \alpha N_1 \cos \Omega t \quad (3)$$

where α is a small dimensionless parameter, N_0 and N_1 are the amplitudes of the static and dynamic load, respectively, and Ω is the frequency of the applied load. Substituting the dimensionless form of equation (3) into (1) yields

$$\frac{\partial^2 \bar{w}}{\partial \bar{t}^2} - (\bar{N}_0 + \alpha \bar{N}_1 \cos \bar{\Omega}\bar{t})\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} - \tau^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}^2} + [\tau^2(\bar{N}_0 + \alpha \bar{N}_1 \cos \bar{\Omega}\bar{t}) + 1]\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} - 2\tau^2 \frac{\partial^6 \bar{w}}{\partial \bar{x}^6} = 0 \quad (4)$$

where the dimensionless quantities are $\bar{\Omega} = \Omega\sqrt{\frac{\rho A L^4}{EI}}$, $\bar{N}_0 = \frac{N_0 L^2}{EI}$ and $\bar{N}_1 = \frac{N_1 L^2}{EI}$.

Similarly, the non-dimensional forms of nonlocal bending moment $\bar{M}$ and effective nonlocal bending moment $\bar{M}_{\mathrm{ef}}$ are given by, respectively,

$$\bar{M} = \tau^2 \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + [1 - \tau^2(\bar{N}_0 + \alpha \bar{N}_1 \cos \bar{\Omega}\bar{t})]\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + 2\tau^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} \quad (5)$$

$$\bar{M}_{\mathrm{ef}} = \tau^2 \frac{\partial^2 \bar{w}}{\partial \bar{t}^2} + [1 - \tau^2(\bar{N}_0 + \alpha \bar{N}_1 \cos \bar{\Omega}\bar{t})]\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + 2\tau^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^4} - 2\tau^2 \left\{ \tau^2 \frac{\partial^4 \bar{w}}{\partial \bar{x}^2 \partial \bar{t}^2} + [1 - \tau^2(\bar{N}_0 + \alpha \bar{N}_1 \cos \bar{\Omega}\bar{t})]\frac{\partial^4 \bar{w}}{\partial \bar{x}^4} + 2\tau^2 \frac{\partial^6 \bar{w}}{\partial \bar{x}^6} \right\} \quad (6)$$

where $\bar{M} = \frac{ML}{EI}$ and $\bar{M}_{\mathrm{ef}} = \frac{M_{\mathrm{ef}} L}{EI}$, in which M and M_{ef} are the physical nonlocal bending moment and effective nonlocal bending moment, respectively. It is noticed that the effective nonlocal bending moment, first proposed by Lim [21, 23], contains the nonlocal effects through an infinite series of the nonlocal bending moment as

$$M_{\mathrm{ef}} = M - 2\sum_{n=1}^{\infty} (e_0 a)^{2n} \frac{\partial^{2n} M}{\partial x^{2n}} \qquad (n = 0, 1, 2, \ldots) \quad (7)$$

where the nonlocal bending moment can be obtained from the differential nonlocal stress relation [11] as

$$M-(e_0a)^2\frac{\partial^2 M}{\partial x^2}=-M_{\text{cla}} \tag{8}$$

where the classical bending moment is

$$M_{\text{cla}}=EI\frac{\partial^2 w}{\partial x^2}. \tag{9}$$

3. Perturbation method

Because analytical solutions are almost impossible, the perturbation method is employed to solve equation (4) for asymptotic approximate solutions. A first-order uniform approximation of the transverse displacement is assumed as [27]

$$\bar{w}(\bar{x},\bar{t};\alpha)=\bar{w}_0(\bar{x},\bar{T}_0,\bar{T}_1)+\alpha\bar{w}_1(\bar{x},\bar{T}_0,\bar{T}_1)+\cdots \tag{10}$$

where $\bar{T}_0=\bar{t}$ is a fast timescale characterizing the motion occurring at a specific natural frequency of the corresponding unperturbed system (with an invariant axial force) while $\bar{T}_1=\alpha\bar{t}$ is a slow timescale characterizing the modulation of the amplitudes and phases due to possible resonance caused by the variation of the initial axial force. Hence, it is obvious that

$$\frac{\partial}{\partial\bar{t}}=\frac{\partial}{\partial\bar{T}_0}+\alpha\frac{\partial}{\partial\bar{T}_1}+\mathrm{O}(\alpha^2) \tag{11a}$$

$$\frac{\partial^2}{\partial\bar{t}^2}=\frac{\partial^2}{\partial\bar{T}_0^2}+2\alpha\frac{\partial^2}{\partial\bar{T}_0\partial\bar{T}_1}+\mathrm{O}(\alpha^2) \tag{11b}$$

where terms of the order of $\mathrm{O}(\alpha^2)$ and higher are neglected. By use of equations (10), (11*a*) and (11*b*), and equating coefficients of like powers of the small parameter α in the resultant equations from the governing equation (4), one obtains

$$\frac{\partial^2\bar{w}_0}{\partial\bar{T}_0^2}-\bar{N}_0\frac{\partial^2\bar{w}_0}{\partial\bar{x}^2}-\tau^2\frac{\partial^4\bar{w}_0}{\partial\bar{x}^2\partial\bar{T}_0^2}+(\tau^2\bar{N}_0+1)\frac{\partial^4\bar{w}_0}{\partial\bar{x}^4}-2\tau^2\frac{\partial^6\bar{w}_0}{\partial\bar{x}^6}=0 \tag{12a}$$

$$\begin{aligned}\frac{\partial^2\bar{w}_1}{\partial\bar{T}_0^2}-\bar{N}_0\frac{\partial^2\bar{w}_1}{\partial\bar{x}^2}-\tau^2\frac{\partial^4\bar{w}_1}{\partial\bar{x}^2\partial\bar{T}_0^2}+(\tau^2\bar{N}_0+1)\frac{\partial^4\bar{w}_1}{\partial\bar{x}^4}\\-2\tau^2\frac{\partial^6\bar{w}_1}{\partial\bar{x}^6}=\bar{N}_1\cos\bar{\Omega}\bar{t}\frac{\partial^2\bar{w}_0}{\partial\bar{x}^2}-\tau^2\bar{N}_1\cos\bar{\Omega}\bar{t}\frac{\partial^4\bar{w}_0}{\partial\bar{x}^4}\\+2\tau^2\frac{\partial^4\bar{w}_0}{\partial\bar{x}^2\partial\bar{T}_0\partial\bar{T}_1}-2\frac{\partial^2\bar{w}_0}{\partial\bar{T}_0\partial\bar{T}_1}.\end{aligned} \tag{12b}$$

In the vibration theory, it is well known that, if the variable frequency $\bar{\Omega}$ is close to twice the natural frequency of unperturbed system (12*a*), the sub-harmonic resonance may occur [28], namely

$$\bar{\Omega}=2\bar{\omega}_n+\alpha\upsilon \tag{13}$$

where $\bar{\omega}_n$ is the dimensionless natural frequency for the nanobeam subjected to a constant initial axial force, υ represents a detuning parameter which quantifies the deviation of $\bar{\Omega}$ from twice the frequency $2\bar{\omega}_n$. The solution of equation (12*a*) can be approximated as

$$\bar{w}_0=A_n(\bar{T}_1)\exp(\mathrm{i}\bar{\omega}_n\bar{T}_0)\varphi_n(\bar{x})+\mathrm{cc} \tag{14}$$

where i is the imaginary unit, $n=1,2,3,\ldots$ denotes the mode number, A_n is a complex function related to $\bar{T}_1$ and cc stands for the complex conjugate of all the preceding terms on the right-hand side.

For free vibration, the natural frequency and vibration mode function of a simply supported nanobeam can be expressed as [26]

$$\varphi_n(\bar{x})=\sin(n\pi\bar{x}) \tag{15}$$

$$\bar{\omega}_n=n\pi\sqrt{\frac{\bar{N}_0+(\bar{N}_0\tau^2+1)n^2\pi^2+2\tau^2n^4\pi^4}{1+n^2\pi^2\tau^2}}. \tag{16}$$

Substituting equations (13) and (14)–(16) into equation (12*b*) and expressing the trigonometric functions in exponential form yields

$$\begin{aligned}&\frac{\partial^2\bar{w}_1}{\partial\bar{T}_0^2}-\bar{N}_0\frac{\partial^2\bar{w}_1}{\partial\bar{x}^2}-\tau^2\frac{\partial^4\bar{w}_1}{\partial\bar{x}^2\partial\bar{T}_0^2}+(\tau^2\bar{N}_0+1)\frac{\partial^4\bar{w}_1}{\partial\bar{x}^4}\\&-2\tau^2\frac{\partial^6\bar{w}_1}{\partial\bar{x}^6}=\mathrm{cc}+\mathrm{NST}+\left[2\mathrm{i}\bar{\omega}_n\frac{\mathrm{d}A_n}{\mathrm{d}\bar{T}_1}\left(\tau^2\frac{\mathrm{d}^2\varphi_n}{\mathrm{d}\bar{x}^2}-\varphi_n\right)\right.\\&\left.+\frac{\bar{N}_1}{2}\hat{A}_n\left(\frac{\mathrm{d}^2\varphi_n}{\mathrm{d}\bar{x}^2}-\tau^2\frac{\mathrm{d}^4\varphi_n}{\mathrm{d}\bar{x}^4}\right)\exp(\mathrm{i}\upsilon\bar{T}_1)\right]\exp(\mathrm{i}\bar{\omega}_n\bar{T}_0)\end{aligned} \tag{17}$$

where $\hat{A}_n$ is the complex conjugate of A_n and NST represents the terms that will not bring secular terms into the solution. Equation (17) has a bounded solution only if a solvability condition holds. The solvability condition demands the orthogonal relations as [27]

$$\begin{aligned}&\left\langle 2\mathrm{i}\bar{\omega}_n\frac{\mathrm{d}A_n}{\mathrm{d}\bar{T}_1}\left(\tau^2\frac{\mathrm{d}^2\varphi_n}{\mathrm{d}\bar{x}^2}-\varphi_n\right)+\frac{\bar{N}_1}{2}\hat{A}_n\left(\frac{\mathrm{d}^2\varphi_n}{\mathrm{d}\bar{x}^2}-\tau^2\frac{\mathrm{d}^4\varphi_n}{\mathrm{d}\bar{x}^4}\right)\right.\\&\left.\times\exp(\mathrm{i}\upsilon\bar{T}_1),\varphi_n\right\rangle=0\end{aligned} \tag{18}$$

where the inner product is defined as

$$\langle u,v\rangle=\int_0^1 u\hat{v}\,\mathrm{d}\bar{x}. \tag{19}$$

Application of the distributive law of the inner product to equation (18) yields

$$-(\tau^2n^2\pi^2+1)\left[\mathrm{i}\bar{\omega}_n\frac{\mathrm{d}A_n}{\mathrm{d}\bar{T}_1}+\frac{n^2\pi^2}{4}\bar{N}_1\hat{A}_n\exp(\mathrm{i}\upsilon\bar{T}_1)\right]=0. \tag{20}$$

3.1. Natural frequency of perturbed system

The solution of equation (20) can be determined as

$$\begin{aligned}A_n(\bar{T}_1)=&\left[(1+\mathrm{i})\exp\left(\frac{1}{4\bar{\omega}_n}\sqrt{n^4\pi^4\bar{N}_1^2-4\upsilon^2\bar{\omega}_n^2}\bar{T}_1\right)\right]\\&\times\exp(\mathrm{i}\upsilon\bar{T}_1/2).\end{aligned} \tag{21}$$

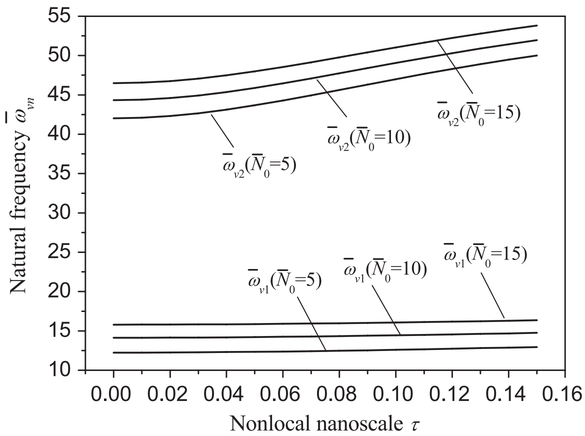

Figure 1. The first two mode natural frequencies with respect to nonlocal nanoscale and static tension for $\bar{N}_1 = 1$.

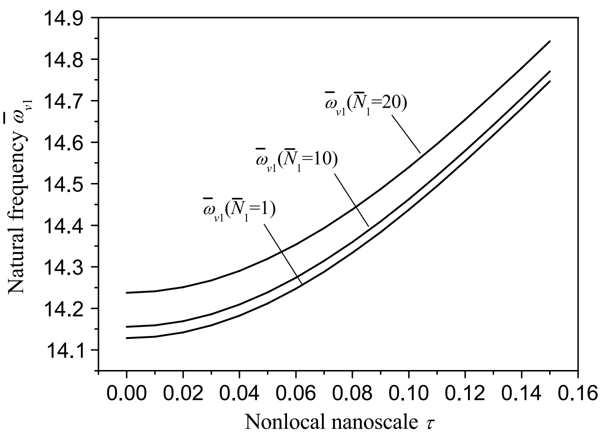

Figure 2. The first mode natural frequency with respect to nonlocal nanoscale and dynamic tension for $\bar{N}_0 = 10$.

Furthermore, a particular solution of equation (12*b*) without the secular terms may be given by

$$\bar{w}_1 = 0. \tag{22}$$

By combining equations (10), (14), (21) and (22), the natural frequency of such a perturbed system is obtained as

$$\varpi = \bar{\omega}_n + \frac{\alpha\upsilon}{2} - \mathrm{i}\frac{\alpha}{4\bar{\omega}_n}\sqrt{n^4\pi^4\bar{N}_1^2 - 4\upsilon^2\bar{\omega}_n^2}. \tag{23}$$

It is a complex frequency and the natural frequency is adopted as the extraction of the quadratic sum of the real part and imaginary part, namely

$$\bar{\omega}_{vn} = \sqrt{\bar{\omega}_n^2 + \alpha\upsilon\bar{\omega}_n + \frac{\alpha^2 n^4\pi^4\bar{N}_1^2}{16\bar{\omega}_n^2}} \tag{24}$$

where $\bar{\omega}_{vn}$ represents the n-mode dimensionless natural frequency of the nanobeam subjected to a variable tension.

Effects of nonlocal nanoscale and the axial tension on the frequency (24) are shown in figures 1–3, where $\alpha = \upsilon = 0.5$ is assumed.

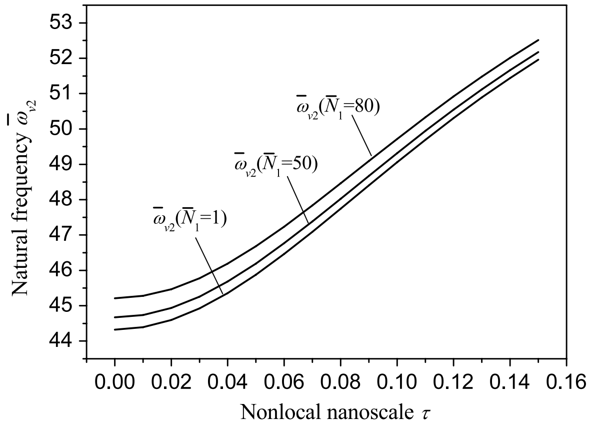

Figure 3. The second mode natural frequency with respect to nonlocal nanoscale and dynamic tension for $\bar{N}_0 = 10$.

It is observed that the natural frequencies increase with increasing nonlocal nanoscale, namely the stiffness of nanobeams increases for higher nonlocal nanoscales. The rate of increase is particularly marked for higher vibration modes. A possible explanation is that a larger nonlocal nanoscale indicates stronger intermolecular interaction constraints and thus higher stiffness. On the other hand, the natural frequencies increase with a larger initial axial tension, for both static and dynamic tensions. However, the effects of dynamic force are weaker than those of static force. Through a similar analytical procedure, it can be predicted that the vibration frequencies should decrease if an axial compression is present. Furthermore, the vibration frequencies decrease with increasing compression.

When the nanobeam is subjected to a variable axial compression, defined as $P = P_0 + \alpha P_1 \cos\Omega t$ and substituting the dimensionless compression $\bar{P} = -\bar{N}$ into equations (16) and (24) directly, one gains the relation between critical dynamic compression and nonlocal nanoscale as

$$\bar{P}_{1\mathrm{cr}} = 4\sqrt{4\left(\frac{(1-\bar{P}_0\tau^2)\pi^2+2\tau^2\pi^4-\bar{P}_0}{1+\pi^2\tau^2}\right)^2+\frac{1}{\pi}\left(\frac{(1-\bar{P}_0\tau^2)\pi^2+2\tau^2\pi^4-\bar{P}_0}{1+\pi^2\tau^2}\right)^{\frac{3}{2}}}. \tag{25}$$

The critical dynamic compression versus the nonlocal nanoscale under some different static compression is shown in figure 4, where $\alpha = \upsilon = 0.5$ is assumed. It implies that stronger nonlocal effects increase the critical dynamic compression while a larger static compression causes the critical dynamic compression to decrease.

3.2. Stability analysis

To analyze the stability of the perturbed system, we expand the complex function in equation (14) as

$$A_n(\bar{T}_1) = a_n(\bar{T}_1)\exp(\mathrm{i}\beta_n(\bar{T}_1)) \tag{26}$$

where a_n and β_n, represent, respectively, amplitude and angle of the response and they are real functions related to $\bar{T}_1$.

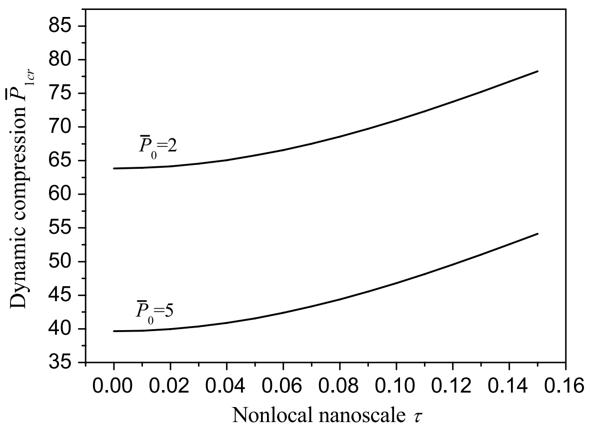

Figure 4. Effects of nonlocal nanoscale on the critical dynamic compression.

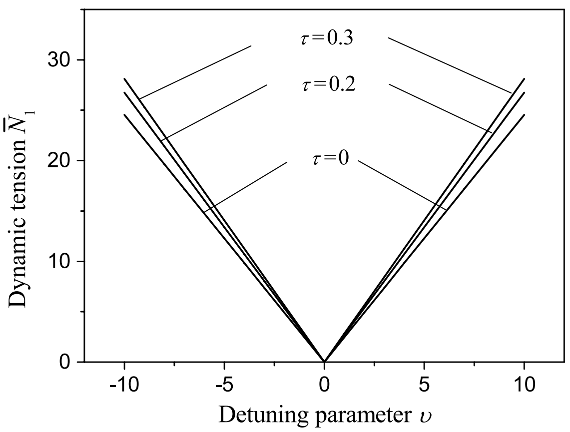

Figure 5. Effects of detuning parameter and nonlocal nanoscale on an unstable region for the first mode sub-harmonic resonance with $\bar{N}_0 = 5$.

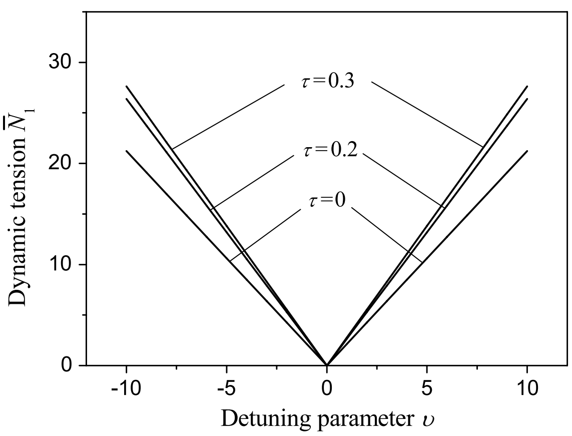

Figure 6. Effects of detuning parameter and nonlocal nanoscale on an unstable region for the second mode sub-harmonic resonance with $\bar{N}_0 = 5$.

Substituting equation (26) into (20) yields

$$\frac{\mathrm{d}a_n}{\mathrm{d}\bar{T}_1} = -\frac{\bar{N}_1}{4\bar{\omega}_n} n^2\pi^2 a_n \sin\theta_n \tag{27}$$

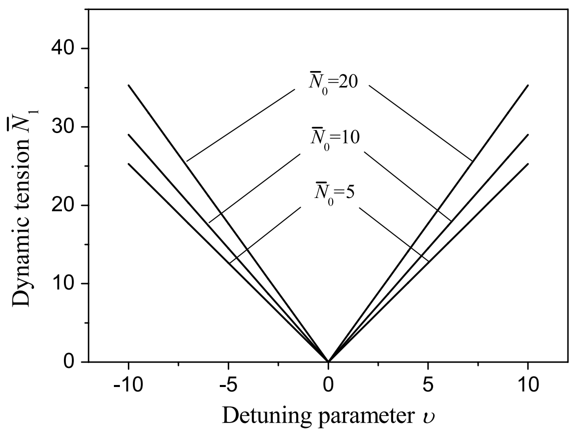

Figure 7. Effects of detuning parameter and static tension on an unstable region for the first mode sub-harmonic resonance with $\tau = 0.1$.

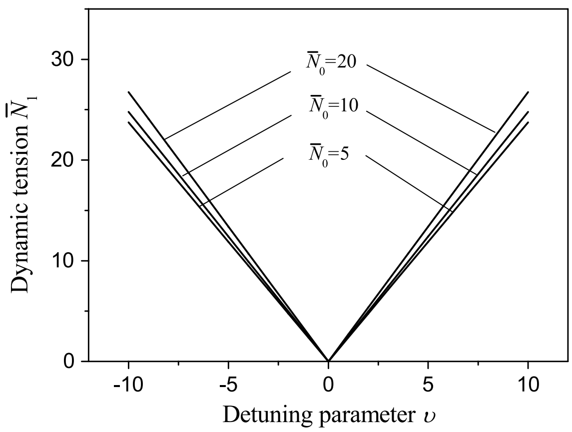

Figure 8. Effects of detuning parameter and static tension on an unstable region for the second mode sub-harmonic resonance with $\tau = 0.1$.

$$\frac{\mathrm{d}\theta_n}{\mathrm{d}\bar{T}_1} = \upsilon - \frac{\bar{N}_1}{2\bar{\omega}_n} n^2\pi^2 \cos\theta_n \tag{28}$$

where $\theta_n = \upsilon\bar{T}_1 - 2\beta_n$ is the new phase angle.

For the steady-state response, the amplitude and new phase angle in equations (27) and (28) should be constant and they result in

$$\upsilon^2 = \frac{\bar{N}_1^2}{4\bar{\omega}_n^2} n^4\pi^4. \tag{29}$$

Therefore, the unstable regions are determined by the stability boundaries as

$$\upsilon = \pm\frac{\bar{N}_1}{2\bar{\omega}_n} n^2\pi^2 \tag{30}$$

where the areas surrounded by the two boundaries are the unstable regions.

From figures 5 to 8, the unstable regions are shown to reduce with increasing nonlocal nanoscale or static tension. With respect to the nonlocal elasticity theory, it can be deduced that stronger nonlocal nanoscale effects result in larger stable regions for nanobeam structures.

In fact, the sub-harmonic resonance can be extended to summation or difference parametric resonance when the variable frequency $\bar{\Omega}$ is related to any two natural frequencies of the unperturbed system [28], namely

$$\bar{\Omega} = \bar{\omega}_m + \bar{\omega}_n + \alpha\upsilon \tag{31a}$$

$$\bar{\Omega} = \bar{\omega}_m - \bar{\omega}_n + \alpha\upsilon. \tag{31b}$$

In equation (31*a*), the summation parametric resonance may occur for $m \neq n$ while the sub-harmonic resonance is recovered for $m = n$. Similarly, equation (31*b*) may result in a different parametric resonance assuming $m > n$ without loss of generality.

Taking the summation parametric resonance as an example, the solution of equation (12*a*) may be expressed as

$$\bar{w}_0 = A_m(\bar{T}_1)\exp(\mathrm{i}\bar{\omega}_m\bar{T}_0)\varphi_m(\bar{x}) + A_n(\bar{T}_1) \times \exp(\mathrm{i}\bar{\omega}_n\bar{T}_0)\varphi_n(\bar{x}) + \mathrm{cc}. \tag{32}$$

Through a similar process, the solvability condition for summation parametric resonance is derived as

$$\left\langle 2\mathrm{i}\bar{\omega}_n\frac{\mathrm{d}A_n}{\mathrm{d}\bar{T}_1}\left(\tau^2\frac{\mathrm{d}^2\varphi_n}{\mathrm{d}\bar{x}^2} - \varphi_n\right) + \frac{\bar{N}_1}{2}\hat{A}_m\left(\frac{\mathrm{d}^2\varphi_m}{\mathrm{d}\bar{x}^2} - \tau^2\frac{\mathrm{d}^4\varphi_m}{\mathrm{d}\bar{x}^4}\right) \times \exp(\mathrm{i}\upsilon\bar{T}_1), \varphi_n\right\rangle = 0 \tag{33a}$$

$$\left\langle 2\mathrm{i}\bar{\omega}_m\frac{\mathrm{d}A_m}{\mathrm{d}\bar{T}_1}\left(\tau^2\frac{\mathrm{d}^2\varphi_m}{\mathrm{d}\bar{x}^2} - \varphi_m\right) + \frac{\bar{N}_1}{2}\hat{A}_n\left(\frac{\mathrm{d}^2\varphi_n}{\mathrm{d}\bar{x}^2} - \tau^2\frac{\mathrm{d}^4\varphi_n}{\mathrm{d}\bar{x}^4}\right) \times \exp(\mathrm{i}\upsilon\bar{T}_1), \varphi_m\right\rangle = 0. \tag{33b}$$

Thus the unstable regions can be determined similarly.

4. Conclusions

By use of nonlocal elasticity theory, the dynamics and stability of transverse vibrations of nanobeams with variable initial axial forces are presented. The nonlocal nanoscale and dimensionless axial static and dynamic forces induce significant effects on vibration frequencies and the unstable regions. Increases in nonlocal effects and axial tensile force cause the vibration frequencies to increase, which imply higher nanobeam stiffness. At the same time, they reduce the unstable regions which implies larger stable regions for the vibrating nanobeam. Additionally, an increase in axial compression results in a lower natural frequency. The critical dynamic compression is proved to increase with increasing nonlocal nanoscale while it decreases with increasing static compression. Although this paper only concerns simply supported nanobeams, the identical approach is also applicable to nanobeams with other boundary conditions. The nonlocal elasticity approach is very efficient and, to a certain extent, indispensable in the continuum approach to model, simulate and analyze the dynamics and stability of size-dependent nanostructures.

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References

[1] Wickert J A and Mote C D Jr 1988 Current research on the vibration and stability of axially-moving materials *Shock Vib. Dig.* **20** 3–13
[2] Wickert J A and Mote C D Jr 1990 Classical vibration analysis of axially moving continua *ASME J. Appl. Mech.* **57** 738–44
[3] Öz H R, Pakdemirli M and Boyaci H 2001 Non-linear vibrations and stability of an axially moving beam with time-dependent velocity *Int. J. Non-Linear Mech.* **36** 107–15
[4] Chen L Q, Zu J W and Wu J 2004 Principal resonance in transverse nonlinear parametric vibration of an axially accelerating viscoelastic string *Acta Mech. Sin.* **20** 307–16
[5] Chen L Q and Yang X 2005 Steady-state response of axially moving viscoelastic beams with pulsating speed: comparison of two nonlinear models *Int. J. Solids Struct.* **42** 37–50
[6] Ghayesh M H and Balar S 2010 Non-linear parametric vibration and stability analysis for two dynamic models of axially moving Timoshenko beams *Appl. Math. Modelling* **34** 2850–9
[7] Ukita H, Tanabe Y and Okada A 2004 Improved photothermal deflection design of an integrated microbeam for application to an external-cavity tunable laser diode *Smart Mater. Struct.* **13** 970–5
[8] Guo J G and Zhao Y P 2007 The size-dependent bending elastic properties of nanobeams with surface effects *Nanotechnology* **18** 295701
[9] Chen Y, Dick A J and Ghorbel F H 2009 Tip trajectories of a smart micro-cantilever beam: analysis and design *Smart Mater. Struct.* **18** 115012
[10] Eringen A C 1972 Nonlocal polar elastic continua *Int. J. Eng. Sci.* **10** 1–16
[11] Eringen A C 1983 On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves *J. Appl. Phys.* **54** 4703–10
[12] Peddieson J, Buchanan G G and McNitt R P 2003 Application of nonlocal continuum models to nanotechnology *Int. J. Eng. Sci.* **41** 305–12
[13] Zhao M H, Cheng G J, Liu G N and Shen Y P 1999 The analysis of crack problems with non-local elasticity *Appl. Math. Mech.* **20** 135–43
[14] Krommer M 2003 The significance of non-local constitutive relations for composite thin plates including piezoelastic layers with prescribed electric charge *Smart Mater. Struct.* **12** 318–30
[15] Wang Q and Varadan V K 2006 Vibration of carbon nanotubes studied using nonlocal continuum mechanics *Smart Mater. Struct.* **15** 659–66
[16] Lu P, Lee H P, Lu C and Zhang P Q 2006 Dynamic properties of flexural beams using a nonlocal elasticity model *J. Appl. Phys.* **99** 073510
[17] Xie G Q, Han X, Liu G R and Long S Y 2006 Effect of small size-scale on the radial buckling pressure of a simply supported multi-walled carbon nanotube *Smart Mater. Struct.* **15** 1143–9
[18] Lim C W and Wang C M 2007 Exact variational nonlocal stress modeling with asymptotic higher-order strain gradients for nanobeams *J. Appl. Phys.* **101** 054312

[19] Wang Q and Varadan V K 2007 Application of nonlocal elastic shell theory in wave propagation analysis of carbon nanotubes *Smart Mater. Struct.* **16** 178–90

[20] Hu Y G, Liew K M, Wang Q, He X Q and Yakobson B I 2008 Nonlocal shell model for elastic wave propagation in single-and double-walled carbon nanotubes *J. Mech. Phys. Solids* **56** 3475–85

[21] Lim C W 2009 Equilibrium and static deflection for bending of a nonlocal nanobeam *Adv. Vib. Eng.* **8** 277–300

[22] Lim C W, Li C and Yu J L 2010 Dynamic behaviour of axially moving nanobeams based on nonlocal elasticity approach *Acta Mech. Sin.* **26** 755–65

[23] Lim C W 2010 On the truth of nanoscale for nanobeams based on nonlocal elastic stress field theory: equilibrium, governing equation and static deflection *Appl. Math. Mech.* **31** 37–54

[24] Lim C W 2010 A nanorod (or nanotube) with lower Young's modulus is more stiff? Is not Young's modulus a stiffness indicator? *Sci. China* G **53** 712–24

[25] Lim C W and Yang Y 2010 New predictions of size-dependent nanoscale based on nonlocal elasticity for wave propagation in carbon nanotubes *J. Comput. Theor. Nanosci.* **7** 988–95

[26] Lim C W, Li C and Yu J L 2011 Analytical solutions for vibration of simply supported nonlocal nanobeams with an axial force *Int. J. Struct. Stab. Dyn.* **11** at press

[27] Nayfeh A H 1981 *Introduction to Perturbation Techniques* (New York: Wiley)

[28] Thomson W T 1981 *Theory of Vibration with Applications* (Englewood Cliffs, NJ: Prentice-Hall)