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Transverse vibration of pre-tensioned nonlocal nanobeams with precise internal axial loads

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This paper investigates the transverse vibration of a simply supported nanobeam with an initial axial tension based on the nonlocal stress field theory with a nonlocal size parameter. Considering an axial elongation due to transverse vibration, the internal axial tension is not precisely equal to the external initial tension. A sixth-order nonlinear partial differential equation that governs the transverse vibration for such nonlocal nanobeam is derived. Using a perturbation method, the relation between natural frequency and nonlocal nanoscale parameter is derived and the transverse vibration mode is solved. The external axial tension and nonlocal nanoscale parameter are proven to play significant roles in the nonlinear vibration behavior of nonlocal nanobeams. Such effects enhance the natural frequency and stiffness as compared to the predictions of the classical continuum mechanics models. Additionally, the frequency is higher if the precise internal axial load is considered with respect to that when only the approximate internal axial tension is assumed.

nonlocal stress, natural frequency, free vibration, nonlocal nanoscale

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1 Introduction

With rapid development in nanotechnology, nanobeams have great potential for wide applications as components in nano-electronic-mechanical systems. Such nanostructures have received growing interest recently. Very often these components are subjected to external loadings during work operation and their resonant properties are of much concern. As a result, nanotechnological research on free vibration properties of nanobeams under certain support conditions is important because such components can be used as design components in nano-sensors and nano-actuators.

At nanoscale, the mechanical characteristics of nanostructures are often significantly different from their behavior at macroscopic scale due to the inherent size effects. Such effects are essential for nanoscale materials or structures and the influence on nano-instruments is great [1]. Size effects exist not only for mechanical properties but also for electronic, optical and some other fields [2]. Generally, theoretical studies on size effects at nanoscale are by means of surface effects [3, 4], strain gradients in elasticity [5] and plasticity [6], as well as nonlocal stress field theory [7, 8], etc. Although surface effects can be neglected at macroscale, the effects could play significant roles at nanoscale because of the very high surface-to-volume ratio [9, 10]. Strain gradient theories for elasticity and plasticity have been proposed in various studies following the works of Mindlin [5]

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and Aifantis [6]. The strain gradient theory consists of two groups including the higher-order and lower-order theories. In the higher-order theory, higher-order stresses are defined to be the work-conjugate to strain gradient, thus leading to the necessity of using higher-order governing equations and boundary conditions.

The nonlocal elasticity theory was first developed by Eringen [7] and it assumes that the stress tensor at a point is a function of strains at all points in the continuum. It is different from the classical continuum theory because the latter is based on constitutive relation which states that the stress at a point is a function of strain at that particular point. This nonlocal theory is proved to be in accordance with atomic model of lattice dynamics and with experimental observations on phonon dispersion [8]. In nonlocal theory, the nonlocal nanoscale in the constitutive equation could be considered simply as a material-dependent parameter. The ratio of internal characteristic scale (such as lattice parameter, C-C bond length, granular distance, etc.) to external characteristic scale (such as crack length, wave length, etc.) is defined within a nonlocal nanoscale parameter. If the internal characteristic scale is much smaller than the external characteristic scale, the nonlocal nanoscale parameter approaches zero and the classical continuum theory is recovered. Otherwise, the nonlocal nanoscale parameter is finite and the corresponding solutions differ greatly from the classical solutions. For nanostructures, the effects of nonlocal nanoscale parameter on the mechanical behaviors are significant because the external characteristic scale is relatively at the same order of magnitude as the internal characteristic scale.

At present, the nonlocal elasticity theory has been used extensively to study lattice dispersion of elastic waves, wave propagation in composites, dislocation mechanics, static deflection, damage and fracture mechanics, surface tension fluids, piezoelectric materials, etc. [11–15]. The mechanical or thermal properties of nanostructures such as carbon nanotube, nanowire, nanoswitch, nanofilm, nanobeam, nanoplate, etc. are sought [11–14, 16–19].

In the application of nonlocal elasticity models for nanostructure, Lim [20, 21] recently showed that the classical governing equations and equilibrium equations cannot be directly applied to nonlocal stress models even with the relevant quantities replaced by the corresponding nonlocal quantities. He further derived via the variational principle new nonlocal stress models and an effective nonlocal bending moment is derived as an infinite series of higher-order nonlocal bending moments. The analysis were applied to bending [20, 21], vibration [22] and wave propagation [23] for nanobeams. The conclusions reached were contradictory to those based on other nonlocal approaches [11–14, 16, 17].

Based on the previous conclusion that an effective nonlocal bending moment, instead of the nonlocal bending moment, should replace the classical bending moment in the equilibrium condition, an exact nonlocal stress model for a simply supported nanobeam subject to an external axial tension is formulated in this paper. Subsequently, the model is applied using perturbation analysis to free vibration study of a nanobeam with an internal axial tension. Here, the internal axial load is not approximated as equal to the external (initial) axial load in the presence of axial elongation. The precise internal axial load is derived with an additional term due to axial elongation induced by transverse vibration. With the consideration of nonlocal effects, the free vibration frequencies of nanostructures are expected to be higher than the corresponding classical continuum solutions. The solutions disagree with the other published results which predict lower free vibration frequency. Numerical examples for a silicon beam are presented and new results discussed. It is concluded that an external axial tension and the nonlocal nanoscale parameter play significant roles in the free vibration of a simply supported nanobeam.

2 New nonlocal elasticity model

Consider a nanobeam with an external (initial) axial tension and simply supported at both ends, as shown in Figure 1, where the axial and transverse directions are denoted by x and y coordinates, respectively. The equilibrium analysis for an element of the nanobeam is illustrated in Figure 1, where M_{ef} is the effective nonlocal bending moment [20, 21], N the internal axial tension, V the shear force and θ the bending angle with respect to the x-axis.

Here, the Euler-Bernoulli beam model is adopted and shear deformation is thus ignored. With reference to Figure 1, the element equilibrium equation with respect to the y -axis based on the D' Alembert principle is as follows:

$$V-\left(V+\frac{\partial V}{\partial x}\mathrm{d}x\right)+N\theta-N\left(\theta+\frac{\partial \theta}{\partial x}\mathrm{d}x\right)-\rho A\frac{\partial^2 w}{\partial t^2}\mathrm{d}x=0, \quad (1)$$

where ρ is the mass density, w the transverse displacement, A the cross sectional area, and t the time. From the moment

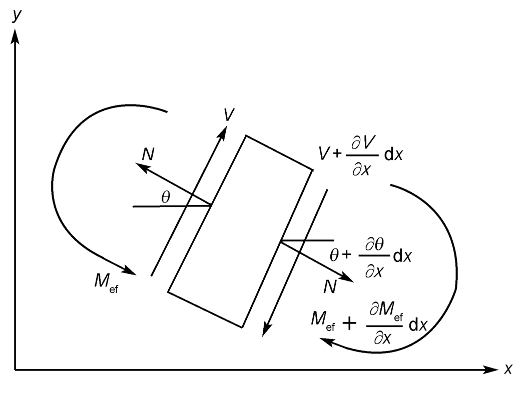

Figure 1 Force equilibrium of an element of the nanobeam.

balance condition, the following condition can be derived

$$\frac{\partial^2 M_{\mathrm{ef}}}{\partial x^2}+N\frac{\partial^2 w}{\partial x^2}-\rho A\frac{\partial^2 w}{\partial t^2}=0, \tag{2}$$

where $V=-\mathrm{d}M_{\mathrm{ef}}/\mathrm{d}x$ and $\theta=-\partial w/\partial x$.

For transverse free vibration, the transverse deformation yields an axial elongation. Thus, the internal axial tension can be expressed as

$$N=P+\sigma_{\mathrm{ef}}A, \tag{3}$$

where P is the external axial tension, σ_{ef} is the effective axial normal stress which is given by [24]

$$\sigma_{\mathrm{ef}}=\sigma-2\sum_{n=1}^{\infty}(e_0a)^{2n}\frac{\mathrm{d}^{2n}\sigma}{\mathrm{d}x^{2n}}, \tag{4}$$

and σ is the nonlocal stress given by

$$\sigma=E\sum_{n=1}^{\infty}(e_0a)^{2(n-1)}\frac{\mathrm{d}^{2(n-1)}\varepsilon}{\mathrm{d}x^{2(n-1)}}, \tag{5}$$

where E is Young's modulus, e_0 is a nonlocal constant dependent on specific material, and a is the internal characteristic length scale. Note that $\sigma_{\mathrm{ef}}A$ in eq. (3) represents the force due to transverse deformation. The normal strain can be deduced from an elastic deformation analysis as

$$\varepsilon=\frac{\int_0^L \Delta_{\mathrm{d}x}\mathrm{d}x}{L}=\frac{1}{L}\int_0^L\left[\sqrt{1+\left(\frac{\partial w}{\partial x}\right)^2}-1\right]\mathrm{d}x, \tag{6}$$

where L is the length of nanobeam and $\Delta_{\mathrm{d}x}$ the deformation of unit length dx. Applying Taylor's expansion to eq. (6) and substituting eqs. (4)–(6) into eq. (3) yield

$$N=P+\frac{EA}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x, \tag{7}$$

where terms of order $O[(e_0a)^2]$ are retained while other higher-order terms are neglected. Substituting eq. (7) into eq. (2), one obtains

$$\frac{\partial^2 M_{\mathrm{ef}}}{\partial x^2}+\left[P+\frac{EA}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right]\frac{\partial^2 w}{\partial x^2}-\rho A\frac{\partial^2 w}{\partial t^2}=0. \tag{8}$$

Based on the variational principle, the exact relationship between bending moment and transverse displacement based on nonlocal stress theory is given by [20, 21]

$$M_{\mathrm{ef}}=-EI\frac{\partial^2 w}{\partial x^2}+EI\sum_{n=1}^{\infty}(2n-1)(e_0a)^{2n}\frac{\partial^{2(n+1)}w}{\partial x^{2(n+1)}}, \tag{9}$$

where I is the second moment of area.

From eqs. (8) and (9), the following nonlinear partial differential equation governing the transverse vibration can be obtained:

$$-EI\frac{\partial^4 w}{\partial x^4}+EI\sum_{n=1}^{\infty}(2n-1)(e_0a)^{2n}\frac{\partial^{2(n+2)}w}{\partial x^{2(n+2)}}-\rho A\frac{\partial^2 w}{\partial t^2}+\left[P+\frac{EA}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right]\frac{\partial^2 w}{\partial x^2}=0. \tag{10}$$

It is obvious to observe in eq. (10) the difference between the nonlocal and classical continuum models. The transverse free vibration of a beam or tube based on classical continuum theory is described by a well-known fourth-order differential equation which can be recovered by dropping the nonlinear terms (axial elongation induced by transverse deformation) and the nonlocal terms containing e_0a in eq. (10). According to some previous nonlocal models [11–14, 16, 17], a fourth-order governing equation was presented. However, the governing equation derived from the variational principle turns out to be an infinite higher-order differential equation in terms of e_0a. The major difference between this new nonlocal model and the previous nonlocal models was highlighted by Lim [20, 21].

Because it is almost impossible to solve the infinite higher-order differential equation, only terms of order $O[(e_0a)^2]$ and lower in the infinite series in eq. (10) are retained. This term is the most significant term which contains nonlocal effect in the governing equation. Hence, eq. (10) becomes a sixth-order partial differential-integral equation as

$$EI\left[-\frac{\partial^4 w}{\partial x^4}+(e_0a)^2\frac{\partial^6 w}{\partial x^6}\right]-\rho A\frac{\partial^2 w}{\partial t^2}+\left[P+\frac{EA}{2L}\int_0^L\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right]\frac{\partial^2 w}{\partial x^2}=0. \tag{11}$$

For transverse free vibration of a simply supported nanobeam considered here, the transverse displacement can be assumed as

$$w(x,t)=W(t)\sin\frac{n\pi x}{L}, \tag{12}$$

where $n=1,2,3,\cdots$ and $W(t)$ indicates the time-dependent vibration mode. Substituting eq. (12) into eq. (11), one has the governing equation

$$-\frac{EIn^4\pi^4W}{L^4}\left[1+(e_0a)^2\frac{n^2\pi^2}{L^2}\right]-\left(P+\frac{EAn^2\pi^2W^2}{4L^2}\right)\frac{n^2\pi^2W}{L^2}-\rho A\frac{\mathrm{d}^2W}{\mathrm{d}t^2}=0. \tag{13}$$

Introduce the non-dimensional variables and parameters as follows:

$$\overline{W}=W\sqrt{\frac{A}{I}},\ \overline{t}=t\sqrt{\frac{EI}{\rho AL^4}},\ \overline{P}=\frac{PL^2}{EI},\ \tau=\frac{e_0a}{L}, \tag{14}$$

where τ is the non-dimensional nonlocal nanoscale parameter. By using these dimensionless variables and parameters, eq. (13) can be cast into the dimensionless form as follows:

$$\frac{\mathrm{d}^2\overline{W}}{\mathrm{d}\overline{t}^2}+\overline{\omega}_n^2\overline{W}+\alpha\overline{W}^3=0, \tag{15}$$

where

$$\overline{\omega}_n=n\pi\sqrt{\overline{P}+n^4\pi^4\tau^2+n^2\pi^2}, \tag{16a}$$

and

$$\alpha=n^4\pi^4/4, \tag{16b}$$

where n=1,2,3,$\cdots$. The dimensionless frequency $\overline{\omega}_n$ in eq. (16a) is related to the dimensional frequency ω_n via

$$\overline{\omega}_n=\omega_n\sqrt{\frac{\rho AL^4}{EI}}. \tag{17a}$$

Because $\alpha\overline{W}^3$ in eq. (15) is originated from $\sigma_{\mathrm{ef}}A$ in eq. (3) it can be considered as a small parameter compared with P. Thus the internal axial force N fluctuates with respect to P by a small force $\sigma_{\mathrm{ef}}A$ due to additional extension caused by transverse deformation. Eq. (15) is a Duffing-type nonlinear equation with dimensionless oscillation frequency $\overline{\Omega}_n$ defined as

$$\overline{\Omega}_n=\Omega_n\sqrt{\frac{\rho AL^4}{EI}}. \tag{17b}$$

If $\sigma_{\mathrm{ef}}A$ is neglected, N=P and the nonlinear term in eq. (15) disappears. The exact frequency in this case is $\overline{\Omega}_n=\overline{\omega}_n=n\pi\sqrt{\left(\overline{P}+n^4\pi^4\tau^2+n^2\pi^2\right)}$. On the other hand, if the nanobeam is subjected into a compression P_{com}, the linear natural frequency is given by $n\pi\sqrt{-\overline{P}_{\mathrm{com}}+n^4\pi^4\tau^2+n^2\pi^2}$. Thus, the initial compression should satisfy

$$\overline{P}_{\mathrm{com}}\leqslant n^2\pi^2\left(n^2\pi^2\tau^2+1\right), \tag{18}$$

and the critical compression, or the critical buckling load, can be approximated as

$$\overline{P}_{\mathrm{cr}}\approx\pi^2\left(\pi^2\tau^2+1\right). \tag{19}$$

Eq. (19) implies that the nonlocal nanoscale effects tend to increase the critical compression for buckling.

3 Solution and discussion

Since eq. (15) is a standard Duffing-type equation [25], its solution can be obtained via a number of approximate methods such as perturbation [26], harmonic balance [25], Newton-harmonic balance [27], Lindstedt–Poincaré method [28], multiple scales [29], etc. According to the method of perturbation analysis, the second-order approximation for eq. (15) can be expressed as [26]

$$\overline{W}\approx a_0\cos\varphi+\frac{\alpha a_0^3}{32\overline{\omega}_n^2}\cos3\varphi-\frac{21\alpha^2a_0^5}{1024\overline{\omega}_n^4}\cos3\varphi+\frac{\alpha^2a_0^5}{1024\overline{\omega}_n^4}\cos5\varphi, \tag{20}$$

where

$$\varphi\approx\left(\overline{\omega}_n+\frac{3\alpha a_0^2}{8\overline{\omega}_n}-\frac{15\alpha^2a_0^4}{256\overline{\omega}_n^3}\right)\overline{t}+\theta_0, \tag{21}$$

where a_0 and θ_0 are integral constants which can be determined from the initial conditions. The frequency can be approximated as

$$\overline{\Omega}_n\approx\overline{\omega}_n+\frac{3\alpha a_0^2}{8\overline{\omega}_n}-\frac{15\alpha^2a_0^4}{256\overline{\omega}_n^3}. \tag{22}$$

The fundamental frequency, n=1, based on the new nonlocal stress model is shown in Figure 2, where numerical solutions for $\overline{\omega}_1$ without nonlinear effect from eq. (16a) and $\overline{\Omega}_1$ with nonlinear effect from eq. (22) are presented, and a_0 =0.5 is assumed. It is seen that the presence of a nonlocal nanoscale τ increases the natural frequency. The result here is in contradiction to the conclusion of many previous studies [11–14, 16, 17]. The critical differences have been fully described in detail by Lim [20, 21]. In brief, as mentioned in the introduction, the disagreement is due to the adoption of a partial nonlocal model in the previous works, in which the classical governing equations of motion were directly applied with relevant classical quantities replaced by the corresponding nonlocal quantities. Such direct replacement is invalid because the governing equation obtained is a mixture of classical and nonlocal models, thus

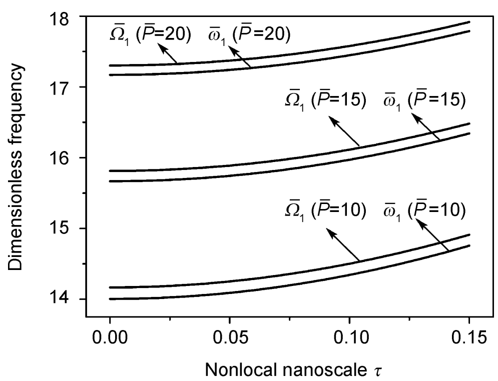

Figure 2 Comparison of approximate natural frequencies for $\overline{\omega}_1$ without nonlinear effect and $\overline{\Omega}_1$ with nonlinear effect for increasing nanoscale τ, a_0=0.5.

they are termed partial nonlocal models. The new conclusion is consistent with the outcome of other non-nonlocal approaches, e.g. Ma et al. [30].

It is also observed from Figure 2 that a larger external tension $\bar{P}$ causes higher $\bar{\omega}_1$ and $\bar{\Omega}_1$ as expected. Moreover, considering precise internal axial tension enhances the stiffness, i.e., nonlinear effects cause a further increase of frequency and stiffness, as can be observed by comparing $\bar{\omega}_1$ with $\bar{\Omega}_1$.

The dimensional natural frequency of a simply supported beam with external axial tension based on the classical continuum theory was given by Thomson [29] and Guede and Elishakoff [31] as

$$(\omega_n)_{\text{cla}} = n\pi\sqrt{\frac{n^2\pi^2 EI}{\rho AL^4}+\frac{P}{\rho AL^2}},\quad (n=1,2,3,\cdots). \tag{23}$$

Considering eqs. (17b) and (22), the dimensional form of nonlocal frequency with nonlinear effect can be derived.

Taking a simply supported silicon beam subjected to external axial tension as an example, the numerical solutions for nonlinear nonlocal (solid curves) and linear classical or local (dotted curves) models are shown in Figures 3 and 4.

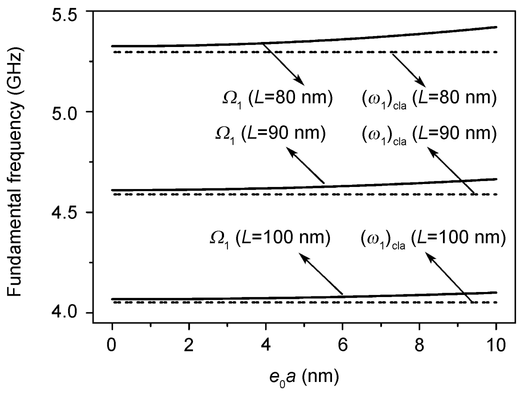

Figure 3 The effect of e_0a on the fundamental frequencies ω_1 and Ω_1 for a silicon nanobeam.

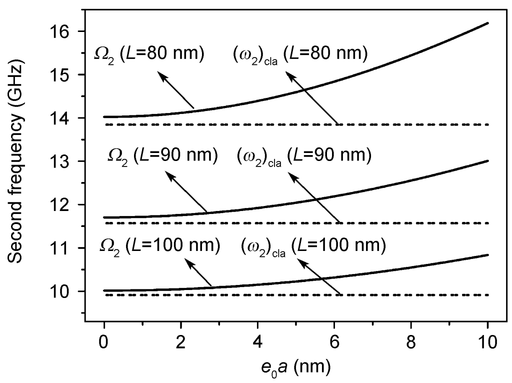

Figure 4 The effect of e_0a on the second model frequencies ω_2 and Ω_2 for a silicon nanobeam.

The cross section of beam is assumed rectangular (a circular cross section is equally valid) with width 3 nm, thickness 1 nm, A=3×10^{-18} m^2, and I=2.5×10^{-37} m^4. Note that the Young's modulus is size-dependent at nanoscale. In this example, E=80.23 GPa which is dependent on the thickness [32], and other parameters are ρ=2400 kg/m^3, P=10^{-4} μN and a_0=0.5.

In these figures, $(\omega_n)_{\text{cla}}$ are the classical continuum results which have no nonlocal effect and the values are not affect by e_0a. Values for Ω_n are higher than $(\omega_n)_{\text{cla}}$ and the difference is much larger for a smaller L or a higher e_0a. The difference cannot be ignored especially for beam at nanoscale. For instance, for L=80 nm and e_0a=5 nm, $(\omega_n)_{\text{cla}}$ is 1.0% and 5.4% lower than Ω_n for the fundamental (n=1) and the second mode (n=2) frequencies, respectively.

The nonlocal model presented in this study can be applied to vibration of carbon nanotubes. Consider another example: a single-walled carbon nanotube with inner and outer radii 0.8 and 1.134 nm, respectively, I=3.11×10^{-37} m^4, A=2.03×10^{-18} m^2, ρ=1300 kg/m^3, P=10^{-4} μN and e_0a=1 nm. Note that Young's modulus varies for nanotubes with different lengths while it tends to be unchanged for sufficiently long nanotubes. In this example, E=1.35 TPa is adopted for the length which exceeds 50 nm [33], where a_0=0.5. Table 1 shows numerical solutions for the first two mode frequencies for the classical, linear nonlocal and nonlinear nonlocal models, respectively.

It is observed in Table 1 that $(\omega_n)_{\text{cla}}$, ω_n and Ω_n decrease with increasing L. In addition, the nonlocal frequencies are higher than the classical frequencies, of which Ω_n is the highest. The difference is particularly significant for a small L. For example, for L=50 nm, $(\omega_1)_{\text{cla}}$ is 0.18% and 2.36% lower than ω_1 and Ω_1, respectively, and for the second mode, the differences are 0.77% and 3.04%, respectively. Furthermore, the nonlinear effect causes the fundamental and second mode frequencies to increase by 2.18% and 2.25%, respectively, with respect to the linear nonlocal frequency for L=50 nm. These differences should be taken into consideration in analyzing and designing nanoscale components for NEMS. With increasing length, the difference becomes

Table 1 Comparison of classical $(\omega_n)_{\text{cla}}$, linear nonlocal ω_n and nonlinear nonlocal Ω_n frequencies for a single-walled carbon nanotube

Frequency (GHz)	Nanotube length L (nm)							
	50	60	70	80	90	100	500	800
$(\omega_1)_{\text{cla}}$	51.28	36.05	26.87	20.90	16.80	13.87	1.32	0.79
ω_1	51.37	36.10	26.89	20.91	16.81	13.88	1.32	0.79
Ω_1	52.49	36.87	27.45	21.33	17.14	14.14	1.32	0.79
$(\omega_2)_{\text{cla}}$	200.68	139.81	103.11	79.29	62.96	51.28	3.15	1.72
ω_2	202.23	140.56	103.52	79.53	63.11	51.37	3.15	1.72
Ω_2	206.79	143.72	105.83	81.30	64.50	52.49	3.18	1.72

smaller and the frequencies have a good agreement.

To analyze vibration of nonlocal nanotubes, it is necessary to determine the internal scale parameter e_0a and this can be done by comparing with some other nanomechanical studies, such as the scanning force microscopy study of Garcia-Sanchez et al. [33]. In the study, a nanotube with L=193 nm, outer radius R_1=1.03 nm, inner radius R_2=0.7 nm, density of graphite ρ=2200 kg/m^3, Young's modulus E= 1.03 TPa was investigated. The first mode frequency was reported as $f_1=\Omega_1/2\pi$=573 MHz. Therefore, the internal scale parameter is matched as e_0a=0.96 nm. If the internal characteristic scale is assumed as the crystal lattice parameter of carbon, i.e., a=0.142 nm, the material constant is obtained as e_0=6.76 in this example.

The axial displacement which varies with time is shown next. The non-dimensional form of eq. (12) is given by

$$\bar{w}(\bar{x},\bar{t}) = \bar{W}(\bar{t})\sin n\pi\bar{x}, \tag{24}$$

where $\bar{w} = w/L$ is the dimensionless transverse displacement and $\bar{x} = x/L$ is the dimensionless coordinate. Applying eq. (20), we obtain the time-dependent nonlinear transverse vibration from eq. (24). The relations of dimensionless displacement, coordinate and time are shown in Figures 5 and 6, for the first and the second modes, respectively, where a_0=0.5, θ_0=0.5, τ=0.1 and $\bar{P}=10$ are assumed.

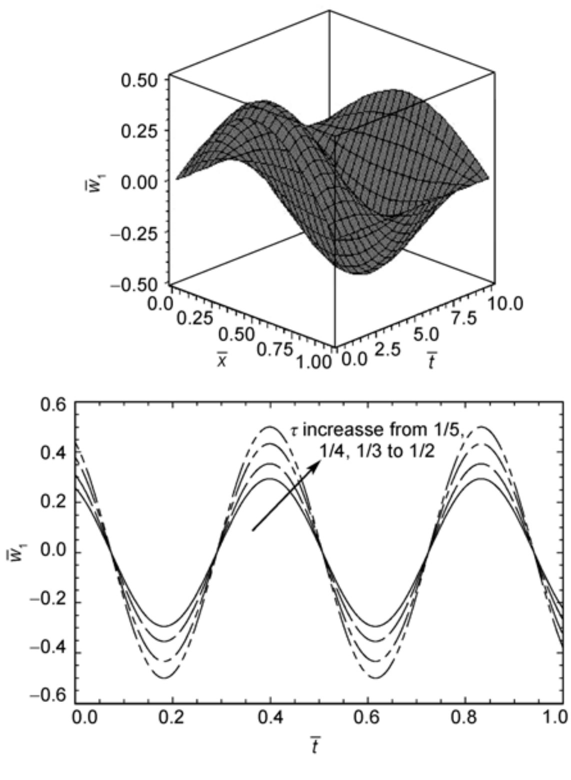

Figure 5 The first vibration mode.

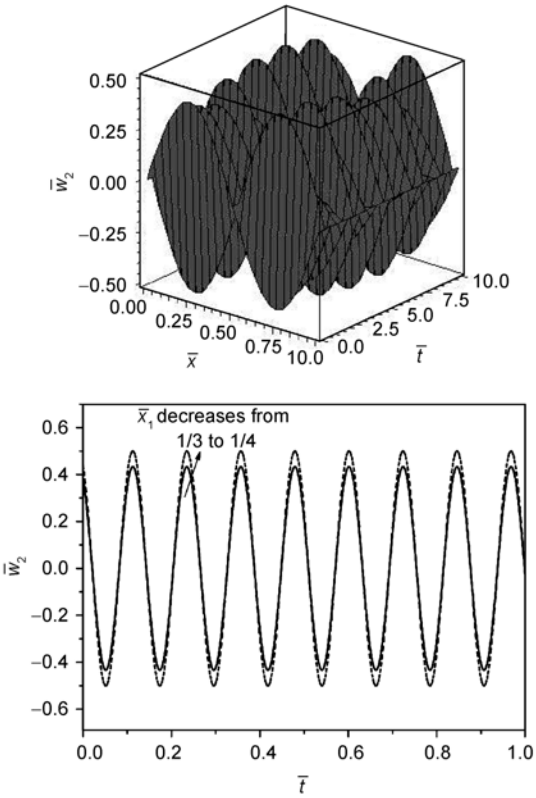

Figure 6 The second vibration mode.

4 Conclusions

Two major points are concluded in this work. Firstly, the nonlocal effects enhance natural frequency of a nanobeam based on the new nonlocal stress model presented here, or the nanobeam stiffness is strengthened in comparison with the classical continuum theory. This is a new conclusion because the previous nonlocal nanobeam studies concluded with reduced stiffness and thus lower vibration frequencies. Secondly, the precise internal axial tension is not equal to the external tension. The nonlinear vibration is equivalent to a Duffing-type oscillation and the consideration of nonlinearity results in higher natural frequencies. External axial tension also has significant influence on the vibration behaviors and the frequency is higher for a larger tension. For a nanobeam with sufficient length, the nonlocal nanoscale effect becomes insignificant and thus the governing equation can be reduced to the classical beam vibration equation. Both nonlocal and classical solutions are in good agreement, and validity of the nonlocal model developed here for vanishing nonlocal effect is established.

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