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TWISTING STATICS AND DYNAMICS FOR CIRCULAR ELASTIC NANOSOLIDS BY NONLOCAL ELASTICITY THEORY**

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ABSTRACT The torsional static and dynamic behaviors of circular nanosolids such as nanoshafts, nanorods and nanotubes are established based on a new nonlocal elastic stress field theory. Based on a new expression for strain energy with a nonlocal nanoscale parameter, new higher-order governing equations and the corresponding boundary conditions are first derived here via the variational principle because the classical equilibrium conditions and/or equations of motion cannot be directly applied to nonlocal nanostructures even if the stress and moment quantities are replaced by the corresponding nonlocal quantities. The static twist and torsional vibration of circular, nonlocal nanosolids are solved and discussed in detail. A comparison of the conventional and new nonlocal models is also presented for a fully fixed nanosolid, where a lower-order governing equation and reduced stiffness are found in the conventional model while the new model reports opposite solutions. Analytical solutions and numerical examples based on the new nonlocal stress theory demonstrate that nonlocal stress enhances stiffness of nanosolids, i.e. the angular displacement decreases with the increasing nonlocal nanoscale while the natural frequency increases with the increasing nonlocal nanoscale.

KEY WORDS angular displacement, nanoscale, nonlocal stress, torsion, vibration

I. INTRODUCTION

Mechanical behaviors of materials or structures at nanoscale are important in the designs of nano-electronic-mechanical-systems (NEMS). Comparing with macro materials and structures, some effects which are not observed at macro-sizes appear at nanoscale, such as size-dependent phenomena. Because of such effects, the classical continuum mechanics fails at nanoscale. For example, according to the classical continuum theory, the stress is singular at a crack tip despite the weak external load. It is strange and cannot be explained from physics because each material has limited fatigue strength. In fact, atomic simulation and experiments have proved nonsingularity of stress at the crack tip[1].

Currently, three main approaches are used to investigate nanomechanics: experiment, molecular dynamic (MD) simulation and the continuum theory. Due to the complexity of instruments, equipment and technology, precise experiments at nanoscale are extremely difficult to be conducted. As MD simu-

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lation considers each individual molecule and its multiple mechanical or chemical web-interactions, its use requires extremely fast computing facilities and thence it is largely confined to relatively restricted systems with a limited number of molecules. That is why many researchers focus on the new continuum theory in nanomechanics. To analytically investigate the size-dependent effect, nonlocal theory[2], surface/interface stress theory[3,4] and strain gradient theory[5] have been proposed. The nonlocal stress field theory states that the stress at a point in a domain is dependent not only on the classical local strain at that particular point, but also on the spatial integrals that represent the weighted averages of the local strain contribution of all other points in the domain. Eringen's nonlocal elasticity[2] allows one to account for the small scale effect that becomes significant when dealing with nanostructures. This theory suits for the nanomechanics. For instance, stress at the crack tip is nonsingularity via nonlocal theory[1]. Surface effect[3,4] is present due to the high surface-to-bulk ratio at nanoscale, and the mechanical properties of atoms at and near a surface differ significantly from those of the internal atoms because of reduced coordination. The similar situation occurs at an interface between two dissimilar media. Investigation of the effects of surface and interface stresses on the mechanical responses of thin films and multilayers has been developing probably since the work by Stoney[3]. A very recent work of Wang et al.[6] reviewed the advances in surface stress effect in mechanical behaviors of nanostructures. The strain gradient theory[5] consists of two groups, the higher-order and lower-order sub-theories. For example, in the higher-order theory, higher-order stresses are defined to be the work-conjugate to strain gradient, thus leading to the necessity of using higher-order governing equations and boundary conditions. These approaches are all available in nanostructures and in fact, some work was presented based on different methods. For example, Maranganti and Sharma[7] characterized the mechanical property of the nanostructured materials by using the surface stress model and nonlocal stress theory. They explained the length scales at which nonlocal effects manifest for different materials. In this study, the nonlocal stress theory is adopted to investigate the torsional behavior of nanosolids.

Because it is mathematically difficult to obtain the solutions of nonlocal problems due to the spatial integrals in the nonlocal relations proposed originally[2], Eringen[8] established an equivalent differential constitutive equation within a two-dimensional region and under certain conditions using Green's function with a certain approximation error in the year 1983. Subsequently, there were many studies on nonlocal field[9–25] and most of these studies focused on carbon nanotubes (CNTs)[11,12,20–22,25], nanobeams[10,13,15,17,18,23,24] and nanoplates[16]. The main research interests include static behaviors such as bending, buckling, and dynamic behaviors such as free vibration, wave propagation, etc. The previous studies[10–13,15,16] adopted the classical equilibrium conditions and/or equations of motion for nonlocal nanostructures and directly replaced the stress and moment quantities with the corresponding nonlocal quantities. These conventional models are termed the partial nonlocal models by Lim[17–19] and they are questionable because they predict intriguing solutions with respect to the new nonlocal stress model and solutions for bending, buckling and vibration of nanobeams and tensile behavior of nanostructures. For the partial nonlocal model, three critical issues for nanobeam solutions overlooked in the previous studies were identified[17,18]. Yang and Lim[25] further made a convincing comparison with the new, partial nonlocal models and the MD simulation. The conclusion by Yang and Lim[25] is MD and the new nonlocal model showed stiffness enhancement effect while the partial nonlocal model showed otherwise. According to the new nonlocal model, a new effective nonlocal bending moment is defined which can be expressed as an infinite series in terms of the nonlocal bending moment and the nonlocal nanoscale. This exact nonlocal model began to receive considerable attention recently[20–25].

Although there has been research for transverse bending or vibration of nonlocal nanostructures[10–13,15–18,20–25], very limited studies on torsional behaviors are available at present. Torsional deformation and vibration are easily seen in NEMS or some other nano-devices and a new torsional model that considers the true nonlocal effect is necessary. In this paper, a new torsion model is proposed via the variational principle and the angular deformation and free vibration of nanosolids are analyzed. Conclusions of the new nonlocal model are consistent with the solutions for transverse bending, vibration of nanobeams and CNTs presented by Lim et al.[17–25]. In addition, this paper also attempts to make a comparison with the conventional partial nonlocal model, where the nonlocal stress and nonlocal moment quantities are directly substituted into the classical equilibrium equation. Intriguing solutions are concluded that nanostructural stiffness is reduced due to increasing nanoscale size effects. The

intriguing conclusions could be removed if the partial nonlocal model is replaced by a new exact nonlocal stress model as what is presented in this paper.

II. EXACT NONLOCAL SHEAR STRESS MODEL FOR TORSION

For a circular nanosolid with radius R, length L and at a point r from the center, the nonlocal and classical shear stresses s_{xr} and s'_{xr}, respectively, are related through a differential constitutive equation

$$s_{xr} - (e_0 a)^2 \frac{\mathrm{d}^2 s_{xr}}{\mathrm{d}x^2} = s'_{xr} \tag{1}$$

where e_0 is a constant dependent on material, a is an internal characteristic length and x is the coordinate along the nanosolid. Further considering the relation between the classical shear stress s'_{xr} and the classical shear strain γ where $s'_{xr} = G\gamma = Gr\dfrac{\mathrm{d}\theta}{\mathrm{d}x}$, we obtain

$$\bar{s}_{xr} - \tau^2 \frac{\mathrm{d}^2 \bar{s}_{xr}}{\mathrm{d}\bar{x}^2} = \bar{r}\frac{\mathrm{d}\theta}{\mathrm{d}\bar{x}} \tag{2}$$

where $\bar{s}_{xr} = s_{xr}/G$, $\bar{x} = x/L$, $\bar{r} = r/L$, nonlocal nanoscale $\tau = e_0 a/L$, angular displacement θ, and shear modulus G. This is a second-order ordinary differential equation and the general solution can be expressed as

$$\bar{s}_{xr} = \sum_{n=1}^{\infty} \tau^{2n-2} \bar{r} \frac{\mathrm{d}^{2n-1}\theta}{\mathrm{d}\bar{x}^{2n-1}} = \sum_{n=1}^{\infty} \tau^{2n-2} \left(\bar{r}\frac{\mathrm{d}\theta}{\mathrm{d}\bar{x}}\right)^{\langle 2n-2\rangle} = \sum_{n=1}^{\infty} \tau^{2n-2} \gamma^{\langle 2n-2\rangle} \tag{3}$$

where $()^{\langle n\rangle}$ represents the n-order derivative with respect to $\bar{x}$. The strain energy density can be written as

$$u = \int_0^{\gamma} s_{xr}\mathrm{d}\gamma = G\sum_{n=1}^{\infty} \tau^{2n-2} \int_0^{\gamma} \gamma^{\langle 2n-2\rangle}\mathrm{d}\gamma \tag{4}$$

Variation of the total strain energy is given by

$$\begin{aligned}
\delta U = \frac{GJ}{L}\int_0^1 \sum_{n=0}^{\infty} (2n-1)\,\tau^{2n}\theta^{\langle 2n+2\rangle}\delta\theta\,\mathrm{d}\bar{x} + \frac{GJ}{L}\Bigg[& \left(-\sum_{n=0}^{\infty}(2n-1)\,\tau^{2n}\theta^{\langle 2n+1\rangle}\right)\delta\theta \\
& + \left(\sum_{n=1}^{\infty}(2n-1)\,\tau^{2n}\theta^{\langle 2n\rangle}\right)\delta\theta^{\langle 1\rangle} + \left(-\sum_{n=1}^{\infty} 2n\tau^{2n+2}\theta^{<2n+1>}\right)\delta\theta^{\langle 2\rangle} \\
& + \left(\sum_{n=1}^{\infty}(2n-1)\,\tau^{2n+2}\theta^{<2n>}\right)\delta\theta^{\langle 3\rangle} + \left(-\sum_{n=1}^{\infty} 2n\tau^{2n+4}\theta^{<2n+1>}\right)\delta\theta^{\langle 4\rangle} \\
& + \left(\sum_{n=1}^{\infty}(2n-1)\,\tau^{2n+4}\theta^{<2n>}\right)\delta\theta^{\langle 5\rangle} + \left(-\sum_{n=1}^{\infty} 2n\tau^{2n+6}\theta^{<2n+1>}\right)\delta\theta^{\langle 6\rangle} + \cdots\Bigg]_0^1
\end{aligned} \tag{5}$$

where $J = \int_A r^2 \mathrm{d}A$ is the polar moment of inertia over the cross-sectional area A.

III. TORSIONAL BEHAVIOR ANALYSIS

In this section, torsional statics and dynamics are presented and discussed. Various boundary conditions are considered in the analysis of dynamic behaviors. In addition, results of the partial nonlocal model, which has been existing extensively, is also presented for comparison.

3.1. Static Twist Analysis

For nanosolids, the twisting moment per unit length is T_0, variation of the work done by the moment is

$$\delta W_{T_0} = \frac{GJ}{L}\bar{T}_0 \int_0^1 \delta\theta \mathrm{d}\bar{x} \tag{6}$$

where $\bar{T}_0 = T_0 L^2/(GJ)$. For static equilibrium, the variational principle requires that

$$\delta\left(U - W_{T_0}\right) = 0 \tag{7}$$

From Eqs.(5), (6) and (7), one obtains an equilibrium equation. Since $\delta\theta$ in the energy function does not vanish, we have

$$\sum_{n=0}^{\infty}\left(2n-1\right)\tau^{2n}\theta^{\langle 2n+2\rangle} = \bar{T}_0 \tag{8}$$

and the following higher-order boundary conditions

$$\left.\begin{array}{lll} -\sum\limits_{n=0}^{\infty}\left(2n-1\right)\tau^{2n}\theta^{\langle 2n+1\rangle} = 0 & \text{or} & \theta = 0 \\ \sum\limits_{n=1}^{\infty}\left(2n-1\right)\tau^{2n}\theta^{\langle 2n\rangle} = 0 & \text{or} & \theta^{\langle 1\rangle} = 0 \\ -\sum\limits_{n=1}^{\infty} 2n\tau^{2n+2}\theta^{<2n+1>} = 0 & \text{or} & \theta^{\langle 2\rangle} = 0 \\ \sum\limits_{n=1}^{\infty}\left(2n-1\right)\tau^{2n+2}\theta^{<2n>} = 0 & \text{or} & \theta^{\langle 3\rangle} = 0 \\ \vdots & \text{or} & \vdots \end{array}\right\} \text{at } \bar{x} = 0,1 \tag{9}$$

Considering the most significant nonlocal terms in Eqs.(8) and (9) where $n = 1$, the governing equation for a nanosolid fixed at both ends is

$$\tau^2\theta^{\langle 4\rangle} - \theta^{\langle 2\rangle} = \bar{T}_0 \tag{10}$$

and the boundary conditions for a fully fixed nanosolid without angular displacement and higher-order stress are

$$\begin{array}{l} \theta\left(0\right) = 0, \quad \theta^{\langle 2\rangle}\left(0\right) + 3\tau^2\theta^{\langle 4\rangle}\left(0\right) = 0 \\ \theta\left(1\right) = 0, \quad \theta^{\langle 2\rangle}\left(1\right) + 3\tau^2\theta^{\langle 4\rangle}\left(1\right) = 0 \end{array} \tag{11}$$

Solving Eq.(10) using the boundary conditions in Eqs.(11) yields the angular displacement as

$$\theta\left(\bar{x}\right) = \frac{\bar{T}_0\tau^2}{4\left(1+\mathrm{e}^{1/\tau}\right)}\mathrm{e}^{\bar{x}/\tau} + \frac{\bar{T}_0\tau^2\mathrm{e}^{1/\tau}}{4\left(1+\mathrm{e}^{1/\tau}\right)}\mathrm{e}^{-\bar{x}/\tau} - \frac{\bar{T}_0}{2}\bar{x}^2 + \frac{\bar{T}_0}{2}\bar{x} - \frac{\bar{T}_0\tau^2}{4} \tag{12}$$

The classical torsional solution[26] is recovered in the limit when $\tau \to 0$. The effect of τ on the maximum angular displacement at mid-point $\bar{x} = 0.5$ is illustrated in Fig.1. It is observed that the angular displacement decreases with increasing τ. Therefore, the classical continuum theory overestimates the angular displacement at nanoscale and some new continuum theories, such as the nonlocal stress theory, are necessary.

On the other hand, the normalized angular rotation displacement with respect to the maximum classical solution is

$$\frac{\theta}{\left(\theta_{\max}\right)_{\text{classical}}} = \frac{2\left(1+\mathrm{e}^{1/\tau}\right)\left(2\bar{x}-2\bar{x}^2-\tau^2\right) + 2\tau^2\left(\mathrm{e}^{\bar{x}/\tau}+\mathrm{e}^{1/\tau}\mathrm{e}^{-\bar{x}/\tau}\right)}{1+\mathrm{e}^{1/\tau}} \tag{13}$$

A comparison of normalized classical and nonlocal angular rotations is shown in Fig.2. The significant effect of nonlocal nanoscale which yields as much as 5% reduction in angular displacement for $\tau = 0.15$ is observed.

3.2. Torsional Vibration Analysis

The nonlocal torsional dynamics of nanosolids is presented and two kinds of higher-order boundary conditions are taken into account. For comparison, a partial nonlocal model for torsional analysis is also proposed.

For free vibration of a nanosolid, the kinetic energy for one complete vibration cycle is given by

$$\int_0^{P_e} E_k\,\mathrm{d}t = \int_0^{P_e}\int_0^{L}\frac{1}{2}\rho\pi R^2\left(R\frac{\partial\theta}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t = \frac{\pi R^4 L\rho}{2P_e}\int_0^1\int_0^1\left(\frac{\partial\theta}{\partial\bar{t}}\right)^2\mathrm{d}\bar{x}\,\mathrm{d}\bar{t} \tag{14}$$

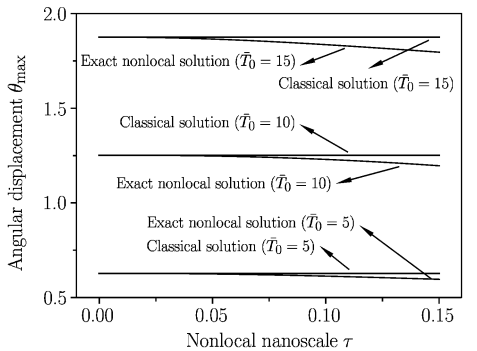

Fig. 1. The maximum angular displacement for varying nonlocal nanoscale.

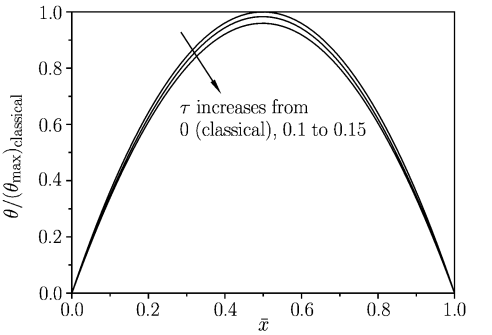

Fig. 2. Reduction of normalized angular displacement due to nonlocal effect.

where ρ is mass density, t is time and $\bar{t} = t/P_e$ is the dimensionless time, in which P_e is a characteristic time. Variation of the kinetic energy yields

$$\delta \int_0^{P_e} E_k \,\mathrm{d}t = \frac{\pi R^4 L \rho}{P_e} \int_0^1 \int_0^1 \frac{\partial \theta}{\partial \bar{t}} \delta \left(\frac{\partial \theta}{\partial \bar{t}} \right) \mathrm{d}\bar{x}\,\mathrm{d}\bar{t} = \frac{\pi R^4 L \rho}{P_e} \left[\int_0^1 \frac{\partial \theta}{\partial \bar{t}} \delta\theta \Big|_{\bar{t}=0}^{\bar{t}=1} \mathrm{d}\bar{x} - \int_0^1 \int_0^1 \frac{\partial^2 \theta}{\partial \bar{t}^2} \delta\theta \,\mathrm{d}\bar{x}\,\mathrm{d}\bar{t} \right] \tag{15}$$

For equilibrium, the variational principle requires that

$$\delta \int_0^{P_e} (U - E_k) \,\mathrm{d}t = 0 \tag{16}$$

Considering

$$\int_0^1 \frac{\partial \theta}{\partial \bar{t}} \delta\theta \Big|_{\bar{t}=0}^{\bar{t}=1} \mathrm{d}\bar{x} = 0 \tag{17}$$

because the motion is periodic and at $\bar{t} = 0$ and $\bar{t} = 1$, the quantity within the integration have the same values, we have from Eq.(16)

$$\begin{aligned} 0 = {} & \frac{GJP_e}{L} \int_0^1 \int_0^1 \left(\sum_{n=0}^{\infty} (2n-1)\, \tau^{2n} \theta^{\langle 2n+2 \rangle} + \bar{\rho} \frac{\partial^2 \theta}{\partial \bar{t}^2} \right) \delta\theta \,\mathrm{d}\bar{x}\,\mathrm{d}\bar{t} \\ & + \frac{GJP_e}{L} \left[\left(- \sum_{n=0}^{\infty} (2n-1)\, \tau^{2n} \theta^{\langle 2n+1 \rangle} \right) \delta\theta \right. \\ & + \left(\sum_{n=1}^{\infty} (2n-1)\, \tau^{2n} \theta^{\langle 2n \rangle} \right) \delta\theta^{\langle 1 \rangle} + \left(- \sum_{n=1}^{\infty} 2n \tau^{2n+2} \theta^{<2n+1>} \right) \delta\theta^{\langle 2 \rangle} \\ & + \left(\sum_{n=1}^{\infty} (2n-1)\, \tau^{2n+2} \theta^{<2n>} \right) \delta\theta^{\langle 3 \rangle} + \left(- \sum_{n=1}^{\infty} 2n \tau^{2n+4} \theta^{<2n+1>} \right) \delta\theta^{\langle 4 \rangle} \\ & \left. + \left(\sum_{n=1}^{\infty} (2n-1)\, \tau^{2n+4} \theta^{<2n>} \right) \delta\theta^{\langle 5 \rangle} + \left(- \sum_{n=1}^{\infty} 2n \tau^{2n+6} \theta^{<2n+1>} \right) \delta\theta^{\langle 6 \rangle} + \cdots \right]_{\bar{x}=0}^{\bar{x}=1} \mathrm{d}\bar{t} \end{aligned} \tag{18}$$

where $\bar{\rho} = \dfrac{\pi R^4 L^2}{GJP_e^2} \rho$ is the dimensionless mass density. Because $\delta\theta$ do not vanish, we have

$$\sum_{n=0}^{\infty} (2n-1)\, \tau^{2n} \theta^{\langle 2n+2 \rangle} + \bar{\rho} \frac{\partial^2 \theta}{\partial \bar{t}^2} = 0 \tag{19}$$

and the following boundary conditions

$$\left.\begin{array}{lll} -\sum\limits_{n=0}^{\infty}(2n-1)\,\tau^{2n}\theta^{\langle 2n+1\rangle}=0 & \text{or} & \theta=0 \\ \sum\limits_{n=1}^{\infty}(2n-1)\,\tau^{2n}\theta^{\langle 2n\rangle}=0 & \text{or} & \theta^{\langle 1\rangle}=0 \\ -\sum\limits_{n=1}^{\infty}2n\tau^{2n+2}\theta^{<2n+1>}=0 & \text{or} & \theta^{\langle 2\rangle}=0 \\ \sum\limits_{n=1}^{\infty}(2n-1)\,\tau^{2n+2}\theta^{<2n>}=0 & \text{or} & \theta^{\langle 3\rangle}=0 \\ \vdots & \text{or} & \vdots \end{array}\right\}\ \text{at } \bar{x}=0,1 \tag{20}$$

Again, it is found that the governing equation (19) and boundary conditions (20) contain higher-order terms. Note that the governing equation (19) is reduced to the classical torsional vibration case when the nonlocal nanoscale vanishes[26].

The simplified governing equation and boundary conditions considering only the most significant nonlocal term $n=1$ in Eqs.(19) and (20) are given by, respectively

$$\tau^2\theta^{\langle 4\rangle}-\theta^{\langle 2\rangle}+\bar{\rho}\frac{\partial^2\theta}{\partial\bar{t}^2}=0 \tag{21}$$

$$\left.\begin{array}{lll} \theta^{\langle 1\rangle}-\tau^2\theta^{\langle 3\rangle}=0 & \text{or} & \theta=0 \\ \theta^{\langle 2\rangle}+3\tau^2\theta^{\langle 4\rangle}=0 & \text{or} & \theta^{\langle 1\rangle}=0 \end{array}\right\}\ \text{at } \bar{x}=0,1 \tag{22}$$

For linear torsional vibration of a nanosolid, the modes are harmonic in time. Hence the time-dependent angular rotation can be represented by

$$\theta(\bar{x},\bar{t})=\Theta(\bar{x})\,\mathrm{e}^{\mathrm{i}\bar{\omega}_n\bar{t}} \tag{23}$$

where $\mathrm{i}=\sqrt{-1}$, Θ is the time-independent angular amplitude, $\bar{\omega}_n=\omega_n P_e$ $(n=1,2,3\cdots)$ is the dimensionless natural frequency in which ω_n is the dimensional frequency.

Substituting Eq.(23) into Eqs.(21) and (22) yields

$$-\Theta^{\langle 2\rangle}+\tau^2\Theta^{\langle 4\rangle}-\bar{\rho}\bar{\omega}_n^2\Theta=0 \tag{24}$$

and the higher-order boundary conditions are

$$\left.\begin{array}{lll} \Theta^{\langle 1\rangle}-\tau^2\Theta^{\langle 3\rangle}=0 & \text{or} & \Theta=0 \\ \Theta^{\langle 2\rangle}+3\tau^2\Theta^{\langle 4\rangle}=0 & \text{or} & \Theta^{\langle 1\rangle}=0 \end{array}\right\}\ \text{at } \bar{x}=0,1 \tag{25}$$

Since Eq.(24) is a fourth-order ordinary differential equation, its solution can be written as

$$\Theta_n(\bar{x})=C_{1n}\mathrm{e}^{\mathrm{i}\lambda_{1n}\bar{x}}+C_{2n}\mathrm{e}^{\mathrm{i}\lambda_{2n}\bar{x}}+C_{3n}\mathrm{e}^{\mathrm{i}\lambda_{3n}\bar{x}}+C_{4n}\mathrm{e}^{\mathrm{i}\lambda_{4n}\bar{x}} \tag{26}$$

where C_{jn} $(j=1,2,3,4)$ are coefficients and λ_{jn} $(j=1,2,3,4)$ are the four roots of the dispersion relation in Eq.(24), or

$$\tau^2\lambda_n^4+\lambda_n^2-\bar{\rho}\bar{\omega}_n^2=0 \tag{27}$$

Equation (27) can also be obtained by substituting $\Theta_n(\bar{x})=C_n\mathrm{e}^{\mathrm{i}\lambda_n\bar{x}}$ into Eq.(24), where C_n is a nonzero constant.

3.2.1. Fully fixed nanosolids

The boundary conditions for a fully fixed nanosolid are given by

$$\begin{array}{ll} \Theta(0)=0, & \Theta^{\langle 2\rangle}(0)+3\tau^2\Theta^{\langle 4\rangle}(0)=0 \\ \Theta(1)=0, & \Theta^{\langle 2\rangle}(1)+3\tau^2\Theta^{\langle 4\rangle}(1)=0 \end{array} \tag{28}$$

Substituting Eq.(26) into Eq.(28) yields

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \mathrm{e}^{\mathrm{i}\lambda_{1n}} & \mathrm{e}^{\mathrm{i}\lambda_{2n}} & \mathrm{e}^{\mathrm{i}\lambda_{3n}} & \mathrm{e}^{\mathrm{i}\lambda_{4n}} \\ k_{1n} & k_{2n} & k_{3n} & k_{4n} \\ k_{1n}\mathrm{e}^{\mathrm{i}\lambda_{1n}} & k_{2n}\mathrm{e}^{\mathrm{i}\lambda_{2n}} & k_{3n}\mathrm{e}^{\mathrm{i}\lambda_{3n}} & k_{4n}\mathrm{e}^{\mathrm{i}\lambda_{4n}} \end{bmatrix} \begin{Bmatrix} C_{1n} \\ C_{2n} \\ C_{3n} \\ C_{4n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{29}$$

where $k_{jn} = -\lambda_{jn}^2 + 3\tau^2\lambda_{jn}^4$, $(j = 1, 2, 3, 4)$. To obtain a nonzero solution of C_{jn}, the coefficient determinant of the matrix should be zero, or

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \mathrm{e}^{\mathrm{i}\lambda_{1n}} & \mathrm{e}^{\mathrm{i}\lambda_{2n}} & \mathrm{e}^{\mathrm{i}\lambda_{3n}} & \mathrm{e}^{\mathrm{i}\lambda_{4n}} \\ k_{1n} & k_{2n} & k_{3n} & k_{4n} \\ k_{1n}\mathrm{e}^{\mathrm{i}\lambda_{1n}} & k_{2n}\mathrm{e}^{\mathrm{i}\lambda_{2n}} & k_{3n}\mathrm{e}^{\mathrm{i}\lambda_{3n}} & k_{4n}\mathrm{e}^{\mathrm{i}\lambda_{4n}} \end{vmatrix} = 0 \tag{30}$$

From Eqs.(27) and (30), the relation between dimensionless natural frequency and nonlocal nanoscale is thus derived. Figures 3 and 4 show the first three mode dimensionless natural frequencies containing the nonlocal effect, where the dotted curves are nonlocal results while solid lines represent the corresponding classical solutions, which can be obtained by dropping the nonlocal terms containing τ in Eqs.(27) and (30). It is observed that the nonlocal nanoscale enhances natural frequency, especially the higher-mode frequencies. The classical solution undervalues the torsional vibration frequency and it fails to predict the free vibration behavior of a nanosolid. It also demonstrates that higher dimensionless density causes lower natural frequency from Figs.3 and 4.

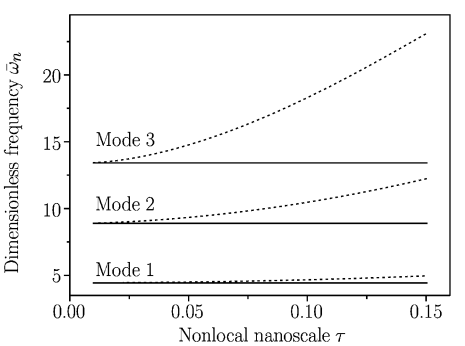

Fig. 3. Effects of nonlocal nanoscale on the first three mode dimensionless frequencies of fully fixed nanosolids for $\bar{\rho} = 0.5$, where the dotted curves and solid lines are the nonlocal and classical results, respectively.

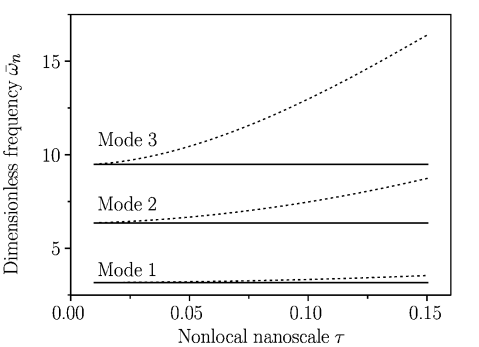

Fig. 4. Effects of nonlocal nanoscale on the first three mode dimensionless frequencies of fully fixed nanosolids for $\bar{\rho} = 1.0$, where the dotted curves and solid lines are the nonlocal and classical results, respectively.

For comparison and to illustrate the major differences, the conventional partial nonlocal theory for torsional analysis of a nanosolid is derived. Firstly, the correlation between nonlocal twisting moment and relative angular rotation is obtained from Eq.(1), as

$$T - (e_0 a)^2 \frac{\partial^2 T}{\partial x^2} = GJ\frac{\partial \theta}{\partial x} \tag{31}$$

where $T = \int\int_A r s_{xr} \mathrm{d}A$ is the nonlocal twisting moment. The conventional nonlocal theory combines the nonlocal relation and classical equilibrium. Subsequently, force equilibrium is analysis applied to an element $\mathrm{d}x$ of the nanosolid, and the equilibrium equation is derived based on D' Alembert principle as

$$\rho J\frac{\partial^2 \theta}{\partial t^2}\mathrm{d}x = \frac{\partial T}{\partial x}\mathrm{d}x \tag{32}$$

Therefore, the conventional nonlocal governing equation of torsional vibration is derived from Eqs.(31) and (32) as

$$\rho\frac{\partial^2 \theta}{\partial t^2} - (e_0 a)^2 \rho\frac{\partial^4 \theta}{\partial x^2 \partial t^2} = G\frac{\partial^2 \theta}{\partial x^2} \tag{33}$$

It is clearly seen that the governing equation (33) is a lower-order equation comparing to that of the new nonlocal stress theory in Eq.(19). Following Eq.(33), one obtains the partial nonlocal effects on natural frequency of a fully fixed nanosolid as shown in Fig.5.

Therefore, the partial (conventional) nonlocal effects cause natural frequency to decrease and the nonlocal frequencies are lower than the corresponding classical solutions. In other words, the partial nonlocal model results in lower stiffness for nanostructures. However, opposite conclusion has been found in the analysis above where a new nonlocal shear stress model was employed.

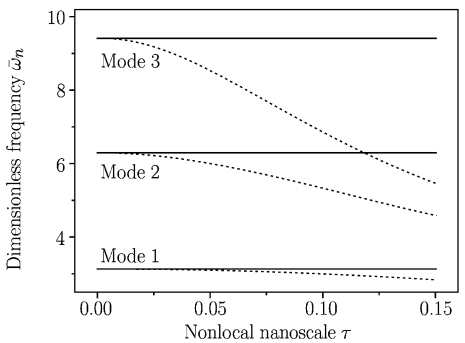

Fig. 5. Effects of nonlocal nanoscale (conventional nonlocal model) on the first three mode natural frequencies for a fully fixed nanosolid, where solid and dotted curves are the classical and nonlocal results, respectively.

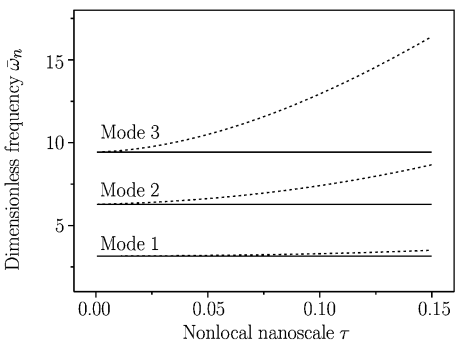

Fig. 6. Effects of nonlocal nanoscale on the first three mode frequencies for completely free nanosolids for $\bar{\rho} = 2.0$, where the dotted curves and solid lines are nonlocal and classical results, respectively.

3.2.2. Completely free nanosolids

Similarly, for a completely free nanosolid without end torsion but with higher-order angular condition, the simplified boundary conditions from Eq.(25) are

$$\begin{aligned} &\Theta^{\langle 3\rangle}(0) = 0, \quad \Theta^{\langle 1\rangle}(0) = 0 \\ &\Theta^{\langle 3\rangle}(1) = 0, \quad \Theta^{\langle 1\rangle}(1) = 0 \end{aligned} \tag{34}$$

Following the same procedure, the relation between $\bar{\omega}_n$ and τ can be obtained. Figure 6 shows the nonlocal effect on the first three mode natural frequencies for $\bar{\rho} = 2.0$. The solid lines represent the corresponding classical solutions[26].

$$\omega_{cn} = \frac{n\pi}{L}\sqrt{\frac{G}{\rho}} \quad (n = 1, 2, 3, \cdots) \tag{35}$$

or in non-dimensional form as

$$\bar{\omega}_{cn} = \frac{n\pi}{L}\sqrt{\frac{G}{\rho}}P_e = \frac{n\pi}{L}\sqrt{\frac{G}{\rho}}\sqrt{\frac{\pi R^4 L^2 \rho}{2GJ}} = n\pi \quad (n = 1, 2, 3, \cdots) \tag{36}$$

where $\bar{\rho} = 2.0$ and $J = \pi R^4/2$ for a circular nanosolid. Again, it is observed that τ enhances the natural frequency especially for higher-mode frequency. Note that the classical solutions (solid lines) in Fig.6, as given in Eq.(36), are identical to the nonlocal solutions when $\tau = 0$. This example proves again the validity and correctness of the nonlocal shear stress model presented in this work.

3.2.3. Torsional vibration of nanosolids subjected to an initial twisting moment

Considering a nanosolid with an initial twisting moment per unit length T_0, the variational principle requires

$$\delta \int_0^{P_e} (U + W_{T_0} - E_k)\,\mathrm{d}t = 0 \tag{37}$$

By substituting Eqs.(5), (6) and (15) into Eq.(37), one obtains

$$
\begin{aligned}
0 = {} & \frac{GJP_e}{L}\int_0^1\int_0^1\left(\sum_{n=0}^{\infty}(2n-1)\,\tau^{2n}\theta^{\langle 2n+2\rangle}+\bar{T}_0+\bar{\rho}\frac{\partial^2\theta}{\partial\bar{t}^2}\right)\delta\theta\,\mathrm{d}\bar{x}\mathrm{d}\bar{t}\\
& +\frac{GJP_e}{L}\left[\left(-\sum_{n=0}^{\infty}(2n-1)\,\tau^{2n}\theta^{\langle 2n+1\rangle}\right)\delta\theta\right.\\
& +\left(\sum_{n=1}^{\infty}(2n-1)\,\tau^{2n}\theta^{\langle 2n\rangle}\right)\delta\theta^{\langle 1\rangle}+\left(-\sum_{n=1}^{\infty}2n\tau^{2n+2}\theta^{<2n+1>}\right)\delta\theta^{\langle 2\rangle}\\
& +\left(\sum_{n=1}^{\infty}(2n-1)\,\tau^{2n+2}\theta^{<2n>}\right)\delta\theta^{\langle 3\rangle}+\left(-\sum_{n=1}^{\infty}2n\tau^{2n+4}\theta^{<2n+1>}\right)\delta\theta^{\langle 4\rangle}\\
& \left.+\left(\sum_{n=1}^{\infty}(2n-1)\,\tau^{2n+4}\theta^{<2n>}\right)\delta\theta^{\langle 5\rangle}+\left(-\sum_{n=1}^{\infty}2n\tau^{2n+6}\theta^{<2n+1>}\right)\delta\theta^{\langle 6\rangle}+\cdots\right]_{\bar{x}=0}^{\bar{x}=1}\mathrm{d}\bar{t}
\end{aligned}
\tag{38}
$$

which results in the higher-order governing equation as

$$
\sum_{n=0}^{\infty}(2n-1)\,\tau^{2n}\theta^{\langle 2n+2\rangle}+\bar{\rho}\frac{\partial^2\theta}{\partial\bar{t}^2}+\bar{T}_0=0 \tag{39}
$$

The simplified governing equation which considers the most significant nonlocal term in Eq.(39) is given by

$$
\tau^2\theta^{\langle 4\rangle}-\theta^{\langle 2\rangle}+\bar{\rho}\frac{\partial^2\theta}{\partial\bar{t}^2}+\bar{T}_0=0 \tag{40}
$$

Using Eq.(23), the simplified governing equation becomes

$$
\tau^2\Theta^{\langle 4\rangle}-\Theta^{\langle 2\rangle}-\bar{\rho}\bar{\omega}_n^2\Theta+\bar{T}_0=0 \tag{41}
$$

and the dispersion relation is

$$
\tau^2\lambda_n^4+\lambda_n^2-\bar{\rho}\bar{\omega}_n^2+\bar{T}_0=0 \tag{42}
$$

Taking a nanosolid with completely free ends as an example, the boundary conditions are shown in Eq.(34). Following the same procedure, the nonlocal nanoscale effect on $\bar{\omega}_n$ is shown in Figs.7 and 8.

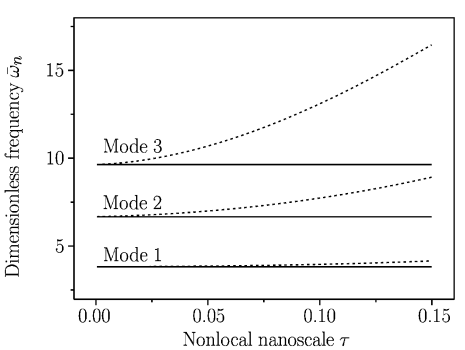

Fig. 7. Effects of nonlocal nanoscale on the first three mode $\bar{\omega}_n$ for completely free nanosolids subjected to an initial twisting moment for $\bar{\rho}=2.0$ and $\bar{T}_0=5$. The dotted curves and solid lines are nonlocal and classical results, respectively.

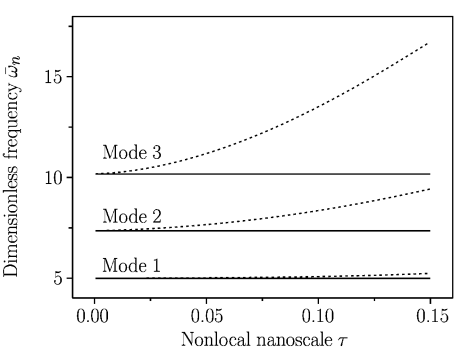

Fig. 8. Effects of nonlocal nanoscale on the first three mode $\bar{\omega}_n$ for completely free nanosolids subjected to an initial twisting moment for $\bar{\rho}=2.0$ and $\bar{T}_0=15$. The dotted curves and solid lines are nonlocal and classical results, respectively.

IV. CONCLUSIONS

The torsional static deformation and free vibration of nanosolids are investigated in this paper. Unlike the conventional nonlocal models, the new nonlocal stress theory reflects the exact nonlocal effects and it is expected to provide better understanding than the partial (conventional) nonlocal model. A nonlocal shear stress model for torsion is derived first and a new higher-order governing equation with the corresponding higher-order non-classical boundary conditions is established by means of the variational and Hamilton's principle. The nonlocal nanoscale is found to enhance the stiffness of nanostructures, thence it yields reduced angular displacement and increased natural frequency for nanosolids. The conclusion is consistent with other studies for transverse bending behavior based on the new nonlocal stress theory. The classical solutions are recovered in the limit of vanishing nonlocal nanoscale and the validity of the new nonlocal stress model is validated.

References

[1] Eringen,A.C. and Kim,B.S., Stress concentration at the tip of the crack. *Mechanics Research Communications*, 1974, 1(4): 233-237.

[2] Eringen,A.C. and Edelen,D.G.B., On nonlocal elasticity. *International Journal of Engineering Science*, 1972, 10(3): 233-248.

[3] Stoney,G.C., The tension of metallic films deposited by electrolysis. *Proceeding of Royal Society of London*, 1909, A82: 172-175.

[4] Cammarata,R.C., Surface and interface stress effects in thin films. *Progress in Surface Science*, 1994, 46(1): 1-38.

[5] Gao,H., Huang,Y., Nix,W.D. and Hutchinson,J.W., Mechanism-based strain gradient plasticity—I theory. *Journal of the Mechanics and Physics of Solids*, 1999, 47(6): 1239-1263.

[6] Wang,J.X., Huang,Z.P., Duan,H.L., Yu,S.W., Feng,X.Q., Wang,G.F., Zhang,W.X. and Wang,T.J., Surface stress effect in mechanics of nanostructured materials. *Acta Mechanica Solida Sinica*, 2011, 24(1): 52-82.

[7] Maranganti,R. and Sharma,P., Length scales at which classical elasticity breaks down for various materials. *Physical Review Letters*, 2007, 98(19): 195504.

[8] Eringen,A.C., On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *Journal of Applied Physics*, 1983, 54(9): 4703-4710.

[9] Zhou,Z.G., Wang,B. and Sun,Y.G., Analysis of the dynamic behavior of a Griffith permeable crack in piezoelectric materials with the non-local theory. *Acta Mechanica Solida Sinica*, 2003, 16(1): 52-60.

[10] Peddieson,J., Buchanan,G.G. and McNitt,R.P., Application of nonlocal continuum models to nanotechnology. *International Journal of Engineering Science*, 2003, 41(3-5): 305-312.

[11] He,X.Q., Kitipornchai,S. and Liew,K.M., Buckling analysis of multi-walled carbon nanotubes: a continuum model accounting for van der Waals interaction. *Journal of the Mechanics and Physics of Solids*, 2005, 53(2): 303-326.

[12] Wang,Q. and Varadan,V.K., Vibration of carbon nanotubes studied using nonlocal continuum mechanics. *Smart Materials and Structures*, 2006, 15(2): 659-666.

[13] Lu,P., Lee,H.P., Lu,C. and Zhang,P.Q., Dynamic properties of flexural beams using a nonlocal elasticity model. *Journal of Applied Physics*, 2006, 99(7): 073510.

[14] Liang,J., Scattering of harmonic anti-plane shear stress waves by a crack in functionally graded piezoelectric/piezomagnetic materials. *Acta Mechanica Solida Sinica*, 2007, 20(1): 75-86.

[15] Wang,C.M., Zhang.Y.Y. and Kitipornchai,S., Vibration of initially stressed micro- and nano-beams. *International Journal of Structural Stability and Dynamics*, 2007, 7(4): 555-570.

[16] Duan,W.H. and Wang,C.M., Exact solutions for axisymmetric bending of micro/nanoscale circular plates based on nonlocal plate theory. *Nanotechnology*, 2007, 18(38): 385704.

[17] Lim,C.W., Equilibrium and static deflection for bending of a nonlocal nanobeam. *Advances in Vibration Engineering*, 2009, 8(4): 277-300.

[18] Lim,C.W., On the truth of nanoscale for nanobeams based on nonlocal elastic stress field theory: equilibrium, governing equation and static deflection. *Applied Mathmatics and Mechanics*, 2010, 31(1): 37-54.

[19] Lim,C.W., A nanorod (or nanotube) with lower Young's modulus is stiffer? Is not Young's modulus a stiffness indicator? *Science in China Series G: Physics, Mechanics & Astronomy*, 2010, 53(4): 712-724.

[20] Lim,C.W. and Yang,Y., New predictions of size-dependent nanoscale based on nonlocal elasticity for wave propagation in carbon nanotubes. *Journal of Computational and Theoretical of Nanoscience*, 2010, 7(6): 988-995.

[21] Lim,C.W. and Yang,Y., Wave propagation in carbon nanotubes: nonlocal elasticity-induced stiffness and velocity enhancement effects. *Journal of Mechanics of Materials and Structures*, 2010, 5(3): 459-476.

[22] Lim,C.W., Niu,J.C. and Yu,Y.M., Nonlocal stress theory for buckling instability of nanotubes: new predictions on stiffness strengthening effects of nanoscales. *Journal of Computational and Theoretical of Nanoscience*, 2010, 7(10): 2104-2111.
[23] Li,C, Lim,C.W., Yu,J.L. and Zeng,Q.C., Analytical solutions for vibration of simply supported nonlocal nanobeams with an axial force. *International Journal of Structural Stability and Dynamics*, 2011, 11(2): 257-271.
[24] Li,C., Lim,C.W. and Yu,J.L., Dynamics and stability of transverse vibrations of nonlocal nanobeams with a variable axial load. *Smart Materials and Structures*, 2011, 20(1): 015023.
[25] Yang,Y. and Lim,C.W., A new nonlocal cylindrical shell model for axisymmetric wave propagation in carbon nanotubes. *Advanced Science Letters*, 2011, 4(1): 121-131.
[26] Ramamurti,V., Mechanical Vibration Practice And Noise Control. Oxford: Alpha Science, 2008.