Free torsional vibration of nanotubes based on nonlocal stress theory

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ABSTRACT

A new elastic nonlocal stress model and analytical solutions are developed for torsional dynamic behaviors of circular nanorods/nanotubes. Unlike the previous approaches which directly substitute the nonlocal stress into the equations of motion, this new model begins with the derivation of strain energy using the nonlocal stress and by considering the nonlinear history of straining. The variational principle is applied to derive an infinite-order differential nonlocal equation of motion and the corresponding higher-order boundary conditions which contain a nonlocal nanoscale parameter. Subsequently, free torsional vibration of nanorods/nanotubes and axially moving nanorods/nanotubes are investigated in detail. Unlike the previous conclusions of reduced vibration frequency, the solutions indicate that natural frequency for free torsional vibration increases with increasing nonlocal nanoscale. Furthermore, the critical speed for torsional vibration of axially moving nanorods/nanotubes is derived and it is concluded that this critical speed is significantly influenced by the nonlocal nanoscale.

1. Introduction

The dynamic characteristics of materials and structures at nanoscale are significantly different from their behaviors at larger scales due to the influence of surfaces stress and size effects which are not present at macroscale. Therefore, a thorough and in-depth understanding of the mechanical behaviors of nanostructures is of paramount importance in the design and analysis of nano-electronic-mechanical-systems. Because the classical continuum mechanics is proved to fail at nanoscale, some new continuum mechanics theories and atomic/molecular dynamic simulations are necessary. On the other hand, due to the minute sizes which require extremely high precision, the conduct of experimental investigation is not only difficult but they often yield significantly varying measurements, for instance, Young's modulus for carbon nanotubes.

The theory of nonlocal elasticity was first proposed and extensively investigated by Eringen [1–5]. Subsequently, increasingly many studies in this field were reported especially in the past several decades [6–30]. The nonlocal stress theories establish the stress at a point in a domain to depend not only on the classical local stress at that particular point, but also on the spatial integrals that represent the weighted averages of the local stress contribution of all other points in the domain. Most of these studies focused on statics and instability analyses of carbon nanotubes [6–13,22,24], nanobeams

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[14–20,23,25,26] and nanoplates [27–30], such as bending and buckling, and some on dynamics analyses on vibration and wave propagation.

Virtually all previous studies [6,9–16,18] since Peddieson et al. [14] follows the similar approach which adopts the classical equilibrium equations or equations of motion and replace the classical stress terms with the corresponding nonlocal terms. These analyses yield a very common conclusion that an increased nonlocal nanoscale results in reduced nanostructural stiffness, i.e., reduced bending deflection and increased vibration frequency, buckling load and wave propagation velocity, with respect to the classical solutions. Such conclusions seem to go against the common belief either from experiment or molecular dynamic simulation that the stiffness and toughness of structures tend to be significantly enhanced at smaller scales due to size effects.

Recently, a new exact nonlocal stress model for bending [19,20], buckling [23], vibration [25,26] and wave propagation [22,24] of nanobeams or nanotubes was developed by Lim and his associates. Unlike the previous approaches which assumed direct replacement of stress quantities, this new model begins with the derivation of strain energy using the nonlocal stress and by considering the nonlinear history of straining. The variational principle is then applied to derive an infinite-order differential nonlocal equation of motion and the corresponding higher-order boundary conditions which contain a nonlocal nanoscale parameter. The new approach and analyses concluded that the nanostructural stiffness should be enhanced at nanoscale and significant increase in stiffness was reported for bending, buckling, vibration and wave propagation.

Lim [19–21] further defined and derived effective nonlocal stress quantities, such as effective nonlocal moment, which could be directly substituted into the classical equilibrium equations or equations of motion, instead of using the nonlocal stress quantities as adopted by the previous studies [6,9–16,18]. The effective nonlocal stress quantities are related to the normal nonlocal stress quantities via an infinite series. Without using these effective nonlocal stress quantities, Lim [19–21] showed that the relevant previous models should be considered as partial nonlocal models because they are derived via a mixture of the classical and nonlocal models. Lim [20] further discussed that the previous results yielded three critical issues in nonlocal elastic stress field theory for nanobeams which were overlooked.

The new nonlocal stress model [19–21] and the variational principle are adopted in this paper to study free torsional vibration of nanorods or nanotubes and axially moving nanorods/nanotubes. In the past, although the literature for transverse vibration of nanobeams is aplenty, very few studies for the corresponding torsional vibration had been presented, and, to the authors' knowledge, no reported research on an axially moving nanorod/nanotube is found at present. Torsional deformation and vibration are easily seen in nano-electro-mechanical systems. For efficient design of such devices, the torsional dynamics of the nano-components are vital. Unlike the previous approach which concluded in reduced free vibration frequency, the solutions here indicate that the natural frequency for free torsional vibration increases with increasing nonlocal nanoscale. Furthermore, the critical speed for torsional vibration of axially moving nanorods/nanotubes is derived and it is concluded that the critical speed is significantly influenced by the nonlocal nanoscale. The conclusions are found to support the new exact nonlocal model, which led to reduced bending deflection and increased transverse vibration frequency for nanobeams, presented by Lim et al. [19–26].

2. Nonlocal governing equation of motion for torsion and higher-order boundary conditions

According to the nonlocal elastic stress theory [1–4] which was developed based on the atomic theory of lattice dynamics and experimental observations on phonon dispersion, the nonlocal stress $\sigma_{ij}(\mathbf{r})$ at a reference point $\mathbf{r}$ within a homogeneous and isotropic solid V, depends not only on the classical local stress $\sigma'_{ij}(\mathbf{r}')$ at $\mathbf{r}$ but also on stress at all other points within the body. It is given by a spatial integration with weighted averages of the contributions of local stress of all points within the body V. The spatial weight is represented by a specific nonlocal modulus $\alpha(|\mathbf{r}'-\mathbf{r}|,\tau)$ which depends on a dimensionless nano-length scale τ of the material of V. Accordingly, the linear nonlocal stress within an elastic, homogeneous and isotropic body can be expressed by [1–4]

$$\sigma_{ij}(\mathbf{r}) = \int_V \alpha(|\mathbf{r}'-\mathbf{r}|,\tau)\sigma'_{ij}(\mathbf{r}')\,\mathrm{d}V(\mathbf{r}') \tag{1}$$

where $i,j=1$ or $i,j=1,2$ or $i,j=1,2,3$ depending on the relevant dimension. It is clear that for nonlocal elasticity, the classical or local constitutive relation has to be replaced by the nonlocal constitutive relation (1).

The size effects of a nanostructure in a nonlocal stress model is represented through the presence of a nonlocal nanoscale parameter

$$\tau = \frac{e_0 a}{L} \tag{2}$$

where a is an internal characteristic length (e.g., lattice parameter, C–C bond length, granular distance, etc.), L is an external characteristic length (e.g. crack length, wavelength, etc.) and e_0 is a material constant. The magnitude of e_0 is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics. In a macroscopic analysis when the effects of nanoscale becomes infinitely insignificant in the limit $\tau \to 0$, the effects of strains at points $\mathbf{r} \neq \mathbf{r}'$ are negligible, the nonlocal modulus approaches the Dirac delta function and hence $\sigma_{ij}(\mathbf{r})=\sigma'_{ij}(\mathbf{r}')$. Consequently, the classical elasticity for continuum mechanics should be recovered in the limit of vanishing
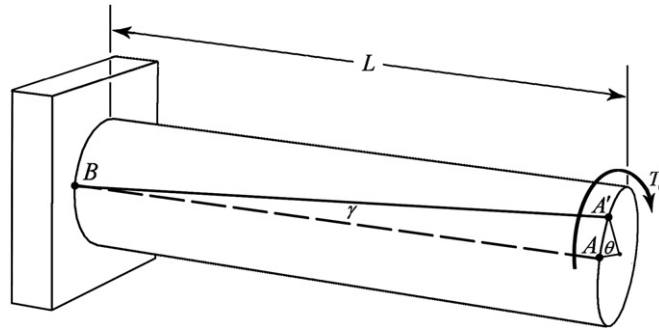
Fig. 1. Torsion of a fixed-free nanorod subjected to an end torque T_0, where a line AB is distorted to $A'B$, with angle of twist θ and shear strain γ.

nonlocal nanoscale. It is noted that the magnitude of τ is extremely small, and usually in most studies τ is from 0 to 0.2 or so.

According to Eringen [4], the integral nonlocal stress in Eq. (1) can be represented by an equivalent differential constitutive equation within a two-dimensional region and under certain conditions using Green's function as

$$s_{xr} - (e_0 a)^2 \frac{d^2 s_{xr}}{dx^2} = s'_{xr} \tag{3}$$

where x is the axial coordinate, r is the distance to the center of nanorod/nanotube, while s_{xr} and s'_{xr} are the nonlocal and classical shear stresses, respectively. For torsional vibration, the classical shear stress and shear strain are related by

$$s'_{xr} = G\gamma = Gr \frac{d\theta}{dx} \tag{4}$$

where G is the shear modulus, γ is the shear strain, θ is the angular twist distribution along the nanorod/nanotube as shown in Fig. 1.

For generality and brevity, the nonlocal constitutive relation is non-dimensionalized as

$$\bar{s}_{xr} - \tau^2 \frac{d^2 \bar{s}_{xr}}{d\bar{x}^2} = \bar{r} \frac{d\theta}{d\bar{x}} \tag{5}$$

using the following dimensionless parameters:

$$\bar{s}_{xr} = \frac{s_{xr}}{G}, \quad \tau = \frac{e_0 a}{L}, \quad \bar{x} = \frac{x}{L}, \quad \bar{r} = \frac{r}{L} \tag{6}$$

Eq. (5) is a second-order ordinary differential equation and it has been shown by Lim et al. [31] that for a nanorod/nanotube without initial shear stress the solution is

$$\bar{s}_{xr} = \sum_{n=1}^{\infty} \tau^{2n-2} \gamma^{\langle 2n-2 \rangle} \tag{7}$$

where $\gamma = \bar{r} d\theta/d\bar{x}$ and $()^{\langle n \rangle} = d^n/d\bar{x}^n$ for $n \geq 0$. For $n = 0$, $\gamma^{\langle 0 \rangle} = \gamma$.

Applying Eq. (7), the nonlocal twisting moment is defined as

$$T = \int_A r s_{xr} \, dA = \frac{GI_P}{L} \sum_{n=1}^{\infty} \tau^{2n-2} \theta^{\langle 2n-1 \rangle} \tag{8}$$

where $I_P = \int_A r^2 \, dA$ is the polar moment of inertia, A is the cross sectional area, and for nanorods $I_P = \pi R^4/2$ while for nanotubes $I_P = \pi(R_1^4 - R_2^4)/2$, in which R is the radius of a circular nanorod, and R_1 and R_2 are the outer and inner radii of a nanotube, respectively. The nonlocal twisting moment in dimensionless terms is

$$\bar{T} = \frac{TL}{GI_P} = \sum_{n=1}^{\infty} \tau^{2n-2} \theta^{\langle 2n-1 \rangle} = \theta^{\langle 1 \rangle} + \tau^2 \theta^{\langle 3 \rangle} + \tau^4 \theta^{\langle 5 \rangle} + \tau^6 \theta^{\langle 7 \rangle} + \tau^8 \theta^{\langle 9 \rangle} + \cdots \tag{9}$$

Considering the straining of a nanorod/nanotube with respect to an unstrained state, the strain energy density at an arbitrary point of a nanorod/nanotube under torsion can be expressed as [31]

$$u = \int_0^\gamma s_{xr} \, d\gamma = G \sum_{n=1}^{\infty} \tau^{2n-2} \int_0^\gamma \gamma^{\langle 2n-2 \rangle} \, d\gamma = u_1 + u_2 + u_3 \tag{10}$$

where the nonlocal shear stresses s_{xr} is work-conjugated to the shear strain γ, and

$$u_1 = \frac{1}{2} G\gamma^2$$

$$u_2 = \frac{1}{2} G \sum_{n=1}^{\infty} (-1)^{n+1} \tau^{2n} (\gamma^{\langle n \rangle})^2$$

$$u_3 = G \sum_{n=1}^{\infty} \left\{ \tau^{2n+2} \sum_{m=1}^{n} [(-1)^{m+1} \gamma^{\langle m \rangle} \gamma^{\langle 2n-m+2 \rangle}] \right\} \tag{11}$$

Thus, the total strain energy for a deformed nanorod/nanotube under torsion is

$$U = \int_V u \, dV = \int_V (u_1 + u_2 + u_3) \, dV = U_1 + U_2 + U_3 \tag{12}$$

Variation of the strain energy components according to the variational principle yields

$$\delta U = \delta U_1 + \delta U_2 + \delta U_3 \tag{13}$$

Details of the variation in terms of θ, equivalently, in terms of the nonlocal twisting moment $\overline{T}$ are available in Lim et al. [31]. The Hamilton's principle requires that

$$\delta \int_{t_1}^{t_2} (E - U) \, dt = 0 \tag{14}$$

where $(E - U)$ is a Lagrangian. With respect to free torsional vibration of a circular nanorod/nanotube in this paper, U is the strain energy given in Eq. (12) and E is the kinetic energy given by

$$E = \int_0^L \int_0^R \int_0^{2\pi} \frac{1}{2} \rho \left(r \frac{\partial \theta}{\partial t} \right)^2 r \, d\alpha \, dr \, dx = \frac{\pi \rho R^4 L}{4 P_e^2} \int_0^1 \left(\frac{\partial \theta}{\partial \overline{t}} \right)^2 d\overline{x} \tag{15}$$

where t and $\overline{t} = t/P_e$ are the dimensional and dimensionless temporal variables, in which P_e is a characteristic time which can be taken as the period of vibration. From Eqs. (14) and (15), the variation of time integral of kinetic energy is

$$\delta \int_{t_1}^{t_2} E \, dt = \frac{\pi \rho R^4 L}{2 P_e} \int_0^1 \int_{\overline{t}_1}^{\overline{t}_2} \frac{\partial \theta}{\partial \overline{t}} \delta \left(\frac{\partial \theta}{\partial \overline{t}} \right) d\overline{t} \, d\overline{x} = \frac{\pi \rho R^4 L}{2 P_e} \int_0^1 \left(\frac{\partial \theta}{\partial \overline{t}} \delta \theta \right) \Big|_{\overline{t}_1}^{\overline{t}_2} d\overline{x} - \frac{\pi \rho R^4 L}{2 P_e} \int_0^1 \int_{\overline{t}_1}^{\overline{t}_2} \frac{\partial^2 \theta}{\partial \overline{t}^2} \delta \theta \, d\overline{t} \, d\overline{x} \tag{16}$$

and the dimensionless limits of integration are from $\overline{t}_1 = t_1/P_e$ to $\overline{t}_2 = t_2/P_e$. According to Hamilton's principle, the initial and final configurations do not vary along the path, thus the first integral in the equation above vanishes automatically.

Substituting Eqs. (13) and (16) into Eq. (14) yields

$$0 = \frac{GI_P P_e}{L} \int_{\overline{t}_1}^{\overline{t}_2} \int_0^1 \left[\sum_{n=0}^{\infty} (2n-1) \tau^{2n} \theta^{\langle 2n+2 \rangle} + \overline{\rho} \frac{\partial^2 \theta}{\partial \overline{t}^2} \right] \delta \theta \, d\overline{x} \, d\overline{t}$$

$$+ \frac{GI_P P_e}{L} \int_{\overline{t}_1}^{\overline{t}_2} \left[\left(-\sum_{n=0}^{\infty} (2n-1) \tau^{2n} \theta^{\langle 2n+1 \rangle} \right) \delta \theta + \left(\sum_{n=1}^{\infty} (2n-1) \tau^{2n} \theta^{\langle 2n \rangle} \right) \delta \theta^{\langle 1 \rangle} + \left(-\sum_{n=1}^{\infty} 2n \tau^{2n+2} \theta^{\langle 2n+1 \rangle} \right) \delta \theta^{\langle 2 \rangle} \right.$$

$$+ \left(\sum_{n=1}^{\infty} (2n-1) \tau^{2n+2} \theta^{\langle 2n \rangle} \right) \delta \theta^{\langle 3 \rangle} + \left(-\sum_{n=1}^{\infty} 2n \tau^{2n+4} \theta^{\langle 2n+1 \rangle} \right) \delta \theta^{\langle 4 \rangle} + \left(\sum_{n=1}^{\infty} (2n-1) \tau^{2n+4} \theta^{\langle 2n \rangle} \right) \delta \theta^{\langle 5 \rangle}$$

$$\left. + \left(-\sum_{n=1}^{\infty} 2n \tau^{2n+6} \theta^{\langle 2n+1 \rangle} \right) \delta \theta^{\langle 6 \rangle} + \cdots \right]_{\overline{x}=0}^{\overline{x}=1} d\overline{t} \tag{17}$$

or

$$0 = \frac{GI_P P_e}{L} \int_{\overline{t}_1}^{\overline{t}_2} \int_0^1 \left(-\overline{T}^{\langle 1 \rangle} + 2 \sum_{n=1}^{\infty} \tau^{2n} \overline{T}^{\langle 2n+1 \rangle} + \overline{\rho} \frac{\partial^2 \theta}{\partial \overline{t}^2} \right) \delta \theta \, d\overline{x} \, d\overline{t}$$

$$+ \frac{GI_P P_e}{L} \int_{\overline{t}_1}^{\overline{t}_2} \left[\left(\overline{T} - 2 \sum_{n=1}^{\infty} \tau^{2n} \overline{T}^{\langle 2n \rangle} \right) \delta \theta + \sum_{n=1}^{\infty} \tau^2 \left(\overline{T}^{\langle 1 \rangle} + 2 \sum_{n=1}^{\infty} \tau^{2n} \overline{T}^{\langle 2n+1 \rangle} \right) \delta \theta^{\langle 1 \rangle} - 2 \sum_{n=0}^{\infty} \tau^{2n+4} \overline{T}^{\langle 2n+2 \rangle} \delta \theta^{\langle 2 \rangle} \right.$$

$$+ \tau^4 \left(\overline{T}^{\langle 1 \rangle} + 2 \sum_{n=1}^{\infty} \tau^{2n} \overline{T}^{\langle 2n+1 \rangle} \right) \delta \theta^{\langle 3 \rangle} - 2 \sum_{n=0}^{\infty} \tau^{2n+6} \overline{T}^{\langle 2n+2 \rangle} \delta \theta^{\langle 4 \rangle} + \tau^6 \left(\overline{T}^{\langle 1 \rangle} + 2 \sum_{n=1}^{\infty} \tau^{2n} \overline{T}^{\langle 2n+1 \rangle} \right) \delta \theta^{\langle 5 \rangle}$$

$$\left. - 2 \sum_{n=0}^{\infty} \tau^{2n+8} \overline{T}^{\langle 2n+2 \rangle} \delta \theta^{\langle 6 \rangle} + \cdots \right]_{\overline{x}=0}^{\overline{x}=1} d\overline{t} \tag{18}$$

where $\overline{\rho} = (\pi R^4 L^2 / 2 GI_P P_e^2) \rho$ is the dimensionless mass density. Because $\delta \theta$ cannot vanish, according to the variational principle, we have

$$\sum_{n=0}^{\infty} (2n-1) \tau^{2n} \theta^{\langle 2n+2 \rangle} + \overline{\rho} \frac{\partial^2 \theta}{\partial \overline{t}^2} = 0 \tag{19}$$

or

$$-\overline{T}^{\langle 1 \rangle} + 2\sum_{n=1}^{\infty} \tau^{2n}\overline{T}^{\langle 2n+1 \rangle} + \overline{\rho}\frac{\partial^2 \theta}{\partial \overline{t}^2} = 0 \tag{20}$$

where $\overline{T}$ is given in Eq. (9). For vanishing nonlocal effect in the limit $\tau \to 0$ and neglecting the higher-order terms associated with τ, the governing Eqs. (19) and (20) are reduced into the classical torsional vibration of a rod or a tube [32,33].

According to Lim et al. [31], a dimensionless effective nonlocal torque is defined as

$$\overline{T}_{\mathrm{ef}} = \overline{T} - 2\sum_{n=1}^{\infty} \tau^{2n}\overline{T}^{\langle 2n \rangle} = -\sum_{n=0}^{\infty}(2n-1)\tau^{2n}\theta^{\langle 2n+1 \rangle} = \theta^{\langle 1 \rangle} - \tau^2\theta^{\langle 3 \rangle} - 3\tau^4\theta^{\langle 5 \rangle} - 5\tau^6\theta^{\langle 7 \rangle} - 7\tau^8\theta^{\langle 9 \rangle} - \cdots \tag{21}$$

where $\overline{T}_{\mathrm{ef}} = T_{\mathrm{ef}}L/GI_P$ in which T_{ef} is the corresponding dimensional quantity. Hence Eq. (20) above can be expressed as

$$\overline{T}_{\mathrm{ef}}^{\langle 1 \rangle} = \overline{\rho}\frac{\partial^2 \theta}{\partial \overline{t}^2} \tag{22}$$

and the corresponding boundary conditions are

$$\left.\begin{array}{l}
-\sum_{n=0}^{\infty}(2n-1)\tau^{2n}\theta^{\langle 2n+1 \rangle} = \overline{T} - 2\sum_{n=1}^{\infty}\tau^{2n}\overline{T}^{\langle 2n \rangle} = 0 \qquad \text{or } \theta = 0 \\[3mm]
\sum_{n=1}^{\infty}(2n-1)\tau^{2n}\theta^{\langle 2n \rangle} = \sum_{n=1}^{\infty}\tau^2\left(\overline{T}^{\langle 1 \rangle} + 2\sum_{n=1}^{\infty}\tau^{2n}\overline{T}^{\langle 2n+1 \rangle}\right) = 0 \quad \text{or } \theta^{\langle 1 \rangle} = 0 \\[3mm]
-\sum_{n=1}^{\infty}2n\tau^{2n+2}\theta^{\langle 2n+1 \rangle} = -2\sum_{n=0}^{\infty}\tau^{2n+4}\overline{T}^{\langle 2n+2 \rangle} = 0 \quad \text{or } \theta^{\langle 2 \rangle} = 0 \\[3mm]
\sum_{n=1}^{\infty}(2n-1)\tau^{2n+2}\theta^{\langle 2n \rangle} = \tau^4\left(\overline{T}^{\langle 1 \rangle} + 2\sum_{n=1}^{\infty}\tau^{2n}\overline{T}^{\langle 2n+1 \rangle}\right) = 0 \quad \text{or } \theta^{\langle 3 \rangle} = 0 \\[3mm]
-\sum_{n=1}^{\infty}2n\tau^{2n+4}\theta^{\langle 2n+1 \rangle} = -2\sum_{n=0}^{\infty}\tau^{2n+6}\overline{T}^{\langle 2n+2 \rangle} = 0 \quad \text{or } \theta^{\langle 4 \rangle} = 0 \\[3mm]
\sum_{n=1}^{\infty}(2n-1)\tau^{2n+4}\theta^{\langle 2n \rangle} = \tau^6\left(\overline{T}^{\langle 1 \rangle} + 2\sum_{n=1}^{\infty}\tau^{2n}\overline{T}^{\langle 2n+1 \rangle}\right) = 0 \quad \text{or } \theta^{\langle 5 \rangle} = 0 \\[3mm]
-\sum_{n=1}^{\infty}2n\tau^{2n+6}\theta^{\langle 2n+1 \rangle} = -2\sum_{n=0}^{\infty}\tau^{2n+8}\overline{T}^{\langle 2n+2 \rangle} = 0 \quad \text{or } \theta^{\langle 6 \rangle} = 0 \\[3mm]
\vdots \hspace{8cm} \vdots
\end{array}\right\}_{\text{at } \overline{x}=0,1} \tag{23}$$

Eq. (22) is similar to the classical equation of motion for torsional vibration of a rod/tube except that the twisting moment is replaced by the effective torsion torque in this case. It is clear here that for analysis of a nanostructure based on the nonlocal stress field theory, the classical equilibrium condition or equation of motion cannot be directly applied by replacing the twisting moment with the nonlocal twisting moment but rather it should be replaced by an effective nonlocal twisting moment as in Eq. (21).

It has been shown by Lim and his associates [19–26,31] that, by direct replacement of classical stress quantities by the corresponding nonlocal quantities, the solutions yield a nanostructure with reduced stiffness while by replacing with an effective nonlocal quantity, the stiffness of the nanostructure is greatly enhanced, i.e., lower nanobeam deflection [19,20], higher transverse nanobeam vibration frequency [25] and buckling load [23] and higher wave propagation velocity [22,24].

For torsional vibration of a nanorod/nanotube, the truncated governing equation of motion considering only the most significant terms of nonlocal effect τ from Eq. (19) is

$$\tau^2\theta^{\langle 4 \rangle} - \theta^{\langle 2 \rangle} + \overline{\rho}\frac{\partial^2 \theta}{\partial \overline{t}^2} = 0 \tag{24}$$

and the corresponding boundary conditions are

$$\left.\begin{array}{ll}
\theta^{\langle 1 \rangle} - \tau^2\theta^{\langle 3 \rangle} = 0 & \text{or } \theta = 0 \\[2mm]
\theta^{\langle 2 \rangle} + 3\tau^2\theta^{\langle 4 \rangle} = 0 & \text{or } \theta^{\langle 1 \rangle} = 0
\end{array}\right\}_{\text{at } \overline{x}=0,1} \tag{25}$$

where the first two are the natural (left) and geometric (right) classical boundary conditions while the other two are the natural (left) and geometric (right) higher-order boundary conditions, respectively. It has been defined and discussed at length by Lim [21] that the two choices of higher-order boundary conditions can be classified as soft and hard boundary conditions with respect to the natural and geometric conditions, respectively. Such definitions are adopted here.

For linear torsional vibration of a nanorod/nanotube, the modes are harmonic in time. Hence the time-dependent angular rotation can be represented by

$$\theta(\overline{x},\overline{t}) = \Theta(\overline{x})e^{i\overline{\omega}_n\overline{t}} \tag{26}$$

where $i=(-1)^{1/2}$, Θ is the time-independent angular amplitude, and $\overline{\omega}_n(n=1,2,3,\dots)$ is the dimensionless natural frequency. Substituting Eq. (26) into (24) and (25) yields

$$\tau^2\Theta^{\langle4\rangle}-\Theta^{\langle2\rangle}-\overline{\rho\omega}_n^2\Theta=0 \tag{27}$$

$$\left.\begin{array}{ll}\Theta^{\langle1\rangle}-\tau^2\Theta^{\langle3\rangle}=0 & \text{or } \Theta=0\\ \Theta^{\langle2\rangle}+3\tau^2\Theta^{\langle4\rangle}=0 & \text{or } \Theta^{\langle1\rangle}=0\end{array}\right\}_{\text{at } \overline{x}=0,1} \tag{28}$$

Since Eq. (27) is a fourth-order ordinary differential equation, its solution can be expressed as

$$\Theta_n(\overline{x})=C_{1n}e^{i\beta_{1n}\overline{x}}+C_{2n}e^{i\beta_{2n}\overline{x}}+C_{3n}e^{i\beta_{3n}\overline{x}}+C_{4n}e^{i\beta_{4n}\overline{x}} \tag{29}$$

where $C_{jn}(j=1,2,3,4)$ are the coefficients and $\beta_{jn}(j=1,2,3,4)$ are the four roots of the dispersion relation of Eq. (27), or

$$\tau^2\beta_n^4+\beta_n^2-\overline{\rho\omega}_n^2=0 \tag{30}$$

Eq. (30) can also be obtained by substituting $\Theta_n(\overline{x})=C_ne^{i\beta_n\overline{x}}$ into Eq. (27), where C_n is a nonzero constant.

It should be emphasized here that Eq. (30) should be applied with extreme care. As discussed in Lim and Yang [22] and Yang and Lim [24], the nonlocal elasticity model is constructed based on a continuum model which simulates a lattice dynamics model with discrete atoms/molecules. For extremely high wavenumber, all the continuum models including the strain gradient model, couple stress model, etc., may not be valid because in such cases, the wavelength could be in the sub-nano scales and spans only a few atoms/molecules. A different and revolutionary continuum model should be devised.

3. Torsional vibration analysis

3.1. Doubly fixed nanorods/nanotubes

For a nanorod/nanotube fixed at both ends, the governing equation of motion considering the most significant terms of nonlocal effect τ is given by Eq. (27) and the general solution by Eq. (29). Considering soft higher-order boundary conditions, the constraints from Eq. (28) are [21,31]

$$\Theta|_{\overline{x}=0}=0; \quad \Theta|_{\overline{x}=1}=0$$
$$(\Theta^{\langle2\rangle}+3\tau^2\Theta^{\langle4\rangle})\big|_{\overline{x}=0}=0; \quad (\Theta^{\langle2\rangle}+3\tau^2\Theta^{\langle4\rangle})\big|_{\overline{x}=1}=0 \tag{31}$$

Substituting Eq. (29) into Eq. (31) yields

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{i\beta_{1n}} & e^{i\beta_{2n}} & e^{i\beta_{3n}} & e^{i\beta_{4n}} \\ k_{1n} & k_{2n} & k_{3n} & k_{4n} \\ k_{1n}e^{i\beta_{1n}} & k_{2n}e^{i\beta_{2n}} & k_{3n}e^{i\beta_{3n}} & k_{4n}e^{i\beta_{4n}} \end{pmatrix}\begin{pmatrix} C_{1n} \\ C_{2n} \\ C_{3n} \\ C_{4n} \end{pmatrix}=0 \tag{32}$$

where $k_{jn}=-\beta_{jn}^2+3\tau^2\beta_{jn}^4$ $(j=1,2,3,4)$. For nontrivial solution of C_{jn}, the determinant of coefficient matrix should be zero, or

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ e^{i\beta_{1n}} & e^{i\beta_{2n}} & e^{i\beta_{3n}} & e^{i\beta_{4n}} \\ k_{1n} & k_{2n} & k_{3n} & k_{4n} \\ k_{1n}e^{i\beta_{1n}} & k_{2n}e^{i\beta_{2n}} & k_{3n}e^{i\beta_{3n}} & k_{4n}e^{i\beta_{4n}} \end{vmatrix}=0 \tag{33}$$

From Eqs. (30) and (33), the relation of dimensionless natural frequency and nonlocal nanoscale can be established. Figs. 2 and 3 show the first three modes of $\overline{\omega}_n(n=1,2,3)$, which are significantly enhanced by the presence of τ. The dotted lines represent the corresponding classical solutions, which can be obtained by omitting the nonlocal terms containing τ in Eqs. (30) and (33). It is observed that the presence of τ enhances the natural frequency, and the effect is greater for higher-mode frequencies. The classical solution undervalues the torsional vibration frequency and it fails to be applicable at nanoscale. It also demonstrates that higher density causes lower $\overline{\omega}_n(n=1,2,3)$ from Figs. 2 and 3.

3.2. Doubly free nanorods/nanotubes

For a doubly free nanorod/nanotube, the governing equation of motion is again given by Eq. (27) and the general solution by Eq. (29). Considering hard higher-order boundary conditions, the constraints from Eq. (28) are [21,31]

$$(\Theta^{\langle1\rangle}-\tau^2\Theta^{\langle3\rangle})\big|_{\overline{x}=0}=0; \quad (\Theta^{\langle1\rangle}-\tau^2\Theta^{\langle3\rangle})\big|_{\overline{x}=1}=0$$
$$\Theta^{\langle1\rangle}\big|_{\overline{x}=0}=0; \quad \Theta^{\langle1\rangle}\big|_{\overline{x}=1}=0 \tag{34}$$
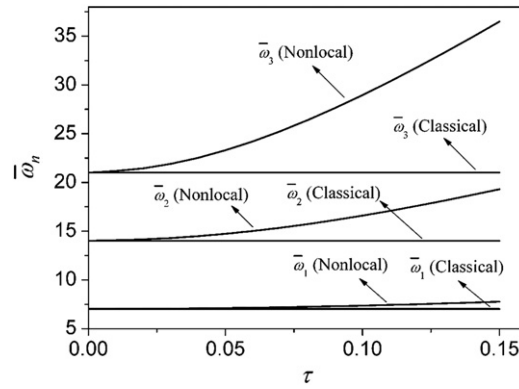
Fig. 2. Effects of τ on $\overline{\omega}_n (n = 1,2,3)$ for a doubly soft fixed nanorod/nanotube with $\overline{\rho} = 0.2$.
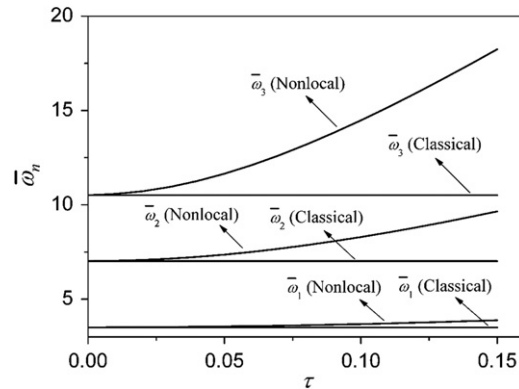
Fig. 3. Effects of τ on $\overline{\omega}_n (n = 1,2,3)$ for a doubly soft fixed nanorod/nanotube with $\overline{\rho} = 0.8$.

or in simplified form

$$\Theta^{\langle 3 \rangle}\Big|_{\overline{x}=0} = 0; \quad \Theta^{\langle 3 \rangle}\Big|_{\overline{x}=1} = 0$$

$$\Theta^{\langle 1 \rangle}\Big|_{\overline{x}=0} = 0; \quad \Theta^{\langle 1 \rangle}\Big|_{\overline{x}=1} = 0 \tag{35}$$

Following the same solution method, the relation between τ and $\overline{\omega}_n (n = 1,2,3)$ is shown in Fig. 4. Again, it is observed that increasing τ causes the natural frequencies to increase.

3.3. Fixed-free nanorods/nanotubes

For a nanotube fixed at $\overline{x} = 0$ and free at $\overline{x} = 1$, the soft-fixed hard-free boundary conditions are

$$\Theta|_{\overline{x}=0} = 0; \quad (\Theta^{\langle 1 \rangle} - \tau^2 \Theta^{\langle 3 \rangle})\Big|_{\overline{x}=1} = 0$$

$$(\Theta^{\langle 2 \rangle} + 3\tau^2 \Theta^{\langle 4 \rangle})\Big|_{\overline{x}=0} = 0; \quad \Theta^{\langle 1 \rangle}\Big|_{\overline{x}=1} = 0 \tag{36}$$

or in simplified form

$$\Theta|_{\overline{x}=0} = 0; \quad \Theta^{\langle 3 \rangle}\Big|_{\overline{x}=1} = 0$$

$$(\Theta^{\langle 2 \rangle} + 3\tau^2 \Theta^{\langle 4 \rangle})\Big|_{\overline{x}=0} = 0; \quad \Theta^{\langle 1 \rangle}\Big|_{\overline{x}=1} = 0 \tag{37}$$

Similarly, the relation between nonlocal nanoscale and natural frequencies is shown in Fig. 5.

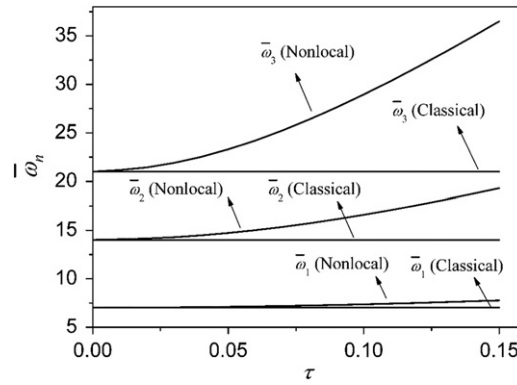**Fig. 4.** Effects of τ on $\overline{\omega}_n (n=1,2,3)$ for a doubly hard free nanorod/nanotube with $\overline{\rho}=0.2$.
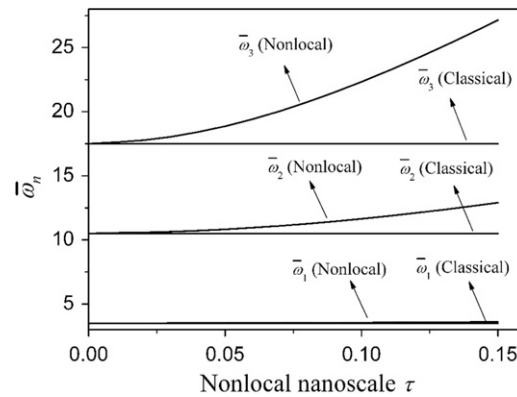

Fig. 5. Effects of τ on $\overline{\omega}_n (n=1,2,3)$ for a soft-fixed hard-free nanorod/nanotube with $\overline{\rho}=0.2$.

4. Torsional vibration of an axially moving nanorod/nanotube

For a moving nanorod/nanotube, the kinetic energy due to the axial velocity v should be taken into consideration. For simplicity, a nanorod with radius R is considered here. The total kinetic energy is

$$\int_0^{P_e} E \, dt = \frac{\rho \pi R^2}{2} \int_0^{P_e} \int_0^L \left[\left(R \frac{\partial \theta}{\partial t} + v\theta \right)^2 + v^2 \right] dx \, dt = \frac{\pi \rho R^4 L}{2P_e} \int_0^1 \int_0^1 \left[\left(\frac{\partial \theta}{\partial \overline{t}} \right)^2 + \overline{v}^2 \theta^2 + 2\overline{v}\theta \frac{\partial \theta}{\partial \overline{t}} + \overline{v}^2 \right] d\overline{x} \, d\overline{t} \quad (38)$$

where $\overline{t} = t/P_e$, $\overline{v} = vP_e/R$. Variation of the kinetic energy above is given by

$$\delta \int_0^{P_e} E \, dt = \frac{\pi \rho R^4 L}{P_e} \int_0^1 \left(-\int_0^1 \frac{\partial^2 \theta}{\partial \overline{t}^2} \delta\theta \, d\overline{x} + \int_0^1 \overline{v}^2 \theta \delta\theta \, d\overline{x} + \overline{v}\theta \delta\theta \Big|_0^1 \right) d\overline{t} \quad (39)$$

Similarly, the variational principle requires that

$$0 = \frac{GI_P P_e}{L} \int_0^1 \int_0^1 \left[\sum_{n=0}^{\infty} (2n-1)\tau^{2n}\theta^{\langle 2n+2 \rangle} + \overline{\rho}\frac{\partial^2 \theta}{\partial \overline{t}^2} - \overline{\rho}\overline{v}^2\theta \right] \delta\theta \, d\overline{x} \, d\overline{t}$$

$$+ \frac{GI_P P_e}{L} \int_0^1 \left[\left(-\sum_{n=0}^{\infty} (2n-1)\tau^{2n}\theta^{\langle 2n+1 \rangle} - \overline{\rho}\overline{v}\theta \right) \delta\theta + \left(\sum_{n=1}^{\infty} (2n-1)\tau^{2n}\theta^{\langle 2n \rangle} \right) \delta\theta^{\langle 1 \rangle} + \left(-\sum_{n=1}^{\infty} 2n\tau^{2n+2}\theta^{\langle 2n+1 \rangle} \right) \delta\theta^{\langle 2 \rangle} \right.$$

$$+ \left(\sum_{n=1}^{\infty} (2n-1)\tau^{2n+2}\theta^{\langle 2n \rangle} \right) \delta\theta^{\langle 3 \rangle} + \left(-\sum_{n=1}^{\infty} 2n\tau^{2n+4}\theta^{\langle 2n+1 \rangle} \right) \delta\theta^{\langle 4 \rangle}$$

$$\left. + \left(\sum_{n=1}^{\infty} (2n-1)\tau^{2n+4}\theta^{\langle 2n \rangle} \right) \delta\theta^{\langle 5 \rangle} + \left(-\sum_{n=1}^{\infty} 2n\tau^{2n+6}\theta^{\langle 2n+1 \rangle} \right) \delta\theta^{\langle 6 \rangle} + \cdots \right]_{\overline{x}=0}^{\overline{x}=1} d\overline{t} \quad (40)$$

where $\overline{\rho} = (\pi R^4 L^2 / P_e^2 GI_P)\rho$. Since $\delta\theta$ cannot vanish, hence the higher-order governing equation and higher-order boundary conditions are, respectively,

$$\sum_{n=0}^{\infty}(2n-1)\tau^{2n}\theta^{\langle 2n+2\rangle} + \overline{\rho}\frac{\partial^2\theta}{\partial\overline{t}^2} - \overline{\rho}\overline{v}^2\theta = 0 \tag{41}$$

$$\left.\begin{array}{ll}
-\sum_{n=0}^{\infty}(2n-1)\tau^{2n}\theta^{\langle 2n+1\rangle} = \overline{\rho}\overline{v}\theta & \text{or } \theta = 0 \\[2mm]
\sum_{n=1}^{\infty}(2n-1)\tau^{2n}\theta^{\langle 2n\rangle} = 0 & \text{or } \theta^{\langle 1\rangle} = 0 \\[2mm]
-\sum_{n=1}^{\infty}2n\tau^{2n+2}\theta^{\langle 2n+1\rangle} = 0 & \text{or } \theta^{\langle 2\rangle} = 0 \\[2mm]
\sum_{n=1}^{\infty}(2n-1)\tau^{2n+2}\theta^{\langle 2n\rangle} = 0 & \text{or } \theta^{\langle 3\rangle} = 0 \\[2mm]
-\sum_{n=1}^{\infty}2n\tau^{2n+4}\theta^{\langle 2n+1\rangle} = 0 & \text{or } \theta^{\langle 4\rangle} = 0 \\[2mm]
\sum_{n=1}^{\infty}(2n-1)\tau^{2n+4}\theta^{\langle 2n\rangle} = 0 & \text{or } \theta^{\langle 5\rangle} = 0 \\[2mm]
-\sum_{n=1}^{\infty}2n\tau^{2n+6}\theta^{\langle 2n+1\rangle} = 0 & \text{or } \theta^{\langle 6\rangle} = 0 \\[2mm]
\qquad\vdots & \qquad\vdots
\end{array}\right\}_{\text{at } \overline{x}=0,1} \tag{42}$$

If only the most significant nonlocal terms having τ are considered, the simplified governing equation for a nanorod/nanotube is

$$\tau^2\theta^{\langle 4\rangle} - \theta^{\langle 2\rangle} + \overline{\rho}\frac{\partial^2\theta}{\partial\overline{t}^2} - \overline{\rho}\overline{v}^2\theta = 0 \tag{43}$$

and the corresponding higher-order boundary conditions are

$$\left.\begin{array}{ll}
\theta^{\langle 1\rangle} - \tau^2\theta^{\langle 3\rangle} = \overline{\rho}\overline{v}\theta & \text{or } \theta = 0 \\[2mm]
\tau^2\theta^{\langle 2\rangle} + 3\tau^4\theta^{\langle 4\rangle} = 0 & \text{or } \theta^{\langle 1\rangle} = 0
\end{array}\right\}_{\text{at } \overline{x}=0,1} \tag{44}$$

Applying Eq. (26), one obtains

$$\tau^2\Theta^{\langle 4\rangle} - \Theta^{\langle 2\rangle} - \overline{\rho}(\overline{\omega}_n^2 + \overline{v}^2)\Theta = 0 \tag{45}$$

$$\left.\begin{array}{ll}
\Theta^{\langle 1\rangle} - \tau^2\Theta^{\langle 3\rangle} = \overline{\rho}\overline{v}\Theta & \text{or } \Theta = 0 \\[2mm]
\tau^2\Theta^{\langle 2\rangle} + 3\tau^4\Theta^{\langle 4\rangle} = 0 & \text{or } \Theta^{\langle 1\rangle} = 0
\end{array}\right\}_{\text{at } \overline{x}=0,1} \tag{46}$$

The general form of solution as expressed in Eq. (29) is still available where $\beta_{jn}(j=1,2,3,4)$ are the four roots of the following dispersion relation:

$$\tau^2\beta^4 + \beta^2 - \overline{\rho}(\overline{\omega}_n^2 + \overline{v}^2) = 0 \tag{47}$$

while the coefficients $C_{jn}(j=1,2,3,4)$ can be solved from the boundary conditions. For a doubly soft-fixed nanorod/nanotube, the higher-order boundary conditions are

$$\Theta|_{\overline{x}=0} = 0; \quad \Theta|_{\overline{x}=1} = 0$$
$$(\Theta^{\langle 2\rangle} + 3\tau^2\Theta^{\langle 4\rangle})\Big|_{\overline{x}=0} = 0; \quad (\Theta^{\langle 2\rangle} + 3\tau^2\Theta^{\langle 4\rangle})\Big|_{\overline{x}=1} = 0 \tag{48}$$

Substituting Eq. (29) into Eq. (30) above yields a similar Eq. (32) where $k_{jn} = -\beta_{jn}^2 + 3\tau^2\beta_{jn}^4$ $(j=1,2,3,4)$. For nontrivial solution, the vibration frequency for an axially moving nanorod/nanotube can be obtained by solving the characteristic equation given in Eq. (33). The relationship between the three mode frequencies $\overline{\omega}_n(n=1,2,3)$ and τ is illustrated Fig. 6 while the effects of $\overline{v}$ for $\tau=0.1$ is presented in Fig. 7.

In Fig. 6, it is again noted that the natural frequency increases with increasing τ, or equivalently, stronger nonlocal effects result in higher shear stiffness. In Fig. 7, meanwhile, the natural frequency drops with increasing axial velocity $\overline{v}$. There exist certain values of $\overline{v}$ where there is no vibration. The axial velocity that corresponds to zero first mode frequency $\overline{\omega}_1$ is called the critical speed $\overline{v}_{cr}$. For instance, when $\tau=0.1$ and $\overline{\rho}=0.2$, the critical speed is given by $\overline{v}_{cr}=7.37$ as shown in Fig. 7. At the critical speed, the angular rotation is independent on time, or the governing equation in Eq. (45) becomes

$$\tau^2\Theta^{\langle 4\rangle} - \Theta^{\langle 2\rangle} - \pi\overline{v}_{cr}^2\Theta = 0 \tag{48}$$

The relation between $\overline{v}_{cr}$ and τ is presented in Fig. 8. It is observed that larger τ results in higher $\overline{v}_{cr}$.
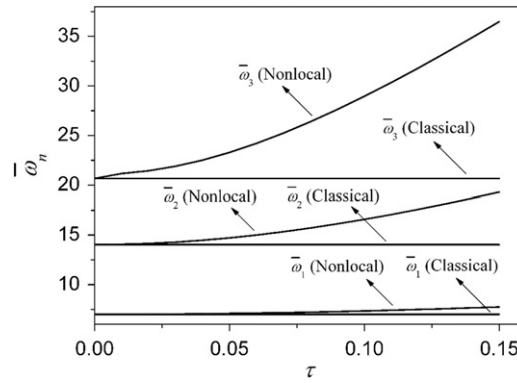
Fig. 6. Effects of τ on $\overline{\omega}_n(n=1,2,3)$ for a doubly soft-fixed nanorod/nanotube with $\overline{v}=0.5$ and $\overline{p}=0.2$.
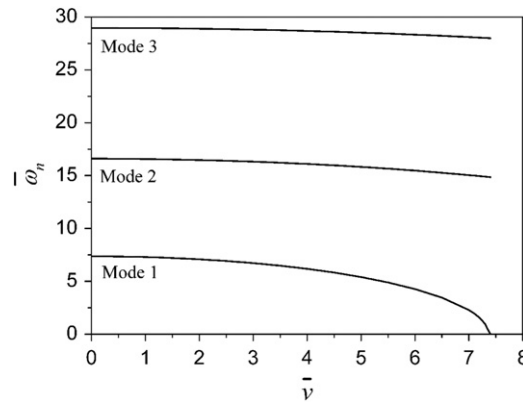

Fig. 7. Effects of dimensionless axial velocity $\overline{v}$ on $\overline{\omega}_n(n=1,2,3)$ of a doubly soft-fixed nanorod/nanotube with $\tau=0.1$ and $\overline{p}=0.2$.
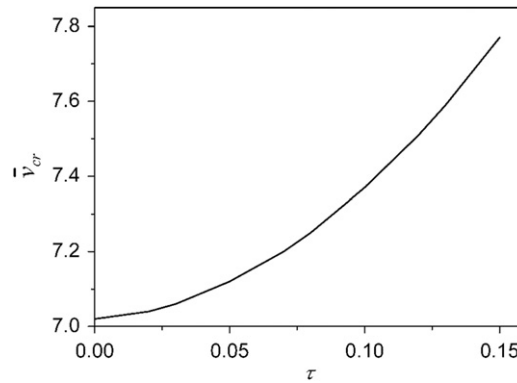

Fig. 8. Relation between critical speed and nonlocal nanoscale of a doubly soft-fixed nanorod/nanotube.

5. Conclusions

Torsional vibration of nanorods/nanotubes and the dynamics of axially moving nanorods/nanotubes are developed based on the nonlocal elastic shear stress theory and the variational principle. This new approach is virtually different from all previous methods where, instead of directly replacing the stress terms in the classical equations of motion by the corresponding nonlocal stress terms, new higher-order governing equations of motion and the higher-order boundary conditions are derived by considering the history of nonlinear straining with reference to an undeformed state. New analytical expressions and numerical solutions for various examples are presented and the effects of various geometric and natural boundary conditions are investigated. It is concluded that the nonlocal nanoscale is found to induce higher torsional stiffness and hence to cause increased free vibration frequency. In addition, the critical velocity of axially moving nanorods/nanotubes is also analyzed where the critical velocities increases with increasing nonlocal nanoscale. In all cases, the solutions are reduced to the classical solutions

of continuum mechanics in the limit of vanishing nonlocal nanoscale effect in which the higher-order terms associated with the nonlocal nanoscale are neglected. The validity of this new nonlocal shear stress model and solutions are verified.

Acknowledgments

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