Buckling analysis of graphene sheets embedded in an elastic medium based on the kp-Ritz method and non-local elasticity theory

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\section*{Abstract}
Graphene finds application in nanoscale electronic devices, nano sensors and nanocomposites in which graphene sheets usually rest on an elastic medium. Buckling of graphene sheets can occur in such structures, which significantly influences its performance. Therefore, it is essential to investigate the buckling behavior of graphene for better engineering design and manufacture. The present work provides an element-free kp-Ritz framework to analyze the buckling behavior of graphene. The element-free kp-Ritz method is first implemented to investigate the buckling behavior of single-layered graphene sheets (SLGSs) embedded in elastic foundations, based on the classical plate theory (CLP), incorporating the non-local elasticity theory, which takes small effects into account when dealing with nanostructures. A Winkler-type model is adopted to simulate the functioning of the elastic medium. Numerical solutions for critical buckling loads of SLGSs are obtained by solving the governing differential equations derived from the principle of minimum potential energy using the element-free kp-Ritz method. The concepts of non-local parameter effects and the Winkler modulus parameter on the buckling patterns are discussed.

\section{1. Introduction}
In recent years, carbon nanostructures have drawn increasing attention from researchers; applications are being extended from superfast microelectronics, modern aerospace and nanocomposites to micro electro-mechanical systems (MEMS), nano electro-mechanical systems (NEMS), biomedical and bioelecticals. Nanostructure-based products contain ultra-capacitors, gas graphene transistors, solar cells, nanoresonators, batteries, mammalian and microbial detection, diagnosis devices and ultra-strength composite materials, etc.\cite{1}. The vast potential for applications lies in the extraordinary properties of nanostructures compared to traditional materials, such as high mechanical strength, extrusive electronic conductivity and outstanding thermal conductivity, etc.\cite{2–4}.

A single-layered graphene sheet\cite{5}, which comprises carbon atoms packed in a hexagonal lattice with a carbon–carbon distance of 0.142 nm, is the basis of nanomaterials since the majority of nanostructures such as carbon nanotubes, carbon nanocones, nanorings, fullerenes etc. are regarded as deformed graphene sheets. Thus, simulations of single-layered graphene sheets is an elementary issue in the analysis of nanostructures\cite{6}. Graphene sheets for MEMS and NEMS applications usually resting on an elastic medium instead of being suspended. Moreover, due to the excellent structural performance and multifunctional characteristics of graphene-polymer nano-composites, it is considered one of the most promising engineering nanostructures\cite{7–10}. Thus, stability response of graphene sheets as MEMS and NEMS components become an important research topic, therefore, it is essential to understand the buckling behavior of graphene sheets on elastic medium so as for better engineering design and manufacture.

Because of very small dimensions involved, it is difficult to experimentally investigate the behavior of nanostructures since there are limits in terms of high experimental cost and time consuming. Moreover, it is difficult to manipulate and measure the response of nano-scale specimens due to the limitation of production process. Therefore, theoretical and numerical approaches are becoming increasingly significant for simulating...
nanomaterials. Among these methods, the atomistic model including the classical molecular dynamics (MD), the density functional theory (DFT) and tight-binding molecular dynamics (TBMD) requires substantial computational resources, although it offers high accuracy and is therefore limited to appropriately small sized materials with finite numbers of atoms. Compared with the atomistic model, the continuum mechanics method, including classical beam, plate and shell theories, is computationally less expensive [1]. Therefore, it can be used to simulate relatively larger nanostructures [11,12]. However, the classical continuum method does not take the effect of the small dimensions into account and thus needs to be modified to reflect relatively precise mechanical behaviors of nanostructures. Thus, some modified continuum methods were adopted to simulate the size-dependent materials. Li [13] utilized the modified couple-stress and sinusoidal plate theories to analyze the static bending and free vibration problems of a simply supported piezoelectric functionally graded microplate. The non-local elasticity theory developed by Eringen [14,15] assumes that the stress at a given point is a function of all other points’ strain state in the body. Thus, it can predict the small scale effect by accounting for information about the long range forces between atoms. Following the development by Peddieson [16] of a non-local Bernoulli/Euler beam model by incorporating non-local elasticity theory into classical beam theory, more and more researchers have adopted it to study the mechanical properties of nanostructures [17–21].

Numerical simulation is a vital tool for theoretical analysis. Different numerical techniques have been used for analyzing the buckling of structures [22,23]. Rouhi et al. [24] developed the atomistic finite element model to analyze the axial buckling behavior of SLGSs. A new version of the differential quadrature method [25] has been used to study the free transverse vibration behavior of circular and annular graphene sheets. The finite difference method has been used to analyze the buckling of rectangular nanoplates based on the non-local theory [26]. Pradhan et al. [27] adopted the differential transformation method (DTM) to predict the buckling behavior of single walled carbon nanotubes on the Winkler medium with different boundary conditions. The semi-analytical finite strip method was used for a detailed study of the buckling characteristics of rectangular single and multi-layered graphene sheets based on the classical plate theory (CPT) [28]. As one of the numerical methods, the element-free kp-Ritz method has wide applications in engineering problems due to its profound formula foundation and its wide-ranging adaptability [29–33]. The element-free kp-Ritz method has been used to simulate the buckling of nanostructures [34]; however, to the best of the author’s knowledge, no element-free buckling analysis of SLGSs in an elastic medium based on the non-local elasticity theory has been reported in published papers. Therefore, it is beneficial to construct an element-free kp-Ritz framework to analyze the buckling behavior of, and gain knowledge about the influence of the non-local parameter on the buckling of SLGSs.

2. Theoretical formulations

2.1. Model of elastic medium

In nanoscale electronic devices, nano sensors and nano-composites, buckling can be observed which significantly influences the performance of graphene. Moreover, in the majority of these products, the graphene sheet is placed on an elastic foundation. Therefore, it is essential to investigate the buckling behavior of graphene sheets embedded in elastic foundations.

For graphene sheets embedded in elastic foundations, two types of models are recommended for describing the mechanical functions of the elastic medium. One is the Pasternak-type medium model [35], which takes both the normal pressure and the transverse shear deformation of elastic foundations into account. Liew et al. [36] adopted this model to investigate the nano-vibration behavior of multi-layered graphene sheets. The other is the Winkler-type medium model [37], which ignores the influence of shear effects and thus reduces the normal pressure term which is regarded as the result of a series of uniformly distributed, mutually independent, vertical linear elastic springs. Thus, the Winkler modulus can be regarded as the stiffness per unit area. Besides these two types of models, there are some other medium models, such as Filonenko–Borodich, generalized and Vlasov models. The mathematical expressions of the recommended foundation models are as follows

\[ q_{\text{pasternak}} = K_{w} w - K_{g} \nu \frac{\partial^2 w}{\partial t^2}, \]  

\[ q_{\text{winkler}} = K_{w} w, \]  

where \( K_{w} \) and \( K_{g} \) are the Winkler modulus and the shear modulus of the elastic medium, respectively. Fig. 1 provides a good illustration of an elastic foundation with a graphene sheet embedded within it.

According to [6], the shear effects have little influence on the buckling load of SLGSs, especially when the Winkler modulus is adequately large. In this paper, the Winkler model is considered to reflect the mechanical function of elastic foundations.

2.2. Classical plate theory (CLP)

The classical plate theory (CPT) assumes that the straight line remains straight and vertical to the mid-plane after deformation, despite the shear deformation and rotational inertia. According to CPT, when the small deflection assumption is adopted, the displacement can be expressed as

\[ U(x, y, z; t) = -\frac{\partial^2 w}{\partial x^2}, \]  

\[ V(x, y, z; t) = -\frac{\partial^2 w}{\partial y^2}, \]

\[ W(x, y, z; t) = w(x, y; t). \]

Subsequently, the strain–displacement relations can be obtained as

\[ \varepsilon_{xx} = -\frac{\partial^2 w}{\partial x^2}, \]  

\[ \varepsilon_{yy} = -\frac{\partial^2 w}{\partial y^2}, \]  

\[ \varepsilon_{zz} = 0, \]
\[
\gamma_{xy} = -22\frac{\partial^2 W}{\partial y^2},
\]
\[
\gamma_{xz} = \gamma_{xy} = 0.
\]

The strain energy can be expressed as
\[
U_{\text{strain}} = \frac{1}{2} \int_{\Gamma} \sigma^T \cdot \sigma \, d\Omega,
\]
where \(\sigma = [\sigma_{xx}, \sigma_{yy}, \sigma_{xy}]^T, \sigma = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}]^T
\).

The energy absorbed in the medium is
\[
U_{\text{medium}} = \frac{1}{2} \int_{\Omega} K_w \cdot \varepsilon^2 \, d\Omega.
\]

where \(K_w\) is the Winkler modulus.

The work done by the axial compressive load can be expressed as
\[
U_p = \int_{\Gamma} \{ \sigma \} \cdot \{ \varepsilon \} \, d\Gamma.
\]

where \(\{ \sigma \} = [p(x), p(y)]^T\).

Therefore, the total potential energy of such a system is
\[
U = U_{\text{strain}} + U_{\text{medium}} - U_p.
\]

where \(U_{\text{strain}}\) is the strain energy of the SLGS, \(U_{\text{medium}}\) is the potential energy of the medium and \(U_p\) is the work done by the external force.

Using the principle of total potential energy, when the first variation of the total potential energy is set to zero, the equilibrium equation can be obtained as
\[
M_{xx,xx} + 2M_{xx,yy} + M_{yy,yy} - K_w \cdot w + p(x)w_{xx} + p(y)w_{yy} = 0,
\]
where \([M_{xx}, M_{yy}, M_{xy}]^T = \int_{x=0}^{h/2} \{ \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy} \} \, dz\).

The most widely used form of non-local constitutive relationship is the differential constitutive equation. According to [15], the differential form obtained from the integrated form is
\[
(1 - (\varepsilon_0 a)^2)\varepsilon_{ij} = C_{ijkl} \varepsilon_{kl},
\]
where \((\varepsilon_0 a)^2\) denotes the non-local parameter, which accounts for the increasingly important long range forces between atoms and the noncontinuity of nanostructures, thus revealing the small-scale effect on the response of nanoscale structures. \(\varepsilon_{ij}\) is the non-local stress, \(\varepsilon_0\) is the non-local material constant ranging from 0 to 14 [38] and \(a\) is internal characteristic lengths (such as lengths of C-C bonds and the lattice parameter). \(C_{ijkl}\) and \(\varepsilon_{ij}\) are the local stiffness tensor and traditional strain, respectively. The non-local theory reduces to local theory when the non-local parameter goes to zero.

The stress–strain relationship of the non-local plate can be described as
\[
\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ - (\varepsilon_0 a)^2 \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} E/(1-\nu^2) & 0 & 0 \\ 0 & E/(1-\nu^2) & 0 \\ 0 & 0 & G(1-\nu)/(1-\nu^2) \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix},
\]
where \(E, \nu\) and \(G\) denote the elastic modulus, Poisson’s ratio and shear modulus, respectively. Thus, the non-local parameters can be expressed as
\[
M_{xx} - (\varepsilon_0 a)^2 \nu^2 M_x = -D \left( \frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} \right),
\]
\[
M_{yy} - (\varepsilon_0 a)^2 \nu^2 M_y = -D \left( \frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \right),
\]
\[
M_{xy} - (\varepsilon_0 a)^2 \nu^2 M_{xy} = -D \left( \frac{\partial^2 W}{\partial x \partial y} + \nu \frac{\partial^2 W}{\partial y \partial x} \right),
\]

where \(D = \frac{Eh^3}{12(1-\nu^2)}\) is the bending rigidity of the plate while \(p(x)\) and \(p(y)\) are axial compressive loads along the length direction (X axis) and width direction (Y axis), respectively.

### 3. Numerical implementation

#### 3.1. Weak form of equilibrium equation

To reduce the differentiability of the shape function used in the element-free method, the original strong form equilibrium equation is multiplied by the virtual displacement \(\delta w\), thus obtaining the following equation
\[
\int_{\Gamma} \delta w \left[ D \nabla^2 w + \left(1 - (\varepsilon_0 a)^2 \nu^2 \right) \left[ K_w w - p(x)w_{xx} - p(y)w_{yy} \right] \right] \, dx \, dy = 0.
\]

The above equation can then be integrated by part as
\[
D \int_{\Gamma} (\delta w_{xx} w_{xx} + 2\delta w_{xy} w_{xy} + \delta w_{yy} w_{yy}) \, dx \, dy
+ K_w \int_{\Gamma} \delta w \, dw \, dx \, dy + (\varepsilon_0 a)^2 K_w \int_{\Gamma} (\delta w_{xx} w_{xx} + \delta w_{yy} w_{yy}) \, dx \, dy
+ p(x) \int_{\Gamma} \delta w \, dw_{x} \, dx \, dy + p(y) \int_{\Gamma} \delta w \, dw_{y} \, dx \, dy
+ (\varepsilon_0 a)^2 p(x) \int_{\Gamma} (\delta w_{xx} w_{xx} + \delta w_{xy} w_{xy}) \, dx \, dy
+ (\varepsilon_0 a)^2 p(y) \int_{\Gamma} (\delta w_{yy} w_{yy} + \delta w_{xy} w_{xy}) \, dx \, dy
+ D \int_{\Gamma} (\delta w_{xx} w_{xx} n_x - \delta w_{x} w_{xx} n_x + 2\delta w_{xy} w_{xy} n_x
- \delta w_{y} w_{xy} n_y - \delta w_{yy} w_{yy} n_y - \delta w_{x} w_{yy} n_y) \, ds
- (\varepsilon_0 a)^2 K_w \int_{\Gamma} (\delta w_{xx} n_x + \delta w_{yy} n_y) \, ds
- p(x) \int_{\Gamma} \delta w_{xx} n_x \, ds - p(y) \int_{\Gamma} \delta w_{yy} n_y \, ds
+ (\varepsilon_0 a)^2 p(x) \int_{\Gamma} (\delta w_{xx} n_x - \delta w_{x} w_{xx} n_x
+ \delta w_{xy} n_y - \delta w_{y} w_{xy} n_y) \, ds
+ (\varepsilon_0 a)^2 p(y) \int_{\Gamma} (\delta w_{yy} n_y - \delta w_{y} w_{yy} n_y
+ \delta w_{xy} n_y - \delta w_{x} w_{xy} n_y) \, ds.
\]

#### 3.2. Kernel particle shape functions

Following the detailed description of the theoretical formulation of the reproducing kernel particle method (RKPM) by Liu et al. [39] and Chen et al. [40], for a domain discretized by a set of nodes \(x_i, i=1,...,NP\), displacement approximations can be expressed as
\[
u^h = \sum_{i=1}^{NP} \psi_i(x) u_i,
\]
where \(\psi_i(x)\) and \(u_i\) denote the shape function and nodal parameter associated with node \(i\), respectively.

With the kernel function \(\Phi_0(x-x_i)\) and the correction function \(C(x; x-x_i)\), the shape function is expressed as
\[
\psi_i(x) = C(x; x-x_i) \Phi_0(x-x_i).
\]
The shape function must satisfy the reproduction conditions
\[ \sum_{l=1}^{NP} \sum_{p+q=0}^{2} \psi_l(x) y^p x^q = x^p y^q \]
for \( p, q = 0, 1, 2 \).
\[ (K - \lambda \bar{K}) w = 0, \]
where
\[ K = K_{\text{SLGS}} + K_{\text{medium}}, \]
\[ K_{\text{SLGS}} = \int_\Omega B^T L S_L B_L d\Omega, \]
\[ K_{\text{medium}} = \int_\Omega B^T S_K B d\Omega, \]
\[ K = \int_\Omega B^T S_L S_B B d\Omega, \]
\[ S_L = \begin{bmatrix} D & 0 & 0 \\ 0 & 2D & 0 \\ 0 & 0 & D \end{bmatrix}, \]
\[ S_K = \begin{bmatrix} K_w & 0 & 0 \\ 0 & (\varepsilon_0 a)^2 K_w & 0 \\ 0 & 0 & (\varepsilon_0 a)^2 K_w \end{bmatrix} + \begin{bmatrix} -p(x) & 0 & 0 \\ 0 & -(\varepsilon_0 a)^2 p(x) & 0 \\ 0 & 0 & -p(y) \end{bmatrix}, \]
\[ B_L = \begin{bmatrix} \psi_{l,x} \\ \psi_{l,xy} \\ \psi_{l,y} \end{bmatrix}, \]
\[ B_K = \begin{bmatrix} \psi_l \\ \psi_{l,x} \\ \psi_{l,y} \end{bmatrix}, \]
\[ B = \begin{bmatrix} \psi_{l,x} \\ \psi_{l,xy} \\ \psi_{l,y} \end{bmatrix}. \]

4. Results and discussion

4.1. Convergence study and validation

The convergence test is performed using classical plate theory on a square SLGS with a four edged simply-supported boundary condition, ignoring elastic foundations. The non-local and mechanical parameters are taken from Ansari [41]: \( (\varepsilon_0 a)^2 = 1.85 \text{ nm}^2 \), the Young’s modulus \( E = 1 \text{ TPa} \), the Poisson’s ratio \( \nu = 0.16 \) and the thickness \( h = 0.34 \text{ nm} \).

Based on the above parameters, a convergence test is carried out to determine the minimum amount of nodes required to obtain accurate and stable results. It should be noted that the scaling parameter is set at 2.3. Fig. 2 shows that a setup of 19 \( \times \) 19 nodes is appropriate for the next computation.

Subsequently, a comparison simulation is performed on an armchair SLGS to validate the whole procedure. Rouhi [24] claimed that the critical buckling force of the armchair SLGS is higher than that of the zigzag SLGS. However, the buckling force of the armchair SLGS is roughly the same as that of the zigzag SLGS, with only small variations. Therefore, the influence of chirality is neglected in the present work.

According to [36], the mechanical parameters of the SLGS are taken as: Young’s modulus \( E = 1.06 \text{ TPa} \), Poisson’s ratio \( \nu = 0.25 \) and thickness \( h = 0.34 \text{ nm} \). Fig. 3 illustrates the comparison of simulation results on a 15 nm wide square armchair SLGS with [6], which uses the same mechanical parameters as used in [36]. The critical load ratio is defined as

Critical buckling load ratio
\[ = \frac{\text{critical buckling load from non local theory}}{\text{critical buckling load from local theory}}. \]
This shows that the results of the present work agree well with results computed using the differential quadrature method (DQM) [6]. The following simulated SLGSs are under biaxial compression if without special descriptions and the mechanical properties are taken from Ansari [41].

4.2. Effect of small scale on the critical buckling load of SLGSs

To investigate the effect of the small dimensions on the critical buckling ratio of SLGSs, a simulation on a square simply-supported SLGS with different non-local parameters was conducted and the results were then compared with those computed using molecular dynamics (MD) [41]. The elastic foundation is not considered, temporarily. Fig. 4 shows the effect of non-local parameters on the critical buckling ratio of simply-supported SLGSs with different widths. This reveals the important phenomenon that the critical buckling loads are sensitive to non-local parameters, especially when the side length of SLGSs is smaller. In addition, with larger non-local parameters, the critical buckling load is smaller, which agrees with [6,42], indicating that the non-local parameters make SLGSs softer.

The differences between non-local parameter effects and non-local effects are examined. The non-local effect refers to the small-scale effect, which is due to the increasingly significant long range forces between atoms and the noncontinuity of the nanoscale structure. In non-local theory, we use non-local parameters to reflect such an effect. The non-local parameter effect is the influence on the mechanical structures when the non-local parameter varies. When the side length of an SLGS is shorter, the critical buckling load is more sensitive to non-local parameters; when the side length is shorter, the non-local parameters effect is larger because even a small change in non-local parameters causes a significant change in critical buckling load. However, the non-local effect originates from the internal physical mechanism, causing the difference between the critical buckling loads of SLGSs and those computed using classical plate theory. The non-local effect can be measured by non-local parameters. Qualitatively, the larger the non-local effect is, the larger the non-local parameters are. The MD curve (Fig. 4) shows that the non-local effect follows the increase-decrease model with the increase in side length.

4.3. Aspect ratio and the critical buckling load of SLGSs

Herein, the aspect ratio is defined as

$$\text{Aspect ratio} = \frac{\text{Length}}{\text{Width}}.$$

A simply-supported SLGS with a constant width of 10 nm and varying length is simulated to explore the effect of aspect ratio on the critical buckling load. The relationship between the non-local parameters effect and aspect ratio can also be obtained.

Fig. 5 establishes that the critical buckling load of an SLGS is larger when it is closer to a square SLGS. The non-local parameters effect is illustrated in Fig. 6, showing that the non-local parameters effect increases as the aspect ratio increases.

4.4. Influence of the Winkler medium on critical buckling load

To study the effect of the Winkler modulus on the critical buckling load of SLGSs, a simulation is performed on a four edged simply-supported square SLGS with a side length of 15 nm. The definition of Winkler modulus parameter $\bar{K}_w$ is taken from Liew et al. [36]

$$\bar{K}_w = \frac{K_w L^4}{D},$$

where $L$ is the side length of the SLGS. The variation of the critical buckling load with the Winkler modulus parameter is plotted in Fig. 7. It is evident from the results that the critical buckling load of SLGSs increases linearly as the Winkler modulus parameter increases. Fig. 8 illustrates the variation of critical buckling load ratio with the Winkler modulus parameter, indicating that the non-
local parameters effect decreases as the Winkler modulus parameter increases. Another study of a four sided simply-supported SLGS under uniaxial compression is carried out to investigate the effect of load type on critical buckling load. Fig. 9 displays the variation of the critical buckling load of the SLGS under uniaxial compression with the Winkler modulus. It can be seen that the critical buckling load is larger when the SLGS is under uniaxial compression than under biaxial compression. Moreover, the trend of the non-local parameters effect with increasing Winkler modulus parameter is also different from that when the SLGS is under biaxial compression. (Fig. 10)

4.5. Effect of the Winkler medium and non-local parameters on the critical buckling patterns

The first eight buckling patterns of a four edged simply-supported square SLGS with a side length of 15 nm are shown in Fig. 11. Herein the value of non-local parameters is zero while the value of the Winkler modulus is 400, which can be described using combination parameters as (0, 400). When the combination parameters are (0, 1200) or (4, 400), the eight buckling modes are as plotted in Fig. 12.

The critical buckling pattern of an SLGS embedded in an elastic medium is the stable form of an SLGS when it buckles from its flat shape, suffering little disturbance under minimum compression. This is the most likely to occur buckling form because external energy absorbed by the SLGS and the elastic foundation is at its lowest. As the Winkler parameter increases, energy absorbed into the elastic foundation is relatively increased and that absorbed into the SLGS is relatively decreased. When the Winkler parameter is large enough, the influence of the elastic foundation on the critical buckling mode becomes dominant, thus contributing to formation of the distinct critical buckling patterns of SLGSs which are quite different to those of SLGSs without elastic foundations.

Comparing the critical buckling patterns of SLGSs under combination parameter (0, 400) with those of SLGSs under combination (0, 1200) and (4, 400), it is found that the Winkler modulus

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Fig. 4. Effect of nonlocal parameter on the critical buckling ratio of simply-supported SLGSs with different width \( u = (\kappa_0 a)^2 \).

Fig. 5. Effect of aspect ratio on critical buckling load of SLGS for different nonlocal parameters \( u = (\kappa_0 a)^2 \).

Fig. 6. Nonlocal effects on critical buckling load of SLGS for different aspect ratio \( u = (\kappa_0 a)^2 \).
The parameter required to shift the critical buckling patterns is smaller when accounting for the non-local parameters. This is regarded as evidence that the non-local parameters make SLGSs softer, thus decreasing the energy absorbed into the SLGS compared with an elastic medium. Therefore, the energy absorbed into an elastic
medium is dominant when non-local parameters are considered, although the Winkler modulus parameter is small.

5. Conclusions

In the present work, buckling analysis of SLGSs embedded in elastic foundations is performed based on the non-local elasticity theory using the element-free kp-Ritz method. The Winkler-type model is adopted to simulate an elastic medium. The concepts of a non-local parameters effect and a non-local effect are distinguished. The influence of non-local parameters, aspect ratio, side length and elastic foundations on the critical buckling load is investigated. It is found that the non-local parameters make SLGSs softer and the non-local effect is related to side length. Liang et al. [38] regarded the non-local parameters as a function of the side length of SLGSs, indicating that non-local parameters have correlations with size and shape (aspect ratio) of SLGSs. Therefore, further simulation using MD to obtain the correlation between critical buckling loads with aspect ratio is needed to calibrate the

Fig. 11. The first eight buckling modes of SLGS \( (u=0, \bar{K}_{\omega} = 400) \) \( (u = c_{0} d^{2}) \).

Fig. 12. The first eight buckling modes of SLGS, \( (u=0, \bar{K}_{\omega} = 1200) \) or \( (u = 4 \ \pi n^{2}, \bar{K}_{\omega} = 400) \) \( (u = c_{0} d^{2}) \).
variation of non-local effects with aspect ratio. Additionally, the influence of boundary conditions on non-local effects should also be investigated. Further it is also concluded that both non-local parameters and elastic medium have influence on the critical buckling patterns of SLGSs. It is evident that the critical buckling pattern is dominated by the proportion of energy absorbed by the SLGS and the elastic foundation.

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