A NOTE ON MORPHISMS DETERMINED BY OBJECTS

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Abstract. We prove that a Hom-finite additive category having determined morphisms on both sides is a dualizing variety. This complements a result by Krause. We prove that in a Hom-finite abelian category having Serre duality, a morphism is right determined by some object if and only if it is an epimorphism. We give a characterization to abelian categories having Serre duality via determined morphisms.

1. Introduction

Let $\mathcal{C}$ be an additive category which is skeletally small, that is, the iso-classes of objects form a set. Let C be an object in $\mathcal{C}$ and denote by $\Gamma(C) = \mathrm{End}_{\mathcal{C}}(C)^{\mathrm{op}}$ the opposite ring of the endomorphism ring of C. For a morphism $\alpha\colon X \to Y$ in $\mathcal{C}$, we may consider its induced map $\mathrm{Hom}_{\mathcal{C}}(C,\alpha)\colon \mathrm{Hom}_{\mathcal{C}}(C,X) \to \mathrm{Hom}_{\mathcal{C}}(C,Y)$ between left $\Gamma(C)$-modules. The image $\mathrm{Im}\ \mathrm{Hom}_{\mathcal{C}}(C,\alpha)$ is a $\Gamma(C)$-submodule of $\mathrm{Hom}_{\mathcal{C}}(C,Y)$.

Recall that for a morphism $\alpha\colon X \to Y$ and an object C, α is said to be *right C-determined* provided that for any morphism $t\colon T \to Y$, $\mathrm{Im}\ \mathrm{Hom}_{\mathcal{C}}(C,t) \subseteq \mathrm{Im}\ \mathrm{Hom}_{\mathcal{C}}(C,\alpha)$ implies that t factors through α, that is, there exists a morphism $t'\colon T \to X$ with $t = \alpha \circ t'$. In the literature, a right C-determined morphism is also called a morphism determined by C, see for example [1, 2].

For a $\Gamma(C)$-submodule H of $\mathrm{Hom}_{\mathcal{C}}(C,Y)$, we say that the pair (C,H) is *right α-represented* provided that α is right C-determined with $\mathrm{Im}\ \mathrm{Hom}_{\mathcal{C}}(C,\alpha) = H$.

The following notion is essentially contained in [6, Definition 2.6].

Definition 1.1. An object Y in $\mathcal{C}$ is *right classified* provided that the following hold:

(RC1) each morphism $\alpha\colon X \to Y$ ending at Y is right C-determined for some C;

(RC2) for any object C and $\Gamma(C)$-submodule H of $\mathrm{Hom}_{\mathcal{C}}(C,Y)$, the pair (C,H) is right α-represented for some $\alpha\colon X \to Y$.

The additive category $\mathcal{C}$ is said to *have right determined morphisms* if each object is right classified. $\square$

Let us justify this terminology. Two morphisms $\alpha_1\colon X_1 \to Y$ and $\alpha_2\colon X_2 \to Y$ are *right equivalent* if α_1 factors through α_2 and α_2 factors through α_1. The corresponding right equivalence class is denoted by $[\alpha_1\rangle = [\alpha_2\rangle$. Following [10], we denote by $[\longrightarrow Y\rangle$ the set of right equivalence classes of morphisms ending at Y. It is indeed a set, since $\mathcal{C}$ is skeletally small.

If two morphisms α_1 and α_2 are right equivalent, then α_1 is right C-determined if and only if so is α_2. So it makes sense to say that the class $[\alpha_1\rangle$ is right C-determined. We denote by ${}^{C}[\longrightarrow Y\rangle$ the subset of $[\longrightarrow Y\rangle$ formed by classes which are right C-determined. Then (RC1) is equivalent to

$$[\longrightarrow Y\rangle = \bigcup {}^{C}[\longrightarrow Y\rangle, \tag{1.1}$$

where C runs over all objects in $\mathcal{C}$.

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We denote by Sub $\mathrm{Hom}_{\mathcal{C}}(C,Y)$ the set of $\Gamma(C)$-submodules of $\mathrm{Hom}_{\mathcal{C}}(C,Y)$. The following map is well-defined

$$\eta_{C,Y}\colon [\longrightarrow Y\rangle \longrightarrow \mathrm{Sub}\ \mathrm{Hom}_{\mathcal{C}}(C,Y), \quad [\alpha\rangle \mapsto \mathrm{Im}\ \mathrm{Hom}_{\mathcal{C}}(C,\alpha).$$

The restriction of $\eta_{C,Y}$ on ${}^{C}[\longrightarrow Y\rangle$ is injective by the following lemma, which is a direct consequence of the definition.

Lemma 1.2. *Let $\alpha_1\colon X_1\to Y$ and $\alpha_2\colon X_2\to Y$ be two right C-determined morphisms. Then α_1 is right equivalent to α_2 if and only if* $\mathrm{Im}\ \mathrm{Hom}_{\mathcal{C}}(C,\alpha_1)=\mathrm{Im}\ \mathrm{Hom}_{\mathcal{C}}(C,\alpha_2)$. □

Then (RC2) is equivalent to the surjectivity of this restriction. In other words, (RC2) is equivalent to the bijection

$$(1.2) \qquad {}^{C}[\longrightarrow Y\rangle \stackrel{\sim}{\longrightarrow} \mathrm{Sub}\ \mathrm{Hom}_{\mathcal{C}}(C,Y), \quad [\alpha\rangle \mapsto \mathrm{Im}\ \mathrm{Hom}_{\mathcal{C}}(C,\alpha).$$

This bijection is known as the *Auslander bijection* at Y; see [10].

In summary, an object Y is right classified if and only if (1.1) and (1.2) hold. In this case, all morphisms ending at Y are classified by the pairs (C,H) of objects C and $\Gamma(C)$-submodules H of $\mathrm{Hom}_{\mathcal{C}}(C,Y)$.

The dual notion is as follows.

Definition 1.3. An object Y in $\mathcal{C}$ is *left classified* if it is right classified as an object in the opposite category $\mathcal{C}^{\mathrm{op}}$. The additive category $\mathcal{C}$ is said to *have left determined morphisms* if each object is left classified.

The additive category $\mathcal{C}$ *has determined morphisms* if it has both right and left determined morphisms. □

One of the fundamental results is that the category A-mod of finitely generated modules over an artin algebra A has determined morphisms; for example, see [4, 9]. This result is extended to dualizing k-varieties for a commutative artinian ring k in [6]. We prove that the converse is true. More precisely, if an additive category $\mathcal{C}$ is k-linear which is Hom-finite and has determined morphisms, then it is a dualizing k-variety; see Proposition 2.1. When the category $\mathcal{C}$ is abelian having Serre duality, we prove that a morphism is right determined by some object if and only if it is an epimorphism, and dually, a morphism is left determined by some object if and only if it is a monomorphism; see Remark 3.5(1). Indeed, we give a characterization to abelian categories having Serre duality via determined morphisms; see Theorem 3.4. In particular, we point out that a non-trivial abelian category having Serre duality is not a dualizing k-variety; see Remark 3.5(2).

2. Categories having determined morphisms

Let k be a commutative artinian ring with a unit, and let $\mathcal{C}$ be a k-linear additive category. We assume that $\mathcal{C}$ is *Hom-finite*, that is, the k-module $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is finitely generated for any objects X and Y in $\mathcal{C}$. We suppose further that $\mathcal{C}$ is skeletally small, meaning that the iso-classes of objects in $\mathcal{C}$ form a set.

We denote by k-mod the abelian category of finitely generated k-modules. Let E be the minimal injective cogenerator of k. Then we have the duality $D=\mathrm{Hom}_k(-,E)\colon k\text{-mod}\to k\text{-mod}$ with $D^2\simeq \mathrm{Id}_{k\text{-mod}}$.

Denote by $(\mathcal{C},k\text{-mod})$ the abelian category of k-linear functors from $\mathcal{C}$ to k-mod. Then D induces a duality

$$(2.1) \qquad D\colon (\mathcal{C},k\text{-mod}) \stackrel{\sim}{\longrightarrow} (\mathcal{C}^{\mathrm{op}},k\text{-mod})^{\mathrm{op}}$$

sending a functor F to DF. Here, $\mathcal{C}^{\mathrm{op}}$ denotes the opposite category of $\mathcal{C}$.

Recall that the Yoneda embedding $\mathcal{C}\to(\mathcal{C}^{\mathrm{op}},k\text{-mod})$ sending X to $\mathrm{Hom}_{\mathcal{C}}(-,X)$. Then we have the following natural isomorphisms

$$(2.2) \qquad \mathrm{Hom}_{(\mathcal{C}^{\mathrm{op}},k\text{-mod})}(\mathrm{Hom}_{\mathcal{C}}(-,C'),F) \stackrel{\sim}{\longrightarrow} F(C') \stackrel{\sim}{\longrightarrow} \mathrm{Hom}_{\Gamma(C)}(\mathrm{Hom}_{\mathcal{C}}(C,C'),F(C))$$

for any $F \in (\mathcal{C}^{\rm op}, k\text{-mod})$ and $C, C' \in \mathcal{C}$ with $C' \in \operatorname{add} C$. Here, $\operatorname{add} C$ denotes the full subcategory formed by direct summands of finite direct sums of C, and $\Gamma(C) = \operatorname{End}_{\mathcal{C}}(C)^{\rm op}$. This composite sends a morphism ξ to ξ_C. The left isomorphism is known as Yoneda Lemma, from which it follows that $\operatorname{Hom}_{\mathcal{C}}(-, C')$ is a projective object in $(\mathcal{C}^{\rm op}, k\text{-mod})$.

By (2.2) and the duality (2.1), we have the following natural isomorphisms

$$\operatorname{Hom}_{(\mathcal{C}^{\rm op}, k\text{-mod})}(F, D\operatorname{Hom}_{\mathcal{C}}(C', -)) \xrightarrow{\sim} DF(C') \xrightarrow{\sim} \operatorname{Hom}_{\Gamma(C)}(F(C), D\operatorname{Hom}_{\mathcal{C}}(C', C)) \tag{2.3}$$

for any $F \in (\mathcal{C}^{\rm op}, k\text{-mod})$ and $C, C' \in \mathcal{C}$ with $C' \in \operatorname{add} C$. The composite sends ξ to ξ_C.

A functor $F\colon \mathcal{C}^{\rm op} \to k\text{-mod}$ is *finitely generated* if there is an epimorphism $\operatorname{Hom}_{\mathcal{C}}(-, Y) \to F$ for some object Y; it is *finitely cogenerated* if there is a monomorphism $F \to D\operatorname{Hom}_{\mathcal{C}}(Y, -)$ for some object Y, or equivalently, its dual DF is finitely generated. The functor $F\colon \mathcal{C}^{\rm op} \to k\text{-mod}$ is *finitely presented* if there is an exact sequence of functors

$$\operatorname{Hom}_{\mathcal{C}}(-, X) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(-, Y) \longrightarrow F \longrightarrow 0.$$

We denote by $\operatorname{fp}(\mathcal{C})$ the full subcategory of $(\mathcal{C}^{\rm op}, k\text{-mod})$ consisting of finitely presented functors.

Following [3, Section 2], the category $\mathcal{C}$ is a *dualizing k-variety* provided that any functor $F\colon \mathcal{C}^{\rm op} \to k\text{-mod}$ is finitely presented if and only if so is its dual DF. In this case, the subcategory $\operatorname{fp}(\mathcal{C}) \subseteq (\mathcal{C}^{\rm op}, k\text{-mod})$ is *exact abelian*, meaning that it is closed under kernels, cokernels and images; consult [3, Theorem 2.4]. We mention that by definition $\mathcal{C}$ is a dualizing k-variety if and only if so is $\mathcal{C}^{\rm op}$.

The aim of this section is to prove the following result. The implication "(3) $\Rightarrow$ (1)" is given in [6, Corollary 2.13]. We mention that the implication "(1) $\Rightarrow$ (3)" is somewhat implicit in the argument in [6, Sections 3 and 5]. Hence, Proposition 2.1 is simply missed in [6]. Here we make this result explicit.

Proposition 2.1. *Let $\mathcal{C}$ be a Hom-finite k-linear additive category which is skeletally small. Then the following statements are equivalent:*

1. *the category $\mathcal{C}$ has determined morphisms;*
2. *for any functor F in $(\mathcal{C}, k\text{-mod})$ or $(\mathcal{C}^{\rm op}, k\text{-mod})$, F is finitely presented if and only if F is finitely generated and finitely cogenerated;*
3. *the category $\mathcal{C}$ is a dualizing k-variety.*

Proof. The equivalence between (1) and (2) follows from Corollary 2.6 and its dual, while the equivalence between (2) and (3) follows from Lemma 2.2. $\square$

The following result is well-known and implicit in [3, Proposition 3.1].

Lemma 2.2. *Let $\mathcal{C}$ be as above. Then $\mathcal{C}$ is a dualizing k-variety if and only if the following two conditions hold:*

1. *any functor $F\colon \mathcal{C}^{\rm op} \to k\text{-mod}$ is finitely presented $\Longleftrightarrow$ it is finitely generated and finitely cogenerated;*
2. *any functor $F\colon \mathcal{C} \to k\text{-mod}$ is finitely presented $\Longleftrightarrow$ it is finitely generated and finitely cogenerated;*

Proof. We observe that the duality (2.1) preserves the functors that are both finitely generated and finitely cogenerated. Then the "if" part follows.

For the "only if" part, we assume that $\mathcal{C}$ is a dualizing k-variety and we only prove (1). Indeed, if F is finitely presented, then DF is finitely presented, in particular, DF is finitely generated. Hence F is finitely cogenerated. This yields the direction "$\Longrightarrow$". Conversely, if F is finitely generated and finitely cogenerated, then F is the image of some morphism $\theta\colon \operatorname{Hom}_{\mathcal{C}}(-, X) \to D\operatorname{Hom}_{\mathcal{C}}(Z, -)$. The morphism θ is in the category $\operatorname{fp}(\mathcal{C})$. Recall that for a dualizing k-variety $\mathcal{C}$, the subcategory $\operatorname{fp}(\mathcal{C}) \subseteq (\mathcal{C}^{\rm op}, k\text{-mod})$ is closed under images. We infer that F is finitely presented. $\square$

For each morphism $\alpha\colon X\to Y$ in $\mathcal{C}$, we may define a finitely presented functor F^α by the exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(-,X)\xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,\alpha)}\mathrm{Hom}_{\mathcal{C}}(-,Y)\longrightarrow F^\alpha\to 0.$$

By Yoneda Lemma, every finitely presented functor arises in this way.

The following result is contained in [6, Proposition 5.2]. We give a proof for completeness.

Lemma 2.3. *The morphism α is right C-determined if and only if there is a monomorphism $F^\alpha\to D\mathrm{Hom}_{\mathcal{C}}(C',-)$ for some $C'\in\mathrm{add}\,C$.*

Proof. For the "only if" part, we assume that $\alpha\colon X\to Y$ is right C-determined. Take an exact sequence of $\Gamma(C)$-modules for some $C'\in\mathrm{add}\,C$

$$\mathrm{Hom}_{\mathcal{C}}(C,X)\xrightarrow{\mathrm{Hom}_{\mathcal{C}}(C,\alpha)}\mathrm{Hom}_{\mathcal{C}}(C,Y)\xrightarrow{\theta_C}D\mathrm{Hom}_{\mathcal{C}}(C',C).$$

Indeed, we may take an injective map $\mathrm{Cok}\,\mathrm{Hom}_{\mathcal{C}}(C,\alpha)\hookrightarrow D\mathrm{Hom}_{\mathcal{C}}(C',C)$ for some $C'\in\mathrm{add}\,C$; here, we use the fact that $D\mathrm{Hom}_{\mathcal{C}}(C,C)$ is an injective cogenerator as a $\Gamma(C)$-module. By the isomorphism (2.3), the map θ_C induces a morphism $\theta\colon\mathrm{Hom}_{\mathcal{C}}(-,Y)\to D\mathrm{Hom}_{\mathcal{C}}(C',-)$. We claim that the following sequence of functors is exact, which yields the required monomorphism.

$$\mathrm{Hom}_{\mathcal{C}}(-,X)\xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,\alpha)}\mathrm{Hom}_{\mathcal{C}}(-,Y)\xrightarrow{\theta}D\mathrm{Hom}_{\mathcal{C}}(C',-)\tag{2.4}$$

The composite is zero by the isomorphism (2.3). Take an arbitrary $t\colon T\to Y$ in $\mathrm{Ker}\,\theta_T$. For any morphism $\psi\colon C\to T$, the morphism $t\circ\psi$ lies in $\mathrm{Ker}\,\theta_C$ by the naturalness of θ, and thus in $\mathrm{Im}\,\mathrm{Hom}_{\mathcal{C}}(C,\alpha)$. In other words, $\mathrm{Im}\,\mathrm{Hom}_{\mathcal{C}}(C,t)\subseteq\mathrm{Im}\,\mathrm{Hom}_{\mathcal{C}}(C,\alpha)$. Since α is right C-determined, we infer that t factors through α. This proves that the above sequence is exact.

For the "if" part, we may assume that we have an exact sequence as (2.4). Take an arbitrary morphism $t\colon T\to Y$ with $\mathrm{Im}\,\mathrm{Hom}_{\mathcal{C}}(C,t)\subseteq\mathrm{Im}\,\mathrm{Hom}_{\mathcal{C}}(C,\alpha)$. Then $\theta_C\circ\mathrm{Hom}_{\mathcal{C}}(C,t)=0$. By the isomorphism (2.3), we have $\theta\circ\mathrm{Hom}_{\mathcal{C}}(-,t)=0$. Note that $\mathrm{Hom}_{\mathcal{C}}(-,T)$ is a projective object in $(\mathcal{C}^{\mathrm{op}},k\text{-mod})$. Then the exact sequence (2.4) yields that $\mathrm{Hom}_{\mathcal{C}}(-,t)$ factors through $\mathrm{Hom}_{\mathcal{C}}(-,\alpha)$. Thus t factors through α, by Yoneda Lemma, and we are done. □

Let Y be an object. Consider a pair (C,H) with C an object and $H\subseteq\mathrm{Hom}_{\mathcal{C}}(C,Y)$ a $\Gamma(C)$-submodule. Recall that $D\mathrm{Hom}_{\mathcal{C}}(C,C)$ is an injective cogenerator as a $\Gamma(C)$-module. Take an embedding of $\Gamma(C)$-modules

$$\mathrm{Hom}_{\mathcal{C}}(C,Y)/H\hookrightarrow D\mathrm{Hom}_{\mathcal{C}}(C',C)$$

for some $C'\in\mathrm{add}\,C$. This gives rise to a map $\theta_C\colon\mathrm{Hom}_{\mathcal{C}}(C,Y)\to D\mathrm{Hom}_{\mathcal{C}}(C',C)$, which corresponds via (2.3) to a morphism $\theta\colon\mathrm{Hom}_{\mathcal{C}}(-,Y)\to D\mathrm{Hom}_{\mathcal{C}}(C',-)$. Denote its image by $F^{(C,H)}$; it is a finitely generated and finitely cogenerated functor. Indeed, all functors in $(\mathcal{C}^{\mathrm{op}},k\text{-mod})$ that are finitely generated and finitely cogenerated arise in this way.

Lemma 2.4. *The pair (C,H) is right α-represented if and only if the functor $F^{(C,H)}$ is finitely presented.*

Proof. For the "only if" part, assume that (C,H) is right α-represented for some $\alpha\colon X\to Y$; in particular, $H=\mathrm{Im}\,\mathrm{Hom}_{\mathcal{C}}(C,\alpha)$. By the proof of Lemma 2.3, the functor F^α is the image of the morphism $\theta\colon\mathrm{Hom}_{\mathcal{C}}(-,Y)\to D\mathrm{Hom}_{\mathcal{C}}(C',-)$. It follows that $F^{(C,H)}=F^\alpha$. In particular, it is finitely presented.

For the "if" part, assume that $F^{(C,H)}$ is finitely presented. Then the kernel of the epimorphism $\mathrm{Hom}_{\mathcal{C}}(-,Y)\to F^{(C,H)}$ is finitely generated. Thus there exists a map $\alpha\colon X\to Y$ such that the following sequence is exact

$$\mathrm{Hom}_{\mathcal{C}}(-,X)\xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,\alpha)}\mathrm{Hom}_{\mathcal{C}}(-,Y)\xrightarrow{\theta}D\mathrm{Hom}_{\mathcal{C}}(C',-).\tag{2.5}$$

Hence, $\mathrm{Im}\,\mathrm{Hom}_{\mathcal{C}}(C,\alpha)=H$ and $F^{(C,H)}\simeq F^\alpha$. By Lemma 2.3 the map α is right C-determined, and (C,H) is right α-represented. □

Corollary 2.5. *Let Y be an object in $\mathcal{C}$. Then the following statements are equivalent:*

(1) *the object Y is right classified;*
(2) *for any quotient functor F of* $\mathrm{Hom}_{\mathcal{C}}(-,Y)$*, F is finitely presented if and only if F is finitely cogenerated.*

Proof. Observe that the quotient functor F is finitely presented if and only if $F = F^{\alpha}$ for some morphism $\alpha\colon X \to Y$, and that F is finitely cogenerated if and only if $F = F^{(C,H)}$ for a pair (C,H). Then the result follows from Lemmas 2.3 and 2.4. □

The following is an immediate consequence of the above result.

Corollary 2.6. *Let $\mathcal{C}$ be as above. Then the following statements are equivalent:*
(1) *the additive category $\mathcal{C}$ has right determined morphisms;*
(2) *for any functor F in* $(\mathcal{C}^{\mathrm{op}}, k\text{-mod})$*, F is finitely presented if and only if F is finitely generated and finitely cogenerated.* □

Example 2.7. Let $\mathcal{C}$ be a Hom-finite k-linear additive category which is skeletally small and has split idempotents. Hence $\mathcal{C}$ is Krulll-Schmidt. Denote by $\mathrm{ind}\,\mathcal{C}$ the set of iso-classes of indecomposable objects in $\mathcal{C}$. We assume that for each object Y, there are only finitely many $X \in \mathrm{ind}\,\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X,Y) \neq 0$, and that there exists an object C_0 such that $\mathrm{Hom}_{\mathcal{C}}(C_0,X) \neq 0$ for infinitely many $X \in \mathrm{ind}\,\mathcal{C}$. For example, the category of preprojective modules over a tame hereditary algebra satisfies this condition.

In this case, every finitely generated functor F in $(\mathcal{C}^{\mathrm{op}}, k\text{-mod})$ has finite length, and it follows that F is finitely presented and finitely cogenerated. However, every finitely cogenerated functor in $(\mathcal{C}, k\text{-mod})$ has finite length, and thus the functor $\mathrm{Hom}_{\mathcal{C}}(C_0,-)$ is not finitely cogenerated. It follows from Corollary 2.6 that $\mathcal{C}$ has right determined morphism, but does not have left determined morphisms. Indeed, by the dual of Corollary 2.5, an object C is left classified if and only if there are only finitely many $X \in \mathrm{ind}\,\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(C,X) \neq 0$.

3. Abelian categories having Serre duality

Let $\mathcal{C}$ be a Hom-finite k-linear abelian category. Recall that $\mathcal{C}$ is said to *have Serre duality* provided that there exists a k-linear auto-equivalence $\tau\colon \mathcal{C} \to \mathcal{C}$ with a functorial isomorphism

$$D\mathrm{Ext}^1_{\mathcal{C}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Y,\tau(X)) \tag{3.1}$$

for any objects X, Y in $\mathcal{C}$. The functor τ is called the *Auslander-Reiten translation* of $\mathcal{C}$.

The following notion is modified from Definition 1.1.

Definition 3.1. An object Y in $\mathcal{C}$ is *right epi-classified* provided that the following hold:
(REC1) each epimorphism $\alpha\colon X \to Y$ ending at Y is right C-determined for some C;
(REC2) for any object C and $\Gamma(C)$-submodule H of $\mathrm{Hom}_{\mathcal{C}}(C,Y)$, the pair (C,H) is right α-represented for some epimorphism $\alpha\colon X \to Y$.

If each object in $\mathcal{C}$ is right epi-classified, then $\mathcal{C}$ is said to *have right determined epimorphisms.* □

We observe the following fact.

Lemma 3.2. *Let $\alpha\colon X \to Y$ be a morphism in $\mathcal{C}$ with Y right epi-classified. Then α is right C-determined for some C if and only if α is an epimorphism.*

Consequently, if $\mathcal{C}$ has right determined epimorphisms, then a morphism is right determined by some object if and only if it is an epimorphism.

Proof. We only need to prove the necessity. Recall that for two right equivalent maps $\alpha_1\colon X_1 \to Y$ and $\alpha_2\colon X_2 \to Y$, α_1 is epic if and only if so is α_2. By (REC2) the pair $(C, \mathrm{Im}\,\mathrm{Hom}_{\mathcal{C}}(C,\alpha))$ is right α'-represented for an epimorphism $\alpha'\colon X' \to Y$. Lemma 1.2 implies that α and α' are right equivalent, which follows that α is epic. □

We denote by $[\longrightarrow Y\rangle_{\rm epi}$ the subset of $[\longrightarrow Y\rangle$ formed by epimorphisms. As in Introduction, an object Y being right epi-classified implies that ${}^C[\longrightarrow Y\rangle = {}^C[\longrightarrow Y\rangle_{\rm epi}$ and $[\longrightarrow Y\rangle_{\rm epi} = \bigcup {}^C[\longrightarrow Y\rangle_{\rm epi}$ where C runs over all objects in $\mathcal{C}$, and the Auslander bijection (1.2) at Y.

Following [7], a morphism $f\colon Z \to Y$ is *projectively trivial* if $\mathrm{Ext}^1_{\mathcal{C}}(f,-)=0$. For any objects Z and Y, denote by $\mathcal{P}(Z,Y)$ the subset of $\mathrm{Hom}_{\mathcal{C}}(Z,Y)$ formed by projectively trivial morphisms. This gives rise to an ideal $\mathcal{P}$ of $\mathcal{C}$ and the corresponding factor category is denoted by $\underline{\mathcal{C}}$. Dually, one defines *injectively trivial* morphisms and the factor category $\overline{\mathcal{C}}$. For almost split sequences, we refer to [4].

Proposition 3.3. *Let $\mathcal{C}$ be a Hom-finite k-linear abelian category, and let Y be right epi-classified. Then we have the following statements:*

(1) *if Y is indecomposable, then there is an almost split sequence $0 \to K \to X \to Y \to 0$ for some objects K and X;*
(2) *$\mathcal{P}(Z,Y)=0$ for any object Z.*

In particular, if the abelian category $\mathcal{C}$ has right determined epimorphisms, we have $\mathcal{C} = \underline{\mathcal{C}}$.

Proof. Denote by $\mathrm{rad}\ \mathrm{End}_{\mathcal{C}}(Y)$ the Jacobson radical of $\mathrm{End}_{\mathcal{C}}(Y)$. We apply (REC2) to the pair $(Y, \mathrm{rad}\ \mathrm{End}_{\mathcal{C}}(Y))$, and assume that it is right α-represented with $\alpha\colon X \to Y$ an epimorphism; moreover, we may assume that α is right minimal. It follows from [4, Proposition V.1.14] that $0 \to \mathrm{Ker}\ \alpha \to X \xrightarrow{\alpha} Y \to 0$ is an almost split sequence.

For (2), let $f\colon Z \to Y$ be a projectively trivial morphism. Then from the definition, one infers that f factors through any epimorphism $\alpha\colon X \to Y$. In particular, by (REC2) we may take α to be an epimorphism which is right Z-determined with $\mathrm{Im}\ \mathrm{Hom}_{\mathcal{C}}(Z,\alpha)=0$. This implies that $f=0$. $\square$

The dual of Definition 3.1 is as follows: an object Y in $\mathcal{C}$ is *left mono-classified* if it is right epi-classified in the opposite category $\mathcal{C}^{\rm op}$; the abelian category $\mathcal{C}$ has *left determined monomorphisms* if each object is left mono-classified.

The following result is an abelian analogue of [6, Theorem 4.2]. The proof relies on the results in [7].

Theorem 3.4. *Let $\mathcal{C}$ be a Hom-finite k-linear abelian category. Then $\mathcal{C}$ has Serre duality if and only if $\mathcal{C}$ has right determined epimorphisms and left determined monomorphisms.*

Proof. For the "only if" part, we assume that $\mathcal{C}$ has Serre duality with its Auslander-Reiten translation τ. We only prove that $\mathcal{C}$ has right determined epimorphisms. Fix an object Y in $\mathcal{C}$. For an epimorphism $\alpha\colon X \to Y$, denote its kernel by K. Then we have an exact sequence in $(\mathcal{C}^{\rm op}, k\text{-mod})$

$$\mathrm{Hom}_{\mathcal{C}}(-,X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,\alpha)} \mathrm{Hom}_{\mathcal{C}}(-,Y) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}}(-,K).$$

By Serre duality, $\mathrm{Ext}^1_{\mathcal{C}}(-,K) \simeq D\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}(K),-)$. It follows that there is a monomorphism $F^\alpha \to D\mathrm{Hom}_{\mathcal{C}}(\tau^{-1}(K),-)$. By Lemma 2.3 the morphism α is right $\tau^{-1}(K)$-determined, proving (REC1).

For (REC2), let C be an object and $H \subseteq \mathrm{Hom}_{\mathcal{C}}(C,Y)$ be a $\Gamma(C)$-submodule. Consider the morphism $\theta\colon \mathrm{Hom}_{\mathcal{C}}(-,Y) \to D\mathrm{Hom}_{\mathcal{C}}(C',-)$ with $C' \in \mathrm{add}\ C$ and $\mathrm{Im}\ \theta = F^{(C,H)}$; see Section 2. Combining θ with the isomorphism $D\mathrm{Hom}_{\mathcal{C}}(C',-) \simeq \mathrm{Ext}^1_{\mathcal{C}}(-,\tau(C'))$ we obtain a morphism

$$\theta'\colon \mathrm{Hom}_{\mathcal{C}}(-,Y) \longrightarrow \mathrm{Ext}^1_{\mathcal{C}}(-,\tau(C'))$$

with $\mathrm{Im}\ \theta' \simeq F^{(C,H)}$. Consider the extension $\rho\colon 0 \to \tau(C') \to X \xrightarrow{\alpha} Y \to 0$ corresponding to $\theta'_Y(\mathrm{Id}_Y)$, which induces an exact sequence in $(\mathcal{C}^{\rm op}, k\text{-mod})$

$$\mathrm{Hom}_{\mathcal{C}}(-,X) \xrightarrow{\mathrm{Hom}_{\mathcal{C}}(-,\alpha)} \mathrm{Hom}_{\mathcal{C}}(-,Y) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{C}}(-,\tau(C')).$$

Observe that $\delta = \theta'$. This is because $\delta_Y(\mathrm{Id}_Y) = \theta'_Y(\mathrm{Id}_Y)$ and by Yoneda Lemma. Thus $F^{(C,H)} \simeq \mathrm{Im}\ \delta$, and (C,H) is right α-represented.

For the "if" part, we assume that $\mathcal{C}$ has right determined epimorphisms and left determined monomorphisms. By Proposition 3.3 and its dual, we infer that $\underline{\mathcal{C}} = \mathcal{C} = \overline{\mathcal{C}}$, and that for any indecomposable object Y, there exist an almost split sequence ending at Y and an almost split sequence starting at Y. Then $\mathcal{C}$ has Serre duality by [7, Propositions (3.1) and (3.3)]. □

Remark 3.5. Let $\mathcal{C}$ be a Hom-finite k-linear abelian category having Serre duality, whose Auslander-Reiten translation is denoted by τ.

(1) By Theorem 3.4 and Lemma 3.2, a morphism $\alpha\colon X \to Y$ is right determined by some object if and only if it is an epimorphism, in which case α is right $\tau^{-1}(\mathrm{Ker}\ \alpha)$-determined; dually, a morphism $\beta\colon Y \to Z$ is left determined by some object if and only if it is a monomorphisms, in which case β is left $\tau(\mathrm{Cok}\ \beta)$-determined.

(2) We assume that $\mathcal{C}$ is not zero. Then a morphism that is not epic is not right determined by any object, and thus $\mathcal{C}$ does not have right determined morphisms in the sense of Definition 1.1. By Proposition 2.1, the category $\mathcal{C}$ is not a dualizing k-variety. However, its bounded derived category $\mathbf{D}^b(\mathcal{C})$ has Serre duality [8] and thus is a dualizing k-variety; see [6, Theorem 4.2] or [5, Corollary 2.6].

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