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Auslander–Reiten theory on the homotopy category of projective modules

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ABSTRACT

An Auslander–Reiten formula in the homotopy category of complexes of projective modules is presented. This formula guarantees the explicit description of Auslander–Reiten triangles in the homotopy category. Furthermore, almost split sequences in a module category can be deduced from these Auslander–Reiten triangles.

1. Introduction

In the early seventies, Auslander and Reiten introduced almost split sequences (or Auslander–Reiten sequences) in their fundamental paper [1] for the category Λ-mod of finitely generated modules over an artin algebra Λ. The Auslander–Reiten formula was established, which describes the relation between the two end terms of an almost split sequence. These results are central to the classical Auslander–Reiten theory and were later generalized to other abelian categories [2–4].

The Auslander–Reiten theory for complexes was initiated by Happel. In [5], he introduced the notion of Auslander–Reiten triangles in triangulated categories, and characterized their existence in the bounded derived category $\mathbf{D}^b(\Lambda\text{-mod})$ over an artin algebra. Krause proved in [6] the existence of Auslander–Reiten triangles in compactly generated triangulated categories, by using the Brown representability theorem. This work makes it possible to establish Auslander–Reiten theory in compactly generated triangulated categories. However, it is difficult to describe explicitly the end terms of these Auslander–Reiten triangles.

One kind of compactly generated triangulated category $\mathbf{K}(\Lambda\text{-Inj})$, the homotopy category of complexes of injective Λ-modules, was investigated in [7]. An Auslander–Reiten formula in $\mathbf{K}(\Lambda\text{-Inj})$ was presented. This formula contains as a special case the classical Auslander–Reiten formula; and the Auslander–Reiten triangle ending in the injective resolution of a finitely presented indecomposable non-projective module induces the classical almost split sequence in a module category.

In this paper we work on another kind of homotopy category $\mathbf{K}(\Lambda\text{-Proj})$, the homotopy category of complexes of projective Λ-modules. Jørgensen proved in [8] that $\mathbf{K}(\Lambda\text{-Proj})$ is compactly generated when the ring Λ satisfies some hypotheses. Neeman improved this result and showed that $\mathbf{K}(\Lambda\text{-Proj})$ is compactly generated when the ring Λ is right coherent [9, Proposition 7.14]. He even characterized all compact objects in this category [9, Proposition 7.12]. In this paper we assume that Λ is a noetherian k-algebra, where k is a commutative noetherian ring which is complete and local. In this case $\mathbf{K}(\Lambda\text{-Proj})$ is compactly generated. We present an Auslander–Reiten formula in $\mathbf{K}(\Lambda\text{-Proj})$, which can be used to describe the end terms of Auslander–Reiten triangles explicitly. Moreover, an Auslander–Reiten triangle starting from the projective resolution of an indecomposable non-injective artin module induces an almost split sequence in a module category.

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Let us add two remarks. First we point out that one cannot deduce the results in $\mathbf{K}(\Lambda\text{-Proj})$ from the results in $\mathbf{K}(\Lambda\text{-Inj})$ just by duality. For example, we would like to emphasize that in the proof of the Auslander–Reiten formula in $\mathbf{K}(\Lambda\text{-Proj})$, we make essential use of the right adjoint of the inclusion functor $\mathbf{K}(\Lambda\text{-Proj}) \longrightarrow \mathbf{K}(\Lambda\text{-Flat})$. Note that this adjoint was constructed in recent work of Neeman [9]. Secondly, we remark that if there is a dualizing complex for Λ, then there is an equivalence of triangulated categories $\mathbf{K}(\Lambda\text{-Proj}) \cong \mathbf{K}(\Lambda\text{-Inj})$ (see [10]), and this equivalence preserves Auslander–Reiten triangles. Consequently, it will be interesting to compare the Auslander–Reiten triangles in $\mathbf{K}(\Lambda\text{-Proj})$ and $\mathbf{K}(\Lambda\text{-Inj})$.

Before ending the introduction, let us fix some notations in this paper. All rings Λ in this paper will be assumed associative, with unit. We will consider the category Λ-Mod of left Λ-modules and the following full subcategories:

Λ-mod = the finitely presented Λ-modules,
Λ-Flat = the flat Λ-modules,
Λ-Inj = the injective Λ-modules,
Λ-Proj = the projective Λ-modules,
Λ-proj = the finitely generated projective Λ-modules.

We view right Λ-modules as left Λ^{op}-modules where Λ^{op} is the opposite ring of Λ.

Given any additive category $\mathcal{A}$, we denote by $\mathbf{C}(\mathcal{A})$ the category of cochain complexes in $\mathcal{A}$, and we write $\mathbf{K}(\mathcal{A})$ for the category of cochain complexes up to homotopy. If $\mathcal{A}$ is abelian, the derived category is denoted by $\mathbf{D}(\mathcal{A})$. We denote by $\mathbf{C}^+(\mathcal{A})$, $\mathbf{C}^-(\mathcal{A})$ and $\mathbf{C}^b(\mathcal{A})$ the full subcategories of $\mathbf{C}(\mathcal{A})$ consisting of the bounded below, bounded above and bounded cochain complexes. For $* \in \{+, -, b\}$, let $\mathbf{K}^*(\mathcal{A})$ ($\mathbf{D}^*(\mathcal{A})$, respectively) be the full subcategory of $\mathbf{K}(\mathcal{A})$ ($\mathbf{D}(\mathcal{A})$, respectively) corresponding to $\mathbf{C}^*(\mathcal{A})$.

Usually we use $\mathcal{T}$ to denote a triangulated category, and use Σ to denote its shift functor. For homotopy categories and derived categories, we also use [1] to denote their shift functors.

2. The homotopy category of projectives

Let Λ be a ring. We recall some basic properties of the homotopy category $\mathbf{K}(\Lambda\text{-Proj})$ in this section. Detailed proof of these properties can be found in the recent paper of Neeman [9].

First let $X = (X^n, d_X^n)$ be a complex of Λ-modules. Denote by X^* the complex of Λ^{op}-modules with $(X^*)^n = \mathrm{Hom}_\Lambda(X^{-n}, \Lambda)$ and the differential $d_{X^*}^n = \mathrm{Hom}_\Lambda(d_X^{-n-1}, \Lambda)$.

Recall that an object X in an additive category is *compact* if every map $X \to \coprod_{i\in I} Y_i$ factors through $\coprod_{i\in J} Y_i$ for some finite $J \subseteq I$. The following lemma characterizes the compact objects in $\mathbf{K}(\Lambda\text{-Proj})$ explicitly.

Lemma 2.1. *An object of* $\mathbf{K}(\Lambda\text{-Proj})$ *is compact if and only if it is isomorphic to a complex* X *satisfying*

(1) *X is a complex of finitely generated projective modules.*
(2) *$X^n = 0$ for $n \ll 0$.*
(3) *$H^n(X^*) = 0$ for $n \ll 0$.*

Proof. See [9, Proposition 7.12]. □

Let $\mathcal{T}$ be a triangulated category which admits coproducts. Then $\mathcal{T}$ is said to be *compactly generated* if it coincides with the smallest full triangulated subcategory closed under all coproducts and containing all compact objects. Note that $\mathbf{K}(\Lambda\text{-Proj})$ is a triangulated category with arbitrary coproducts. It is compactly generated in many cases.

Lemma 2.2. *If Λ is a right coherent ring, then the triangulated category* $\mathbf{K}(\Lambda\text{-Proj})$ *is compactly generated.*

Proof. See [9, Proposition 7.14]. □

Recall that a ring is *right coherent* if every finitely generated right ideal is finitely presented. All right noetherian rings are right coherent.

Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ be a triangulated subcategory. The *right orthogonal subcategory* with respect to $\mathcal{S}$ is the full subcategory $\mathcal{S}^\perp$ with objects

$$\mathrm{Obj}(\mathcal{S}^\perp) = \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(Y, X) = 0, \forall Y \in \mathcal{S}\}.$$

It is easy to check that $\mathcal{S}^\perp$ is a triangulated subcategory. The left orthogonal subcategory ${}^\perp\mathcal{S}$ is defined similarly. It is also a triangulated subcategory of $\mathcal{T}$.

The following lemma is well known, see for example [11, Section 3].

Lemma 2.3. *The following statements are equivalent:*

(1) *The inclusion functor* $\mathrm{inc}_{\mathcal{S}}\colon \mathcal{S} \longrightarrow \mathcal{T}$ *has a right adjoint G.*
(2) *The inclusion functor* $\mathrm{inc}_{\mathcal{S}^\perp}\colon \mathcal{S}^\perp \longrightarrow \mathcal{T}$ *has a left adjoint F, and ${}^\perp(\mathcal{S}^\perp) = \mathcal{S}$.*

(3) *For each object X in $\mathcal{T}$, there is a unique (up to a unique isomorphism) exact triangle*

$$\varepsilon_X : X' \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} X'' \to \Sigma(X')$$

with $X' \in \mathcal{S}$ and $X'' \in \mathcal{S}^{\perp}$.

Actually if $\mathcal{S}$ satisfies the equivalent conditions above, then ε_X has the form

$$\varepsilon_X : G(X) \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} F(X) \to \Sigma \circ G(X).$$

We describe two examples which will be used in the later sections.

Example 2.4. Let Λ be a ring. Denote by $\mathbf{K}_{\text{proj}}(\Lambda\text{-Mod})$ the smallest triangulated subcategory of $\mathbf{K}(\Lambda\text{-Mod})$ which is closed under coproducts and contains all projective Λ-modules. Note that $\mathbf{K}^{-}(\Lambda\text{-Proj}) \subseteq \mathbf{K}_{\text{proj}}(\Lambda\text{-Mod}) \subseteq \mathbf{K}(\Lambda\text{-Proj})$. Complexes in $\mathbf{K}_{\text{proj}}(\Lambda\text{-Mod})$ are said to be *K-projective*. Consider the inclusion $\mathbf{K}_{\text{proj}}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}(\Lambda\text{-Mod})$. This inclusion has a right adjoint which we denote by $\mathbf{p}$. For each complex X we call $\mathbf{p}X$ its *projective resolution*. The full subcategory $\mathbf{K}_{\text{proj}}(\Lambda\text{-Mod})^{\perp}$ is equal to $\mathbf{K}_{\text{ac}}(\Lambda\text{-Mod})$, which is formed by all acyclic complexes. The inclusion $\mathbf{K}_{\text{ac}}(\Lambda\text{-Mod}) \longrightarrow \mathbf{K}(\Lambda\text{-Mod})$ has a left adjoint which we denote by $\mathbf{a}$. By Lemma 2.3, for each complex X of Λ-modules there exists a unique exact triangle

$$\mathbf{p}X \xrightarrow{\alpha_X} X \xrightarrow{\beta_X} \mathbf{a}X \to \mathbf{p}X[1],$$

where $\mathbf{p}X$ is K-projective and $\mathbf{a}X$ is acyclic. Note that $\mathbf{p}$ preserves coproducts. Moreover, the composite $\mathbf{K}_{\text{proj}}(\Lambda\text{-Mod}) \xrightarrow{\text{inc}} \mathbf{K}(\Lambda\text{-Mod}) \xrightarrow{\text{can}} \mathbf{D}(\Lambda\text{-Mod})$ is an equivalence of triangulated categories. Hence for any complex X and K-projective complex Y, there are natural isomorphisms

$$\text{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}(Y, X) \cong \text{Hom}_{\mathbf{K}_{\text{proj}}(\Lambda\text{-Mod})}(Y, \mathbf{p}X) \cong \text{Hom}_{\mathbf{D}(\Lambda\text{-Mod})}(Y, \mathbf{p}X) \cong \text{Hom}_{\mathbf{D}(\Lambda\text{-Mod})}(Y, X).$$

We refer to [12, Section 8.1.1 and Section 8.1.2] and [13, Section 5] for the detailed proof.

Example 2.5. Let Λ be a ring. Consider the inclusion $\mathbf{K}(\Lambda\text{-Proj}) \longrightarrow \mathbf{K}(\Lambda\text{-Flat})$. This inclusion admits a right adjoint which we denote by j^*, and j^* preserves coproducts, see [9, Proposition 8.1]. Note that the existence of j^* was shown in [10, Proposition 2.4] in a special case. Complexes in $\mathbf{K}(\Lambda\text{-Proj})^{\perp}$ can be characterized as follows: a complex $X = (X^n, d_X^n)$ of flat Λ-modules lies in $\mathbf{K}(\Lambda\text{-Proj})^{\perp}$ if and only if X is acyclic and the image of d_X^n is a flat module for each $n \in \mathbb{Z}$ [9, Theorem 8.6]. The inclusion $\mathbf{K}(\Lambda\text{-Proj})^{\perp} \longrightarrow \mathbf{K}(\Lambda\text{-Flat})$ has a left adjoint i^*. Thus by Lemma 2.3 for each complex X of flat Λ-modules there is a unique exact triangle

$$j^*(X) \xrightarrow{\alpha'_X} X \xrightarrow{\beta'_X} i^*(X) \to j^*(X)[1].$$

3. On Neeman's adjoint functor

Let Λ be a ring. Neeman [9] pointed out the existence of the right adjoint j^* of the inclusion functor $\mathbf{K}(\Lambda\text{-Proj}) \longrightarrow \mathbf{K}(\Lambda\text{-Flat})$, but did not say too much about this functor. Instead, as we just saw in Example 2.5, he gave a characterization of the orthogonal category $\mathbf{K}(\Lambda\text{-Proj})^{\perp}$. In this section, using his characterization, we calculate the image of some special complexes under j^*, and we find out that they are just the projective resolutions of these complexes.

We introduce an easy lemma first which will be used frequently.

Lemma 3.1. *Let ε and ε' be two exact triangles in a triangulated category $\mathcal{T}$. Consider a morphism $g : Y \to Y'$. If $\text{Hom}_{\mathcal{T}}(X, Z') = 0$, then there are morphisms f and h completing g to a morphism of triangles. Furthermore, if $\text{Hom}_{\mathcal{T}}(X, \Sigma^{-1}Z') = 0$, then such f and h are unique.*

$$\begin{array}{llcccccc}
\varepsilon: & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma(X) \\
& \Big\downarrow{\scriptstyle \exists f} & (\star) & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle \exists h} & & \Big\downarrow{\scriptstyle \Sigma(f)} \\
\varepsilon': & X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma(X').
\end{array}$$

Proof. Applying $\text{Hom}_{\mathcal{T}}(X, -)$ to ε' we get a long exact sequence

$$\text{Hom}_{\mathcal{T}}(X, \Sigma^{-1}(Z')) \xrightarrow{(-\Sigma^{-1}(\gamma))_*} \text{Hom}_{\mathcal{T}}(X, X') \xrightarrow{\alpha'_*} \text{Hom}_{\mathcal{T}}(X, Y') \xrightarrow{\beta'_*} \text{Hom}_{\mathcal{T}}(X, Z').$$

Note that α'_* is an epimorphism because $\text{Hom}_{\mathcal{T}}(X, Z') = 0$. Thus $g \circ \alpha$ is in the image of α'_*, and there is some f such that the square $(\star)$ is commutative. If moreover $\text{Hom}_{\mathcal{T}}(X, \Sigma^{-1}Z') = 0$, then α'_* is an isomorphism, hence such f is unique. The proof of the existence and the uniqueness of h is similar. □

Given a complex X of flat Λ-modules, we have two exact triangles as in Examples 2.4 and 2.5. Since $i^*(X)$ is acyclic, we know that $i^*(X) \in \mathbf{K}_{\mathrm{proj}}(\Lambda\text{-Mod})^{\perp}$. From Lemma 3.1 there is a unique map ξ_X such that the following diagram is commutative

$$\begin{array}{ccccccc} \mathbf{p}X & \xrightarrow{\alpha_X} & X & \xrightarrow{\beta_X} & \mathbf{a}X & \longrightarrow & \mathbf{p}X[1] \\ \downarrow{\scriptstyle \xi_X} & & \| & & \downarrow & & \downarrow \\ j^*(X) & \xrightarrow{\alpha'_X} & X & \xrightarrow{\beta'_X} & i^*(X) & \longrightarrow & j^*(X)[1]. \end{array}$$

Define a functor $\mathbf{r}$ to be the composite

$$\mathbf{K}(\Lambda\text{-Flat}) \xrightarrow{\mathbf{p}|_{\mathbf{K}(\Lambda\text{-Flat})}} \mathbf{K}_{\mathrm{proj}}(\Lambda\text{-Mod}) \xrightarrow{\mathrm{inc}} \mathbf{K}(\Lambda\text{-Proj}).$$

Note that $\mathbf{r}(X) = \mathbf{p}X$.

Lemma 3.2. *The map ξ_X induces a morphism of triangle functors $\xi : \mathbf{r} \to j^*$.*

Proof. Straightforward. □

Lemma 3.3. *Let M be a flat Λ-module. Consider M as a complex concentrated in degree zero. Then ξ_M is an isomorphism.*

Proof. First note that $\mathbf{a}M$ is an acyclic complex $\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$, where all P_i are projective Λ-modules. Thus it is an acyclic complex of flat modules. Moreover, it is not difficult to see that for each i, $\mathrm{Im}\, d_i$ is a flat module by considering the short exact sequence

$$0 \to \mathrm{Im}\, d_i \to P_{i-1} \to \mathrm{Im}\, d_{i-1} \to 0$$

and using induction. Therefore $\mathbf{a}M \in \mathbf{K}(\Lambda\text{-Proj})^{\perp}$, and by Lemma 3.1 we get a unique map ξ'_M such that the following diagram is commutative

$$\begin{array}{ccccccc} j^*(M) & \xrightarrow{\alpha'_M} & M & \xrightarrow{\beta'_M} & i^*(M) & \longrightarrow & j^*(M)[1] \\ \downarrow{\scriptstyle \xi'_M} & & \| & & \downarrow & & \downarrow \\ \mathbf{p}M & \xrightarrow{\alpha_M} & M & \xrightarrow{\beta_M} & \mathbf{a}M & \longrightarrow & \mathbf{p}M[1]. \end{array}$$

Again the uniqueness in Lemma 3.1 implies that $\xi'_M \circ \xi_M = \mathrm{id}_{\mathbf{p}(M)}$ and $\xi_M \circ \xi'_M = \mathrm{id}_{j^*(M)}$. Thus ξ_M is an isomorphism. □

The following lemma is not difficult to prove.

Lemma 3.4. *Let $\mathcal{S}$ and $\mathcal{T}$ be two triangulated categories which admit coproducts, and $G, G' : \mathcal{T} \longrightarrow \mathcal{S}$ be two triangle functors preserving coproducts. Suppose that $\xi : G \longrightarrow G'$ is a morphism of triangle functors. Consider the full subcategory*

$$\mathcal{T}_{\xi} := \{X \in \mathcal{T} \mid \xi_X \text{ is an isomorphism}\}.$$

Then $\mathcal{T}_{\xi}$ is a triangulated subcategory, closed under coproducts.

Proposition 3.5 (*cf.* [10, Theorem 2.7(2)]). *Let X be a bounded above complex of flat modules. Then ξ_X is an isomorphism.*

Proof. Take $\mathbf{K}_{\xi}(\Lambda\text{-Flat}) = \{X \in \mathbf{K}(\Lambda\text{-Flat}) \mid \xi_X \text{ is an isomorphism}\}$. By Lemma 3.2 we know that $\xi : \mathbf{r} \to j^*$ is a morphism of triangle functors. Hence Lemma 3.4 implies that $\mathbf{K}_{\xi}(\Lambda\text{-Flat})$ is a triangulated subcategory of $\mathbf{K}(\Lambda\text{-Flat})$, closed under coproducts.

If X is a complex concentrated in one degree, then from Lemma 3.3 we know that ξ_X is an isomorphism. Therefore the statement holds for any bounded complex by using induction. In unbounded case, without loss of generality, we may assume that X has the form

$$X : \cdots \xrightarrow{d^{-3}} X^{-2} \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \to 0.$$

Define

$$X_{\geq n} : 0 \to X^{-n} \xrightarrow{d^{-n}} \cdots \xrightarrow{d^{-2}} X^{-1} \xrightarrow{d^{-1}} X^0 \to 0.$$

Then there is a sequence of complexes

$$X_{\geq 0} \xrightarrow{\varphi_0} X_{\geq 1} \xrightarrow{\varphi_1} X_{\geq 2} \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} X_{\geq n} \xrightarrow{\varphi_n} \cdots$$

where φ_n is the canonical map. Note that $X = \varinjlim X_{\geq n}$, and we have an exact sequence of chain maps

$$\coprod_{n=0}^{\infty} X_{\geq n} \xrightarrow{1-\varphi} \coprod_{n=0}^{\infty} X_{\geq n} \to X \to 0,$$

where φ is induced by φ_n's. It is easy to see that $1 - \varphi$ is a monomorphism and split in each degree. Recall that a short exact sequence of complexes which is split in each degree induces an exact triangle in the homotopy category. Hence we get an exact triangle in $\mathbf{K}(\Lambda\text{-Flat})$

$$\coprod_{n=0}^{\infty} X_{\geq n} \xrightarrow{1-\varphi} \coprod_{n=0}^{\infty} X_{\geq n} \to X \to \left(\coprod_{n=0}^{\infty} X_{\geq n}\right)[1].$$

Since $X_{\geq n}$ is a bounded complex and $\mathbf{K}_\xi(\Lambda\text{-Flat})$ is closed under coproducts, the complex $\coprod_{n=0}^{\infty} X_{\geq n}$ is an object in $\mathbf{K}_\xi(\Lambda\text{-Flat})$. Therefore X is also an object in $\mathbf{K}_\xi(\Lambda\text{-Flat})$ because $\mathbf{K}_\xi(\Lambda\text{-Flat})$ is a triangulated subcategory of $\mathbf{K}(\Lambda\text{-Flat})$. □

4. The Auslander–Reiten formula

In this section, the Auslander–Reiten formula for complexes is proved.

Let k be a commutative noetherian ring which is complete and local. Throughout this section, we fix a noetherian k-algebra Λ. By a noetherian k-algebra Λ we mean a k-algebra which is finitely generated as a module over k. We denote by Λ-Noeth and Λ-Art the full subcategories of Λ-Mod consisting of the noetherian and artinian Λ-modules, respectively. Note that the assumptions on Λ imply that Λ-Noeth is equal to Λ-mod, the category of finitely presented Λ-modules.

In addition, we fix an injective envelope $E = E(k/\mathfrak{m})$, where $\mathfrak{m}$ denotes the unique maximal ideal of k. It is not difficult to see that E is an injective cogenerator for k-Mod. We have a functor

$$D = \mathrm{Hom}_k(-, E) \colon k\text{-Mod} \longrightarrow k\text{-Mod}$$

which induces a functor between Λ-Mod and Λ^{op}-Mod, and this functor restricts to a duality $D\colon \Lambda\text{-Art} \to \Lambda^{\mathrm{op}}\text{-Noeth}$. Furthermore, we have the following observation.

Lemma 4.1. *Let M be an injective Λ^{op}-module. Then the Λ-module DM is flat.*

Proof. The proof is similar to [14, Corollary 3.2.17]. □

We are now in a position to present the Auslander–Reiten formula in $\mathbf{K}(\Lambda\text{-Proj})$.

Theorem 4.2. *Let Z and Y be complexes of projective Λ-modules, and suppose that Z is a compact object in $\mathbf{K}(\Lambda\text{-Proj})$. Then we have an isomorphism*

$$D\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Z, Y) \cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Y, \mathbf{p}DZ^*) \tag{4.1}$$

which is natural in Z and Y.

To prove the theorem, we need several lemmas.

Lemma 4.3 *([7, Lemma 3.5]). Let Z, Z' be objects in a k-linear compactly generated triangulated category $\mathcal{T}$, and suppose that Z is compact. If there is a natural isomorphism*

$$D\mathrm{Hom}_{\mathcal{T}}(Z, Y) \cong \mathrm{Hom}_{\mathcal{T}}(Y, Z')$$

for all compact $Y \in \mathcal{T}$, then $D\mathrm{Hom}_{\mathcal{T}}(Z, -) \cong \mathrm{Hom}_{\mathcal{T}}(-, Z')$.

Lemma 4.4. *Let Z and Y be complexes of finitely generated projective Λ-modules. Then*

$$\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Z, Y) \cong \mathrm{Hom}_{\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Proj})}(Y^*, Z^*). \tag{4.2}$$

Given a pair of complexes X, Y of modules over Λ or Λ^{op}, we denote by $\mathrm{Hom}_\Lambda(X, Y)$ and $X \otimes_\Lambda Y$ the total Hom and the total tensor product respectively, which are complexes of k-modules. We denote by H^0X the cohomology group of X in degree zero.

Lemma 4.5. *Let X, Y be complexes of Λ-modules. Then we have an isomorphism*

$$H^0\mathrm{Hom}_\Lambda(X, Y) \cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}(X, Y). \tag{4.3}$$

Proof. See [15, *Section* 10.7]. □

Lemma 4.6. *Let X be a complex of Λ-modules. Given an injective Λ-module I, we have an isomorphism*

$$\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Mod})}(X, I) \cong \mathrm{Hom}_{\Lambda}(H^0X, I). \tag{4.4}$$

Proof. See [13, Example 1.5]. □

Lemma 4.7 ([7, Lemma 3.1]). *Let Z, Y be complexes in $\mathbf{C}(\Lambda^{\mathrm{op}}\text{-Mod})$. Then we have in $\mathbf{C}(k\text{-Mod})$ a natural map*

$$Z \otimes_{\Lambda} \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y, \Lambda) \longrightarrow \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y, Z), \tag{4.5}$$

which is an isomorphism if $Y \in \mathbf{C}^-(\Lambda^{\mathrm{op}}\text{-proj})$ and $Z \in \mathbf{C}^+(\Lambda^{\mathrm{op}}\text{-Mod})$.

Proof (*Proof of* Theorem 4.2). We use the fact that $\mathbf{K}(\Lambda\text{-Proj})$ is compactly generated. Therefore by Lemma 4.3 it is sufficient to verify the isomorphism for every compact object Y. Note that this implies Y^* and Z^* to be complexes of finitely generated projective Λ^{op}-modules, and in particular $Y^* \cong \mathbf{p}Y^*$. From Example 2.4 we know that $\mathrm{Hom}_{\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Mod})}(\mathbf{p}Y^*, Z^*) \longrightarrow \mathrm{Hom}_{\mathbf{D}(\Lambda^{\mathrm{op}}\text{-Mod})}(\mathbf{p}Y^*, Z^*)$ is a natural isomorphism. Combining with the quasi-isomorphism $\mathbf{p}Y^* \longrightarrow Y^*$ we get $\mathrm{Hom}_{\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Mod})}(\mathbf{p}Y^*, Z^*) \cong \mathrm{Hom}_{\mathbf{D}(\Lambda^{\mathrm{op}}\text{-Mod})}(Y^*, Z^*)$. Similarly there is another natural isomorphism $\mathrm{Hom}_{\mathbf{D}(\Lambda^{\mathrm{op}}\text{-Mod})}(Y^*, Z^*) \cong \mathrm{Hom}_{\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Mod})}(Y^*, \mathbf{i}Z^*)$, where $\mathbf{i}Z^*$ is the injective resolution of Z^* (see [12, Section 8.1.2]). We obtain the following sequence of isomorphisms, where short arguments are added on the right-hand side.

$$\begin{aligned}
D\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Z, Y) &\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Proj})}(Y^*, Z^*), E) && \text{from (4.2)}\\
&\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Proj})}(\mathbf{p}Y^*, Z^*), E) && Y \text{ compact}\\
&\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathbf{D}(\Lambda^{\mathrm{op}}\text{-Mod})}(Y^*, Z^*), E)\\
&\cong \mathrm{Hom}_k(\mathrm{Hom}_{\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Mod})}(Y^*, \mathbf{i}Z^*), E)\\
&\cong \mathrm{Hom}_k(H^0\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y^*, \mathbf{i}Z^*), E) && \text{from (4.3)}\\
&\cong \mathrm{Hom}_{\mathbf{K}(k\text{-Mod})}(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y^*, \mathbf{i}Z^*), E) && \text{from (4.4)}\\
&\cong H^0\mathrm{Hom}_k(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y^*, \mathbf{i}Z^*), E) && \text{from (4.3)}\\
&\cong H^0\mathrm{Hom}_k(\mathbf{i}Z^* \otimes_{\Lambda} \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(Y^*, \Lambda), E) && \text{from (4.5)}\\
&\cong H^0\mathrm{Hom}_k(\mathbf{i}Z^* \otimes_{\Lambda} Y, E) && Y \text{ compact}\\
&\cong H^0\mathrm{Hom}_{\Lambda}(Y, \mathrm{Hom}_k(\mathbf{i}Z^*, E)) && \text{adjunction}\\
&\cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Flat})}(Y, D\mathbf{i}Z^*) && \text{by Lemma 4.1}\\
&\cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Y, j^*(D\mathbf{i}Z^*)) && \text{adjunction}\\
&\cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Y, \mathbf{p}D\mathbf{i}Z^*) && \text{by Proposition 3.5}\\
&\cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Y, \mathbf{p}DZ^*).
\end{aligned}$$

These isomorphisms complete the proof. □

Remark 4.8. It is interesting to compare the Auslander–Reiten formula in $\mathbf{K}(\Lambda\text{-Proj})$ with the one in $\mathbf{K}(\Lambda\text{-Inj})$. Note that for any compact object Z in $\mathbf{K}(\Lambda\text{-Proj})$ there is an isomorphism $DZ^* \cong D\Lambda \otimes_{\Lambda} Z$ since Z^n is finitely presented over Λ. Hence the isomorphism (4.1) can be written as

$$D\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Z, Y) \cong \mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(Y, \mathbf{p}(D\Lambda \otimes_{\Lambda} Z)).$$

Comparing with the formula (3.4) in [7], we notice that the only difference is the exchange of the order of functors $\mathbf{p}$ and $D\Lambda \otimes_{\Lambda} -$. This phenomenon is well understood when Λ is an artin k-algebra (by an artin k-algebra we mean that k is a commutative artinian ring, and Λ is a k-algebra which is finitely generated over k), because in this case there is an equivalence $D\Lambda \otimes_{\Lambda} - : \mathbf{K}(\Lambda\text{-Proj}) \longrightarrow \mathbf{K}(\Lambda\text{-Inj})$ (see [10, Theorem 4.2]).

5. Auslander–Reiten triangles

Keep the assumptions on Λ as in Section 4. In this section, we produce Auslander–Reiten triangles in the category $\mathbf{K}(\Lambda\text{-Proj})$, using the Auslander–Reiten formula for complexes. In addition, we show that almost split sequences can be obtained from Auslander–Reiten triangles. This yields a simple recipe for the construction of an almost split sequence.

Let us recall the relevant definitions from Auslander–Reiten theory. A map $\alpha : X \to Y$ is said to be *left almost split*, if α is not a section and if every map $X \to Y'$ which is not a section factors through α. Dually, a map $\beta : Y \to Z$ is *right almost split*, if β is not a retraction and if every map $Y' \to Z$ which is not a retraction factors through β.

Definition 5.1. (1) An exact sequence $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ in an abelian category is called an *almost split sequence*, if α is left almost split and β is right almost split.

(2) An exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} X[1]$ in a triangulated category is called an *Auslander–Reiten triangle*, if α is left almost split and β is right almost split.

Krause proved the existence result for Auslander–Reiten triangles in compactly generated triangulated categories; see [6]. This yields the following.

Proposition 5.2. *Let Z be a compact object in* $\mathbf{K}(\Lambda\text{-Proj})$ *which is indecomposable. Then there exists an Auslander–Reiten triangle*

$$\mathbf{p}DZ^*[-1] \to Y \to Z \to \mathbf{p}DZ^*. \tag{5.1}$$

Proof. Denote by $\mathbf{K}^{-,b}(\Lambda^{\mathrm{op}}\text{-proj})$ the full subcategory of $\mathbf{K}(\Lambda^{\mathrm{op}}\text{-proj})$ consisting of the bounded above complexes X with finite non-zero cohomology groups. From Lemma 2.1 we know that Z^* is contained in $\mathbf{K}^{-,b}(\Lambda^{\mathrm{op}}\text{-proj})$, while the assumptions on Λ guarantee that $\mathbf{K}^{-,b}(\Lambda^{\mathrm{op}}\text{-proj}) \cong \mathbf{D}^b(\Lambda^{\mathrm{op}}\text{-mod})$ and $\mathbf{D}^b(\Lambda^{\mathrm{op}}\text{-mod})$ is Krull–Schmidt. Therefore Z indecomposable implies that the endomorphism ring $\Gamma = \mathrm{End}_{\mathbf{K}(\Lambda\text{-Proj})}(Z) \cong \mathrm{End}_{\mathbf{K}(\Lambda^{\mathrm{op}}\text{-Proj})}(Z^*)$ is local. Let $I = E(\Gamma/\mathrm{rad}\,\Gamma)$ and observe that the functor $\mathrm{Hom}_\Gamma(-, I)$ is isomorphic to $D = \mathrm{Hom}_k(-, E)$. Applying formula (4.1) to Theorem 2.2 from [6], we get an Auslander–Reiten triangle in the above form. □

Remark 5.3. Define a functor

$$\mathbf{t}\colon \mathbf{K}(\Lambda\text{-Proj}) \xrightarrow{\mathrm{Hom}_\Lambda(-,\Lambda)} \mathbf{K}(\Lambda^{\mathrm{op}}\text{-Proj}) \xrightarrow{D} \mathbf{K}(\Lambda\text{-Inj}) \xrightarrow{\mathbf{p}} \mathbf{K}(\Lambda\text{-Proj}).$$

This functor restricts to an equivalence

$$\mathbf{t}\colon \mathbf{K}^c(\Lambda\text{-Proj}) \longrightarrow \mathcal{S},$$

where $\mathbf{K}^c(\Lambda\text{-Proj})$ is the full subcategory of $\mathbf{K}(\Lambda\text{-Proj})$ consisting of compact objects, and $\mathcal{S}$ is the full subcategory of $\mathbf{K}(\Lambda\text{-Proj})$ consisting of complexes isomorphic to some X with the form: (1) $X^n = 0$ for $n \gg 0$; (2) $H^n(X) = 0$ for $n \ll 0$; (3) $H^n(X)$ is artinian over Λ for all n. The quasi-inverse of $\mathbf{t}$ maps each complex X to $(\mathbf{p}DX)^*$. Hence for each X in $\mathcal{S}$ which is indecomposable, there is an Auslander–Reiten triangle

$$X[-1] \to Y \to (\mathbf{p}DX)^* \to X. \tag{5.2}$$

Let $X = (X^n, d_X^n)$ be a complex of Λ-modules. Denote by $\mathrm{Cok}^1(X)$ the Λ-module $X^1/\mathrm{Im}\, d_X^0$. For a finitely presented Λ-module we have the transpose construction Tr (see [4, page 7]). The last theorem shows that an Auslander–Reiten triangle starting from the projective resolution of an indecomposable non-injective artinian module induces an almost split sequence. We would like to remind the readers a lemma first.

Lemma 5.4. *Let N be a finitely presented right* Λ*-module. Then N admits a minimal projective resolution*

$$\cdots \to P_2 \xrightarrow{\mu_2} P_1 \xrightarrow{\mu_1} P_0 \to 0$$

with all P_i *finitely generated.*

Proof. First k complete implies that the k-algebra Λ is semiperfect, i.e., every finitely generated Λ-module has a finitely generated projective cover. Combining with the fact that Λ is noetherian we obtain the minimal projective resolution of N with finitely generated components. Compare [4, page 7]. □

Theorem 5.5. *Let M be a left artinian* Λ*-module which is indecomposable and non-injective. Then there exists an Auslander–Reiten triangle*

$$\mathbf{p}M[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} (\mathbf{p}DM)^* \xrightarrow{\gamma} \mathbf{p}M$$

in $\mathbf{K}(\Lambda\text{-Proj})$ *which the functor* Cok^1 *sends to an almost split sequence*

$$0 \to M \xrightarrow{\overline{\alpha^1}} \mathrm{Cok}^1(Y) \xrightarrow{\overline{\beta^1}} \mathrm{Tr}DM \to 0$$

in the category of Λ*-modules.*

Proof. Take $X = \mathbf{p}M$ and observe that $\mathbf{p}D\mathbf{p}M \cong \mathbf{p}DM$ sice $\mathbf{p}M$ is quasi-isomorphic to M and D is an exact functor. The Auslander–Reiten triangle (5.2) in Remark 5.3 has the form now

$$\mathbf{p}M[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} (\mathbf{p}DM)^* \xrightarrow{\gamma} \mathbf{p}M.$$

Note that DM is finitely presented, thus from Lemma 5.4 we may assume that the projective resolution $\mathbf{p}DM$ is minimal with finitely generated components. Using the fact that $\mathbf{K}(\Lambda\text{-Proj})$ is the stable category of the Frobenius category $\mathbf{C}(\Lambda\text{-Proj})$, we may choose a sequence of chain maps

$$0 \to \mathbf{p}M[-1] \xrightarrow{\alpha} Y \xrightarrow{\beta} (\mathbf{p}DM)^* \to 0$$

which is split exact in each degree [16, Section 6].

The functor Cok^1 takes this sequence to an exact sequence

$$\mathrm{Cok}^1(\mathbf{p}M[-1]) \xrightarrow{\overline{\alpha^1}} \mathrm{Cok}^1(Y) \xrightarrow{\overline{\beta^1}} \mathrm{Cok}^1((\mathbf{p}DM)^*) \to 0.$$

Note that $\mathrm{Cok}^1(\mathbf{p}M[-1]) = M$ and $\mathrm{Cok}^1((\mathbf{p}DM)^*) = \mathrm{Tr}DM$, the exact sequence has the simple form as follows

$$M \xrightarrow{\overline{\alpha^1}} \mathrm{Cok}^1(Y) \xrightarrow{\overline{\beta^1}} \mathrm{Tr}DM \to 0. \tag{5.3}$$

It is clear that $\overline{\alpha^1}$ is not a section, otherwise the left inverse of $\overline{\alpha^1}$ can be lifted to the left inverse of α, contrary to the fact that α is left almost split. Observe also that Cok^1 induces a bijection $\mathrm{Hom}_{\mathbf{K}(\Lambda\text{-Proj})}(\mathbf{p}M[-1], \mathbf{p}N[-1]) \xrightarrow{\phi} \mathrm{Hom}_\Lambda(M, N)$ for all N. For every map $h: M \to N$ in Λ-Mod which is not a section, $\phi^{-1}(h)$ is not a section in $\mathbf{K}(\Lambda\text{-Proj})$, hence it factors through α. Therefore h factors through $\phi(\alpha) = \overline{\alpha^1}$, which means that $\overline{\alpha^1}$ is left almost split. In particular, $\overline{\alpha^1}$ is a monomorphism since M is non-injective.

We complete the proof by pointing out that the ring $\mathrm{End}_\Lambda(\mathrm{Tr}DM)$ is local. We use the fact that the noetherian module $\mathrm{Tr}DM$ is indecomposable [4, Theorem 2.4] and $\mathrm{End}_\Lambda(\mathrm{Tr}DM)$ is semiperfect [4, page 6]. We conclude from the following Lemma 5.6 that the sequence (5.3) is almost split. □

Lemma 5.6. *An exact sequence $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ in an abelian category is almost split if and only if α is left almost split and the endomorphism ring of Z is local.*

Proof. See Proposition II.4.4 in [17]. □

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