Communications in Algebra®, 35: 999–1008, 2007

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870601117654

ALMOST SPLIT CONFLATIONS FOR COMPLEXES OF MODULES

Jue Le
School of Mathematical Sciences, Beijing Normal University, Beijing, China

For an Artin algebra Λ, we study the existence of almost split conflations in the Frobenius category of cochain complexes of injective Λ-modules.

Key Words: Almost split conflation; Auslander–Reiten; Exact category.

2000 Mathematics Subject Classification: 16G70; 18G35.

1. INTRODUCTION

Let k be a commutative Artinian ring. Throughout this article, we fix an Artin k-algebra Λ. We denote by $\mathrm{Mod}\,\Lambda$ the category of right Λ-modules, $\mathrm{mod}\,\Lambda$, $\mathrm{Proj}\,\Lambda$, $\mathrm{proj}\,\Lambda$, $\mathrm{inj}\,\Lambda$, and $\mathrm{Inj}\,\Lambda$ the full subcategories of $\mathrm{Mod}\,\Lambda$ consisting of the finitely generated, the projective, the finitely generated projective, the injective, and finitely generated injective Λ-modules, respectively. Let $\mathrm{rad}\,k$ be the radical of k and $\mathrm{I}(k/\mathrm{rad}\,k)$ be the injective envelope of $k/\mathrm{rad}\,k$. The functor $D = \mathrm{Hom}_k(-, \mathrm{I}(k/\mathrm{rad}\,k)) : \mathrm{Mod}\,k \to \mathrm{Mod}\,k$ induces a functor between $\mathrm{Mod}\,\Lambda$ to $\mathrm{Mod}\,\Lambda^{\mathrm{op}}$, which is a duality when it is restricted to $\mathrm{mod}\,\Lambda$.

The concept of an almost split sequence (or Auslander–Reiten sequence) was introduced by Auslander and Reiten (1975), and it plays an important role in the study of module categories. Later, the existence theorem for almost split sequences was generalized in various directions. The corresponding notion for triangulated categories is called Auslander–Reiten triangle. The study of such triangles was initiated by Happel (1987), and the existence in various kinds of triangulated categories has been studied by many authors; see for instance Happel (1987) for the bounded derived category $\mathbf{D}^b(\mathrm{mod}\,\Lambda)$, and Krause (2000), Krause and Le (2006) for compactly generated triangulated categories. In Reiten and Van Den Bergh (2002), the connection between Serre functors and Auslander–Reiten triangles was discussed. For an exact category, almost split conflations are considered in Gabriel and Roiter (1992) (almost split exact pairs in the terminology of Dräxler et al., 1999). A proof for the existence of almost split conflations in the

Received October 31, 2005; Revised February 28, 2006. Communicated by D. Happel.
Address correspondence to Jue Le, School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China; Fax: +86-10-58808202; E-mail: juele1009@hotmail.com

exact category of complexes of fixed size can be found in Bautista et al. (2004). Beligiannis (2002) presented a unified way to prove the existence of almost split morphisms, almost split sequences, and almost split triangles in abstract homotopy categories.

In Krause and Le (2006), Auslander–Reiten triangles in $\mathbf{K}(\operatorname{Inj}\Lambda)$ were studied. We continue this work and lift the existence theorem to $\mathbf{C}(\operatorname{Inj}\Lambda)$. There are two ways to provide such lifting, as the following commutative diagram shows:

$$\begin{array}{ccc} \text{AR-formula in } \mathbf{K}(\operatorname{Inj}\Lambda) & \longrightarrow & \text{AR-triangles in } \mathbf{K}(\operatorname{Inj}\Lambda) \\ \downarrow & & \downarrow \\ \text{AR-formula in } \mathbf{C}(\operatorname{Inj}\Lambda) & \longrightarrow & \text{almost split conflations in } \mathbf{C}(\operatorname{Inj}\Lambda). \end{array}$$

The Auslander–Reiten formula in $\mathbf{K}(\operatorname{Inj}\Lambda)$ guarantees the existence of Auslander–Reiten triangles (**?**). In Section 3, we use the existence theorem in $\mathbf{K}(\operatorname{Inj}\Lambda)$ directly, and describe the case of $\mathbf{C}(\operatorname{Inj}\Lambda)$ by studying the relation between almost split conflations in a Frobenius category and Auslander–Reiten triangles in its stable category. In Section 4, we propose another method. A map τ is defined and an Auslander–Reiten formula in $\mathbf{C}(\operatorname{Inj}\Lambda)$ is deduced from the Auslander–Reiten formula in $\mathbf{K}(\operatorname{Inj}\Lambda)$. Using this formula, the existence of almost split conflations can be proved directly, with τ the Auslander–Reiten translation.

We end this introduction by pointing out that our discussion of the category $\mathbf{C}(\operatorname{Inj}\Lambda)$ of complexes of injective Λ-modules covers the category $\mathbf{C}(\operatorname{Proj}\Lambda)$ of complexes of projective Λ-modules, because the categories $\operatorname{Inj}\Lambda$ and $\operatorname{Proj}\Lambda$ are well known to be equivalent.

2. CATEGORY OF COMPLEXES

This section is devoted to giving the Frobenius structure on $\mathbf{C}(\operatorname{Inj}\Lambda)$, and discussing some other properties of this category.

2.1. Frobenius Categories

Let $\mathscr{C}$ be an additive category. A pair (α, β) of composable morphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in $\mathscr{C}$ is called *exact* if α is the kernel of β and β is the cokernel of α. An *exact category* is an additive category $\mathscr{C}$ endowed with a class $\mathscr{E}$ of exact pairs which is closed under isomorphisms and satisfies some axioms (see Gabriel and Roiter, 1992 or Keller, 1990). The morphisms α and β appearing in a pair (α, β) in $\mathscr{E}$ are called an *inflation* and a *deflation*, and the pair (α, β) is called a *conflation*.

Let $(\mathscr{C}, \mathscr{E})$ be an exact category. An object I of $\mathscr{C}$ is called *$\mathscr{E}$-injective* if each conflation $I \to Y \to Z$ in $\mathscr{E}$ splits. The *$\mathscr{E}$-projective* objects are defined dually. We say *$\mathscr{C}$ has enough injectives* , if for each $X \in \mathscr{C}$ there is an inflation $X \to IX$ with injective IX. If $\mathscr{C}$ also has enough projectives (i.e., for each $x \in \mathscr{C}$, there is a deflation $PX \to X$ with projective PX), and the classes of projectives and injectives coincide, we call $\mathscr{C}$ a *Frobenius category*. In this case the stable category $\underline{\mathscr{C}}$ is a triangulated category (Happel, 1987), and the exact triangle $X \xrightarrow{\bar{\alpha}} Y \xrightarrow{\bar{\beta}} Z \xrightarrow{\bar{\gamma}} \Sigma X$ in $\underline{\mathscr{C}}$ is obtained by the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \\ \| & & \downarrow & & \downarrow{\scriptstyle \gamma} \\ X & \longrightarrow & IX & \longrightarrow & \Sigma X \end{array}$$

where the two rows are conflations of $\mathscr{C}$.

Now we assume that $\mathscr{A}$ is $\operatorname{Inj}\Lambda$ or $\operatorname{Proj}\Lambda$. Then $(\mathbf{C}(\mathscr{A}), \mathscr{E})$, the category of cochain complexes in $\mathscr{A}$, is an exact category, where $\mathscr{E}$ be the class of composable morphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ such that for each $n \in \mathbf{Z}$, the sequence $0 \to X^n \xrightarrow{\alpha^n} Y^n \xrightarrow{\beta^n} Z^n \to 0$ is split exact (Keller, 1996). Moreover, it is a Frobenius category. A complex is an $\mathscr{E}$-*projective*(=$\mathscr{E}$-injective) if and only if it is null homotopic, i.e., the identity morphism of the complex is null homotopic. Hence its stable category coincides with its homotopy category $\mathbf{K}(\mathscr{A})$, which is a triangulated category, and $\Sigma X = X[1]$ is defined in the following way: Let X be an arbitrary complex, we denote by $X[n]$ the shifted complex with $X[n]^i = X^{n+i}$ and differential $d^i_{X[n]} = (-1)^n d_X^{n+i}$.

For $A \in \mathscr{A}$, consider the complex $J_i(A) = (J^s, d^s)$ with $J^s = 0$ if $s \neq i$, $s \neq i+1$, $J^i = J^{i+1} = A$, $d^i = \mathrm{id}_A$. It is not difficult to prove that all the indecomposable $\mathscr{E}$-projectives in $\mathbf{C}(\mathscr{A})$ are the complexes $J_i(A)$ with A indecomposable.

2.2. Homotopically Minimal Complexes

A complex X in some additive category is called *homotopically minimal*, if every map $\phi : X \to X$ of complexes is an isomorphism provided that ϕ is an isomorphism up to homotopy. Let $\mathscr{A}$ be $\operatorname{Inj}\Lambda$ or $\operatorname{Proj}\Lambda$. From Krause (2005, Appendix B) and its dual result, we know that every complex in $\mathbf{C}(\mathscr{A})$ has a decomposition $X = X' \coprod X''$, such that X' is homotopically minimal, and X'' is null homotopic. Moreover, X' is unique up to isomorphism. Denote by $\mathbf{C}_{\mathscr{P}}(\mathscr{A})$ the full subcategory of $\mathbf{C}(\mathscr{A})$ whose objects are the X in $\mathbf{C}(\mathscr{A})$ with $X = X'$. Then we have the following useful lemma.

Lemma 2.1. *Let X be an object in $\mathbf{C}_{\mathscr{P}}(\mathscr{A})$. Then $\operatorname{End}_{\mathbf{C}(\mathscr{A})}(X)$ is a local ring if and only if $\operatorname{End}_{\mathbf{K}(\mathscr{A})}(X)$ is a local ring.*

Proof. Use that X is homotopically minimal. □

2.3. Indecomposable Objects in $\mathbf{C}^{+,b}(\operatorname{inj}\Lambda)$

It is well known that the bounded derived category $\mathbf{D}^b(\operatorname{mod}\Lambda)$ is a Krull–Schmidt category (cf. Balmer and Schlichting, 2001, Corollary 2.10, and note that if Λ is an Artin algebra, then $\mathbf{D}^b(\operatorname{mod}\Lambda)$ has split idempotents implies that $\mathbf{D}^b(\operatorname{mod}\Lambda)$ is Krull–Schmidt). Using the equivalence $\mathbf{K}^{+,b}(\operatorname{inj}\Lambda) \cong \mathbf{D}^b(\operatorname{mod}\Lambda)$ (Krause, 2005, Proposition 2.3), an object Z in $\mathbf{K}^{+,b}(\operatorname{inj}\Lambda)$ is indecomposable if and only if $\operatorname{End}_{\mathbf{K}(\operatorname{Inj}\Lambda)}(Z)$ is a local ring. Furthermore, we have the following proposition.

Proposition 2.2. *Let Z be an object in $\mathbf{C}^{+,b}(\operatorname{inj}\Lambda)$. Then Z is indecomposable if and only if $\operatorname{End}_{\mathbf{C}(\operatorname{Inj}\Lambda)}(Z)$ is a local ring.*

Proof. The sufficiency is obvious. For the necessity, if Z is $\mathscr{E}$-projective, then $Z = J_i(I)$ with I indecomposable, so $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(Z) \cong \mathrm{End}_\Lambda(I)$ is a local ring. If Z is not an $\mathscr{E}$-projective, then Z is homotopically minimal. We claim that Z is indecomposable in $\mathbf{K}^{+,b}(\mathrm{Inj}\,\Lambda)$. Otherwise, there are two nonzero objects Z_1, Z_2 in $\mathbf{K}^{+,b}(\mathrm{inj}\,\Lambda)$, and an isomorphism $Z \cong Z_1 \coprod Z_2$ in $\mathbf{K}^{+,b}(\mathrm{inj}\,\Lambda)$, where Z_1 can be written as $Z_1' \coprod Z_1''$ with Z_1' homotopically minimal and Z_1'' null homotopy. Similarly, Z_2 can be written as $Z_2' \coprod Z_2''$. Hence $Z \cong Z_1' \coprod Z_2'$ in $\mathbf{C}^{+,b}(\mathrm{inj}\,\Lambda)$, a contradiction. Therefore $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(Z)$ is a local ring since $\mathrm{End}_{\mathbf{K}(\mathrm{Inj}\,\Lambda)}(Z)$ is a local ring. □

2.4. Compact Objects

Recall that an object X in an additive category is *compact* if every map $X \to \coprod_{i\in I} Y_i$ factors through $\coprod_{i\in J} Y_i$ for some finite $J \subseteq I$. Let $\mathscr{A}$ be an additive category with arbitrary coproducts. Denote by $\mathscr{A}^c$ ($\mathbf{C}^c(\mathscr{A})$, $\mathbf{K}^c(\mathscr{A})$, resp.) the full subcategory of $\mathscr{A}$ ($\mathbf{C}(\mathscr{A})$, $\mathbf{K}(\mathscr{A})$, resp.) which is formed by all compact objects. We know well that there is an equivalence $\mathbf{K}^c(\mathrm{Inj}\,\Lambda) \cong \mathbf{K}^{+,b}(\mathrm{inj}\,\Lambda)$ (Krause, 2005, Proposition 2.3). In this subsection, we study $\mathbf{C}^c(\mathscr{A})$, and the compact objects in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$ can be described explicitly as a corollary.

Proposition 2.3.

$$\mathbf{C}^c(\mathscr{A}) = \mathbf{C}^b(\mathscr{A}^c).$$

Proof. On the one hand, let X be a bounded complex with compact components and $f : X \to \coprod_{i\in I} Y_i$ be a chain map, then $f^s : X^s \to \coprod_{i\in I} Y_i^s$ factors through a finite subsum $\coprod_{i\in J_s} Y_i^s$. Hence f factors through the subsum indexed by the union of the J_s with $X^s \neq 0$, which implies that X is compact in $\mathbf{C}(\mathscr{A})$.

On the other hand, if $X = (X^s, d^s)$ is a compact object in $\mathbf{C}(\mathscr{A})$, then the morphism $\iota : X \to \coprod_{i=-\infty}^{+\infty} J_i(X^{i+1})$ which is given by

$$\begin{array}{llllll}
X: & \cdots \longrightarrow & X_n & \xrightarrow{d^n} & X^{n+1} & \longrightarrow \cdots \\
 & & \Big\downarrow {\scriptstyle \binom{1}{d^n}} & & \Big\downarrow {\scriptstyle \binom{1}{d^{n+1}}} & \\
\coprod_{i=-\infty}^{+\infty} J_i(X^{i+1}): & \cdots \longrightarrow & X^n \coprod X^{n+1} & \xrightarrow{\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)} & X^{n+1} \coprod X^{n+2} & \longrightarrow \cdots
\end{array}$$

factors through a finite subsum. Hence only finitely many X^i are nonzero. This completes the proof that X is bounded.

Next we show that $X^s \in \mathscr{A}^c$ for each s. In fact, for any morphism $f : X^s \to \coprod_{i\in I} A_i$, we may consider the following chain map

$$\begin{array}{lllllll}
X: & \cdots \longrightarrow & X^{s-1} & \xrightarrow{d^{s-1}} & X^s & \xrightarrow{d^s} & X^{s+1} \longrightarrow \cdots \\
 & & \Big\downarrow {\scriptstyle fd^{s-1}} & & \Big\downarrow {\scriptstyle f} & & \Big\downarrow \\
\coprod_i J_{s-1}(A_i): & 0 \longrightarrow & \coprod_{i\in I} A_i & \xrightarrow{\mathrm{id}} & \coprod_{i\in I} A_i & \longrightarrow & 0 \longrightarrow 0.
\end{array}$$

Again note that X is compact in $\mathbf{C}(\mathscr{A})$, we know f factors through finite subsum. □

Corollary 2.4. $\mathbf{C}^c(\mathrm{Inj}\,\Lambda) = \mathbf{C}^b(\mathrm{inj}\,\Lambda)$

3. ALMOST SPLIT CONFLATION

In this section we consider the existence of almost split conflations in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$. Note that $\mathbf{K}(\mathrm{Inj}\,\Lambda)$ is a compactly generated triangulated category, and the existence of Auslander–Reiten triangles in this category is known (Krause, 2000; Krause and Le, 2006). We study its relation with almost split conflations in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$.

Let us recall the relevant definitions from Auslander–Reiten theory first. A morphism $\alpha : X \to Y$ is called *left almost split*, if α is not a section and if every morphism $X \to W$ which is not a section factors through α. Dually, a morphism $\beta : Y \to Z$ is *right almost split*, if β is not a retraction and if every morphism $W \to Y$ which is not a retraction factors through β.

Definition 3.1.

(1) A conflation $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in an exact category is called an *almost split conflation*, if α is left almost split and β is right almost split.
(2) An exact triangle $X \xrightarrow{\bar{\alpha}} Y \xrightarrow{\bar{\beta}} Z \xrightarrow{\bar{\gamma}} X[1]$ in a triangulated category is called an *Auslander–Reiten triangle*, if $\bar{\alpha}$ is left almost split and $\bar{\beta}$ is right almost split.

Note that the end terms X and Z of an almost split conflation are indecomposable objects with local endomorphism rings. Moreover, each end term determines an almost split conflation up to isomorphism. We have the same remark for the Auslander–Reiten triangles.

Lemma 3.2. *Let $\varepsilon : X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be a nonsplit conflation in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$.*

(1) *If $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(Z)$ is a local ring, then β is right almost split in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$ if and only if $\bar{\beta}$ is right almost split in $\mathbf{K}(\mathrm{Inj}\,\Lambda)$.*
(2) *If $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(X)$ is a local ring, then α is left almost split in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$ if and only if $\bar{\alpha}$ is left almost split in $\mathbf{K}(\mathrm{Inj}\,\Lambda)$.*

Proof. See Bautista et al. (2004, Section 7). □

Using Lemmas 2.1 and 3.2, we immediately get the following proposition.

Proposition 3.3. *If $\varepsilon : X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is an almost split conflation in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$, then $\bar{\varepsilon}$: $X \xrightarrow{\bar{\alpha}} Y \xrightarrow{\bar{\beta}} Z \xrightarrow{\bar{\gamma}} X[1]$ is an AR-triangle in $\mathbf{K}(\mathrm{Inj}\,\Lambda)$. Conversely, if $\varepsilon : X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is a conflation in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$ with $X, Z \in \mathbf{C}_{\mathscr{P}}(\mathrm{Inj}\,\Lambda)$, such that $\bar{\varepsilon}$ is an AR-triangle in $\mathbf{K}(\mathrm{Inj}\,\Lambda)$, then ε is an almost split conflation in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$.*

Lemma 3.4. *Suppose $P \coprod X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \coprod Q$ is a conflation in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$ with P and Q null homotopic. Then it has the form*

$$\eta : P \coprod X \xrightarrow{\Phi} Y_1 \coprod Y_2 \coprod Y_3 \xrightarrow{\Psi} Z \coprod Q,$$

where $\Phi = \begin{pmatrix} \phi & 0 \\ 0 & \alpha_1 \\ 0 & 0 \end{pmatrix}$, $\Psi = \begin{pmatrix} 0 & \beta_1 & 0 \\ 0 & 0 & \psi \end{pmatrix}$ *with* ϕ *and* ψ *isomorphisms and* $\varepsilon : X \xrightarrow{\alpha_1} Y_2 \xrightarrow{\beta_1} Z$ *a conflation.*

Proof. The proof is similar to Bautista et al. (2004, Lemma 7.2). □

Theorem 3.5. *Let* Z *be a nonprojective indecomposable object in* $\mathbf{C}^{+,b}(\operatorname{inj}\Lambda)$. *Then there exists an almost split conflation in* $\mathbf{C}(\operatorname{Inj}\Lambda)$ *ending in* Z.

Proof. Since $\mathbf{K}(\operatorname{Inj}\Lambda)$ is compactly generated and Z is a compact object in $\mathbf{K}(\operatorname{Inj}\Lambda)$ with local endomorphism ring, there exists an Auslander–Reiten triangle

$$\bar{\theta} : X \xrightarrow{\bar{\alpha}} Y \xrightarrow{\bar{\beta}} Z \xrightarrow{\bar{\gamma}} X[1],$$

see Krause (2000, Theorem 2.2). By the definition of exact triangle in $\mathbf{K}(\operatorname{Inj}\Lambda)$, we can find a conflation $\eta : M \xrightarrow{i} N \xrightarrow{p} L$ in $\mathbf{C}(\operatorname{Inj}\Lambda)$ such that $\bar{\theta}$ is isomorphic to $\bar{\eta}$ in $\mathbf{K}(\operatorname{Inj}\Lambda)$. From Lemma 3.4, we can choose a proper η such that $M, L \in \mathbf{C}_{\mathscr{P}}(\operatorname{Inj}\Lambda)$. Hence η is an almost split conflation in $\mathbf{C}(\operatorname{Inj}\Lambda)$, by Proposition 3.3. Since Z is isomorphic to L in $\mathbf{K}(\operatorname{Inj}\Lambda)$, and both Z and L are objects in $\mathbf{C}_{\mathscr{P}}(\operatorname{Inj}\Lambda)$, we know that Z is isomorphic to L in $\mathbf{C}(\operatorname{Inj}\Lambda)$. □

Remark. We fix a locally Noetherian Grothendieck category $\mathscr{A}$. Thus $\mathscr{A}$ is an Abelian category with injective envelopes. It is easy to prove that $\mathbf{C}(\operatorname{Inj}\mathscr{A})$ is a Frobenius category, every object has a decomposition $X = X' \coprod X''$ such that X' is homotopically minimal and X'' is null homotopic, and X' is unique up to isomorphism. Moreover, $\mathbf{K}(\operatorname{Inj}\mathscr{A})$ is compactly generated. Most results in Section 3 can be extended to $\mathbf{C}(\operatorname{Inj}\mathscr{A})$, and we obtain a more general existence theorem: Let Z be a nonprojective object in $\mathbf{C}(\operatorname{Inj}\mathscr{A})$ with local endomorphism ring, such that Z is a compact object in $\mathbf{K}(\operatorname{Inj}\mathscr{A})$. Then there exists an almost split conflation in $\mathbf{C}(\operatorname{Inj}\mathscr{A})$ ending in Z.

4. THE AUSLANDER–REITEN TRANSLATION

The classical Auslander–Reiten formula for modules over an Artin k-algebra Λ says that

$$D\operatorname{Ext}^1_\Lambda(M, N) \cong \overline{\operatorname{Hom}}_\Lambda(N, D\operatorname{Tr} M)$$

whenever M is finitely generated (Auslander and Reiten, 1975). Here, Tr denotes the transpose construction. Auslander and Reiten used this formula to establish the existence of almost split sequences. In Krause and Le (2006), this formula was extended to $\mathbf{K}(\operatorname{Inj}\Lambda)$. In this section, we will define a map τ in $\mathbf{C}(\operatorname{Inj}\Lambda)$, and give an analogous formula. Using this formula, the existence of almost split conflations can be proved directly, with τ the Auslander–Reiten translation.

Let X be a complex in $\mathbf{C}(\operatorname{Inj}\Lambda)$. Denote by $\mathbf{p}X$ its projective resolution, that is, $\mathbf{p}X \in \mathbf{C}(\operatorname{Proj}\Lambda)$ such that each morphism from $\mathbf{p}X$ to an acyclic complex in $\mathbf{C}(\operatorname{Mod}\Lambda)$ is null-homotopic, and there is a quasi-isomorphism $\mathbf{p}X \to X$.

For the existence of $\mathbf{p}X$, see Keller (1994). Then $\mathbf{p}X$ can be decomposed as $(\mathbf{p}X)' \coprod (\mathbf{p}X)''$, where $(\mathbf{p}X)' \in \mathbf{C}_{\mathscr{P}}(\operatorname{Proj}\Lambda)$. Call $(\mathbf{p}X)'$ the *minimal projective resolution* of X. Applying the tensor functor $-\otimes_\Lambda D\Lambda$ to every component of $(\mathbf{p}X)'[-1]$, we obtain a new complex

$$\tau X = (\mathbf{p}X)' \otimes_\Lambda D\Lambda[-1]$$

in $\mathbf{C}(\operatorname{Inj}\Lambda)$. The following proposition implies that τX is homotopically minimal in $\mathbf{C}(\operatorname{Inj}\Lambda)$.

Lemma 4.1. *Let $X \in \mathbf{C}(\operatorname{Proj}\Lambda)$ be homotopically minimal. Then $X \otimes_\Lambda D\Lambda$ is a homotopically minimal object in $\mathbf{C}(\operatorname{Inj}\Lambda)$.*

Proof. Note that the tensor functor $-\otimes_\Lambda D\Lambda$ induces an equivalence from $\operatorname{Proj}\Lambda$ to $\operatorname{Inj}\Lambda$, which can be extended to the equivalence on the categories of complexes $\mathbf{C}(\operatorname{Proj}\Lambda) \cong \mathbf{C}(\operatorname{Inj}\Lambda)$, hence induces further an equivalence $\mathbf{K}(\operatorname{Proj}\Lambda) \cong \mathbf{K}(\operatorname{Inj}\Lambda)$. Then the result is easy to prove. □

Lemma 4.2. *τ induces an endofunctor in $\mathbf{K}(\operatorname{Inj}\Lambda)$. Moreover, the restriction of τ to $\mathbf{K}^c(\operatorname{Inj}\Lambda)$ is fully faithful.*

Proof. The functor induced by τ is the composite

$$\mathbf{K}(\operatorname{Inj}\Lambda) \xrightarrow{can} \mathbf{D}(\operatorname{mod}\Lambda) \xrightarrow{\mathbf{p}} \mathbf{K}(\operatorname{Proj}\Lambda) \xrightarrow{-\otimes_\Lambda D\Lambda} \mathbf{K}(\operatorname{Inj}\Lambda) \xrightarrow{[-1]} \mathbf{K}(\operatorname{Inj}\Lambda)$$

In Krause and Le (2006) it was proved that the restriction of τ to $\mathbf{K}^c(\operatorname{Inj}\Lambda)$ is fully faithful. □

Corollary 4.3. *Let X be a nonprojective indecomposable object in $\mathbf{C}^{+,b}(\operatorname{inj}\Lambda)$. Then τX is indecomposable.*

Proof. From Proposition 2.2 we know that $\operatorname{End}_{\mathbf{C}(\operatorname{Inj}\Lambda)}(X)$, and hence $\operatorname{End}_{\mathbf{K}(\operatorname{Inj}\Lambda)}(X)$ is a local ring. Since X is a compact object in $\mathbf{K}(\operatorname{Inj}\Lambda)$, by Lemma 4.2 $\operatorname{End}_{\mathbf{K}(\operatorname{Inj}\Lambda)}(\tau X)$ is a local ring. Since τX is homotopically minimal, $\operatorname{End}_{\mathbf{C}(\operatorname{Inj}\Lambda)}(\tau X)$ is a local ring, by Lemma 2.1. Hence τX is indecomposable. □

For given objects X and Z in $\mathbf{C}(\operatorname{Inj}\Lambda)$ we denote by $\operatorname{Ext}^1_{\mathscr{E}}(Z, X)$ the set of all exact pairs $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ in $\mathscr{E}$ modulo the equivalence relation which is defined in the following way. Two such pairs, (α, β) and (α', β') are equivalent if there exists a commutative diagram as below:

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \\ \| & & \downarrow & & \| \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' \end{array}$$

then $\mathrm{Ext}^1_{\mathscr{E}}(Z, X)$ becomes an Abelian group under Baer sum. Recall again that $\mathbf{C}(\mathrm{Inj}\,\Lambda)$ is a Frobenius category, so $\overline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(Z, X) \cong \underline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(Z, X) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\,\Lambda)}(Z, X)$.

Lemma 4.4. *For arbitrary X, $Z \in \mathbf{C}(\mathrm{Inj}\,\Lambda)$, there is an isomorphism*

$$\mathrm{Ext}^1_{\mathscr{E}}(Z, X) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\,\Lambda)}(Z, X[1])$$

which is natural in X and Z.

By Krause and Le (2006), we know that for X,Z in $\mathbf{K}(\mathrm{Inj}\,\Lambda)$ with Z compact, there is a natural isomorphism

$$D\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\,\Lambda)}(Z, X) \cong \mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\,\Lambda)}(X, \tau Z[1]).$$

Combining with the isomorphism in 4.4, we immediately get the following proposition.

Proposition 4.5. *Let X be an object in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$ and Z be an object in $\mathbf{C}^{+,b}(\mathrm{inj}\,\Lambda)$. Then we have an isomorphism*

$$\Phi_X : \mathrm{Ext}^1_{\mathscr{E}}(X, \tau Z) \xrightarrow{\sim} D\underline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(Z, X)$$

which is natural in X and Z.

Theorem 4.6. *Let Z be a nonprojective indecomposable object in $\mathbf{C}^{+,b}(\mathrm{inj}\,\Lambda)$. Then there exists an almost split conflation*

$$\tau Z \to Y \to Z$$

in $\mathbf{C}(\mathrm{Inj}\,\Lambda)$.

We introduce some lemmas before proving the theorem.

Lemma 4.7. *Suppose that in an exact category $(\mathscr{C}, \mathscr{E})$ there is a commutative diagram*

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' \end{array}$$

such that (α, β) and (α', β') are conflations in $\mathscr{E}$. Then there exists a morphism $u : Y \to X'$ such that $u\alpha = f$ if and only if there exists a morphism $v : Z \to Y'$ such that $\beta' v = h$.

Lemma 4.8. *Let $\varepsilon : X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z$ be a conflation in $(\mathbf{C}(\mathrm{Inj}\,\Lambda), \mathscr{E})$, then the following are equivalent:*

(1) *ε is an almost split conflation;*

(2) *β is right almost split and* $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(X)$ *is local.*

Proof. See Auslander (1978). □

Proof of Theorem 4.6. Take for X the object Z in Proposition 4.5, we get an isomorphism

$$\Phi_Z : \mathrm{Ext}^1_{\mathscr{E}}(Z, \tau Z) \xrightarrow{\sim} D\underline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(Z, Z).$$

Note that $\Gamma = \underline{\mathrm{End}}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(Z)$ is a local ring, and finitely generated as a k-module. Let f be a nonzero map in $D\underline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(Z, Z)$ such that f vanishes on $\mathrm{rad}\,\Gamma$. Denote by $\eta = \Phi^{-1}(f)$. We claim that η is the almost split conflation.

First, η is not split since it is nonzero. Let $u : W \to Z$ be an arbitrary morphism which is not a retraction, then its composition with any morphism from Z to W is in the radical of Γ, so $D\underline{\mathrm{Hom}}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(Z, u)(f) = 0$. Hence $\Phi_W \mathrm{Ext}^1_{\mathscr{E}}(u, \tau Z)(\eta) = 0$, by the naturalness of Φ. Since Φ_W is an isomorphism, $\mathrm{Ext}^1_{\mathscr{E}}(u, \tau Z)(\eta) = 0$. Lemma 4.7 implies that there exists a morphism $v : W \to Y$ such that $u = \beta v$, hence β is right almost split. Furthermore, in the proof of Corollary 4.3, we have seen that $\mathrm{End}_{\mathbf{C}(\mathrm{Inj}\,\Lambda)}(\tau Z)$ is a local ring. Using Lemma 4.8 we finish the proof. □

ACKNOWLEDGMENTS

The main part of this work was done during a stay at the University of Paderborn. This stay was made possible by a doctoral scholarship from the Asia-Link project "Algebras and Representations in China and Europe". I express my gratitude to Prof. H. Krause for suggesting the topic and his guidance in writing the article. I would like to thank Prof. Y. B. Zhang for her encouragement. Also, I want to show my thanks to the referee for his (her) careful reading and pointing out Proposition 2.3 to me.

REFERENCES

Auslander, M. (1978). Functors and morphisms determined by objects. Representation theory of Artin algebras. *Proceedings of the Philadelphia Conference.* Lecture Notes in Pure and Appl. Math. Vol. 37. New York: Dekker, pp. 1–244.

Auslander, M., Reiten, I. (1975). Representation theory of Artin algebras III. *Comm. Algebra* 3:239–294.

Balmer, P., Schlichting, M. (2001). Idempotent completion of triangulated categories. *J. Algebra* 236(2):819–834.

Bautista, R., Souto Salorio, M. J., Zuazua, R. (2004). Almost split conflations for complexes with fixed size. Preprint.

Beligiannis, A. (2002). Purity and almost split morphisms in abstract homotopy categories: A unified approach via Brown representability. *Algebras and Representation Theory* 5:483–525.

Dräxler, P., Reiten, I., Smalø, S. O. (1999). Ø Solberg and with an appendix by B. Keller. Exact categories and vector space categories. *Trans. of the A.M.S.* 351(2):647–682.

Gabriel, P., Roiter, A. V. (1992). Representations of finite dimensional algebras. In: Kostrikin, A. I., Shafareivch, I. V., eds. *Algebra VIII.* Encyclopaedia of the Mathematical Sciences, 73. Springer.

Happel, D. (1987). On the derived category of a finite-dimensional algebras. *Comment. Math. Helv.* 62:339–389.

Keller, B. (1990). Chain complexes and stable categories. *Manuscripta Math.* 67(4):379–417.

Keller, B. (1994). Deriving DG categories. *Ann. Sci. École. Norm. Sup.* 27:63–102.

Keller, B. (1996). Derived categories and their uses. *Handbook of Algebra*. Vol. 1. North-Holland: Amsterdam, pp. 671–701.

Krause, H. (2000). Auslander–Reiten theory via Brown representability. *K-theory* 20:331–344.

Krause, H. (2005). The stable derived category of a noetherian scheme. *Compos. Math.* 141:1128–1162.

Krause, H., Le, J. (2006). The Auslander–Reiten formula for complexes of modules. *Adv. Math.* 207:133–148.

Reiten, I., Van Den Bergh, M. Noetherian hereditary abelian categories satisfying Serre duality. *J. Amer. Math. Soc.* 15:295–366.