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On the Hochschild cohomology ring of tensor products of algebras

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ABSTRACT

We prove that, as Gerstenhaber algebras, the Hochschild cohomology ring of the tensor product of two algebras is isomorphic to the tensor product of the respective Hochschild cohomology rings of these two algebras, when at least one of them is finite dimensional. In case of finite dimensional symmetric algebras, this isomorphism is an isomorphism of Batalin–Vilkovisky algebras. As an application, we explain by examples how to compute the Batalin–Vilkovisky structure, in particular, the Gerstenhaber Lie bracket, over the Hochschild cohomology ring of the group algebra of a finite abelian group.

1. Introduction

Let A be an associative algebra over a field k. The Hochschild cohomology ring $HH^*(A)$ of A is a graded commutative algebra via the cup product, and in the meantime, it is a graded Lie algebra of degree -1; these make $HH^*(A)$ a Gerstenhaber algebra [4]. If the algebra A is finite dimensional and symmetric, then $HH^*(A)$ has an additional structure, as we now explain.

Let A be a finite dimensional symmetric algebra, for example, the group algebra kG of a finite group G. Then there is a symmetric associative non-degenerate bilinear form $\langle\ ,\ \rangle : A \times A \to k$. This bilinear form gives rise to a duality between Hochschild homology and Hochschild cohomology, i.e. for $n \geqslant 0$,

$$HH^n(A) \simeq \mathrm{Hom}_k\big(HH_n(A), k\big).$$

Recall that there is an operator over the Hochschild homology groups, the so-called Connes' $\mathfrak{B}$-operator [9, Chapter 2]

$$\mathfrak{B} : HH_n(A) \to HH_{n+1}(A), \quad n \geqslant 0.$$

We obtain an operator over the Hochschild cohomology groups by duality

$$\Delta : HH^n(A) \to HH^{n-1}(A), \quad n \geqslant 1.$$

Tradler [15] noticed that the Lie bracket over $HH^*(A)$ can be expressed in terms of this Δ-operator and the cup product. In fact, he proved that $HH^*(A)$ is a Batalin–Vilkovisky algebra [5,11]. For different proofs of this fact, see [13] and [3].

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We are interested in computing the Batalin–Vilkovisky structure over $HH^*(kG)$ for a finite abelian group G. Notice that kG is isomorphic to a tensor product of the group algebras of some cyclic groups, while the Batalin–Vilkovisky structure of $HH^*(kC)$ with C a cyclic group is known [17]. This observation leads us to study the Batalin–Vilkovisky structure of the Hochschild cohomology ring of a tensor product algebra.

Let A and B be two k-algebras such that one of them is finite dimensional. It is a folklore result that there is an isomorphism of graded commutative algebras: $HH^*(A\otimes B)\simeq HH^*(A)\otimes HH^*(B)$ (though we could not find a precise reference in the literature). We prove that it is furthermore an isomorphism of Gerstenhaber algebras (Theorem 3.3). In addition, when both A and B are finite dimensional and symmetric, the mentioned isomorphism is an isomorphism of Batalin–Vilkovisky algebras (Theorem 3.5). The key point of the proof is to use the well-known Alexander–Whitney map and the Eilenberg–Zilber map which are comparison morphisms between the (normalized) bar resolution of $A\otimes B$ and the tensor product of the (normalized) bar resolutions of A and B.

As an application of Theorem 3.5, we explain by examples how to compute the Batalin–Vilkovisky structure over the Hochschild cohomology ring of the group algebra of a finite abelian group. In particular, the Batalin–Vilkovisky structure over the Hochschild cohomology ring of the group algebra of an elementary abelian group is described in Theorem 4.3.

In future work, we shall use these results to compute the Batalin–Vilkovisky structure over the Hochschild cohomology of an algebra with one generator, that is, an algebra of the form $k[X]/(f)$ with f a monic polynomial. We shall also consider the Hochschild cohomology of a group algebra and investigate the behavior of the Δ-operator under the additive decomposition of the Hochschild cohomology of a group algebra; see [8].

The main results of this paper (Theorem 3.3 and Theorem 3.5) should be known to experts in string topology. In fact, Hochschild cohomology corresponds to free loop space homology $H_*(LM)$, our results are thus closed related to some known results in string topology. We are grateful to the referee for this remark. However, it seems that our proof is the first algebraic one of these results.

Throughout this paper, k denotes a field and a k-algebra is always assumed to be associative with unit. The symbol $\otimes$ means $\otimes_k$. For a homogeneous element a in a graded space, $|a|$ denotes its degree.

2. Gerstenhaber vs Batalin–Vilkovisky

In this section, we recall the definitions of Gerstenhaber algebras and Batalin–Vilkovisky algebras. We study the tensor product of two Gerstenhaber algebras and that of Batalin–Vilkovisky algebras, respectively.

Definition 2.1. A *Gerstenhaber algebra* over a field k is a graded k-vector space $A^\bullet=\bigoplus_{n\in\mathbb{Z}}A^n$ equipped with two linear maps: a cup product

$$\smile: A^n\times A^m\to A^{n+m},\qquad (a,b)\mapsto a\smile b$$

and a Lie bracket of degree -1

$$[\ ,\ ]: A^n\times A^m\to A^{n+m-1},\qquad (a,b)\mapsto [a,b]$$

such that

(i) $(A^\bullet,\smile)$ is a graded commutative associative algebra, that is, $a\smile b=(-1)^{|a||b|}b\smile a$;
(ii) $(A^\bullet,[\ ,\ ])$ is a graded Lie algebra of degree -1, i.e.

$$[a,b]=-(-1)^{(|a|-1)(|b|-1)}[b,a]$$

and

$$(-1)^{(|a|-1)(|c|-1)}\big[[a,b],c\big]+(-1)^{(|b|-1)(|a|-1)}\big[[b,c],a\big]+(-1)^{(|c|-1)(|b|-1)}\big[[c,a],b\big]=0;$$

(iii) the cup product and the Lie bracket satisfy the Poisson rule, i.e.

$$[a\smile b,c]=[a,c]\smile b+(-1)^{|a|(|c|-1)}a\smile[b,c],$$

where a,b,c are arbitrary homogeneous elements in $A^\bullet$.

Notice that by (iii), the Lie bracket is a derivation with respect to the first variable. Our definition of Gerstenhaber algebras follows the original work of Gerstenhaber [4] in contrary to [11, (9.14) and (9.18)] and [5, Definition 1.1], where the Lie bracket is a derivation with respect to the second variable.

The cohomology theory of associative algebras was introduced by Hochschild [7]. The Hochschild cohomology ring of a k-algebra is a Gerstenhaber algebra, which was first discovered by Gerstenhaber in [4]. Let us recall his construction here. Given a k-algebra A, its Hochschild cohomology groups are defined as $HH^n(A)\simeq \mathrm{Ext}^n_{A^e}(A,A)$ for $n\geqslant 0$, where $A^e=A\otimes A^{\mathrm{op}}$ is the enveloping algebra of A. There is a projective resolution of A as an A^e-module

$$\mathrm{Bar}_*(A):\cdots\to A^{\otimes(r+2)}\xrightarrow{d_r}A^{\otimes(r+1)}\to\cdots\to A^{\otimes 3}\xrightarrow{d_1}A^{\otimes 2}(\xrightarrow{d_0=\mu}A),$$

where $\mathrm{Bar}_r(A) := A^{\otimes(r+2)}$ for $r \geqslant 0$, the map $\mu : A \otimes A \to A$ is the multiplication of A, and d_r is defined by

$$d_r(a_0 \otimes a_1 \otimes \cdots \otimes a_{r+1}) = \sum_{i=0}^{r}(-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{r+1}$$

for all $a_0, \ldots, a_{r+1} \in A$. This is usually called the (unnormalized) bar resolution of A. The normalized version $\mathbb{B}_*(A)$ is given by $\mathbb{B}_r(A) = A \otimes \overline{A}^{\otimes r} \otimes A$, where $\overline{A} = A/(k \cdot 1_A)$, and with the induced differential from that of $\mathrm{Bar}_*(A)$.

The complex which is used to compute the Hochschild cohomology is $C^*(A) = \mathrm{Hom}_{A^e}(\mathrm{Bar}_*(A), A)$. Note that for each $r \geqslant 0$, $C^r(A) = \mathrm{Hom}_{A^e}(A^{\otimes(r+2)}, A) \simeq \mathrm{Hom}_k(A^{\otimes r}, A)$. We identify $C^0(A)$ with A. Thus $C^*(A)$ has the following form:

$$C^*(A) : A \xrightarrow{\delta^0} \mathrm{Hom}_k(A, A) \to \cdots \to \mathrm{Hom}_k\big(A^{\otimes r}, A\big) \xrightarrow{\delta^r} \mathrm{Hom}_k\big(A^{\otimes(r+1)}, A\big) \to \cdots.$$

Given f in $\mathrm{Hom}_k(A^{\otimes r}, A)$, the map $\delta^r(f)$ is defined by sending $a_1 \otimes \cdots \otimes a_{r+1}$ to

$$a_1 \cdot f(a_2 \otimes \cdots \otimes a_{r+1}) \sum_{i=1}^{r}(-1)^i f(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{r+1})$$
$$+ (-1)^{r+1} f(a_1 \otimes \cdots \otimes a_r) \cdot a_{r+1}.$$

There is also a normalized version $\bar{C}^*(A) = \mathrm{Hom}_{A^e}(\mathbb{B}_*(A), A) \simeq \mathrm{Hom}_k(\overline{A}^{\otimes *}, A)$.

The cup product $\alpha \smile \beta \in C^{n+m}(A) = \mathrm{Hom}_k(A^{\otimes(n+m)}, A)$ for $\alpha \in C^n(A)$ and $\beta \in C^m(A)$ is given by

$$(\alpha \smile \beta)(a_1 \otimes \cdots \otimes a_{n+m}) := \alpha(a_1 \otimes \cdots \otimes a_n) \cdot \beta(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

This cup product induces a well-defined product in Hochschild cohomology

$$\smile : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m}(A)$$

which turns the graded k-vector space $HH^*(A) = \bigoplus_{n\geqslant 0} HH^n(A)$ into a graded commutative algebra [4, Corollary 1].

The Lie bracket is defined as follows. Let $\alpha \in C^n(A)$ and $\beta \in C^m(A)$. If $n, m \geqslant 1$, then for $1 \leqslant i \leqslant n$, set $\alpha \mathbin{\bar{\circ}_i} \beta \in C^{n+m-1}(A)$ by

$$(\alpha \mathbin{\bar{\circ}_i} \beta)(a_1 \otimes \cdots \otimes a_{n+m-1}) := \alpha\big(a_1 \otimes \cdots \otimes a_{i-1} \otimes \beta(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_{n+m-1}\big);$$

if $n \geqslant 1$ and $m = 0$, then $\beta \in A$ and for $1 \leqslant i \leqslant n$, set

$$(\alpha \mathbin{\bar{\circ}_i} \beta)(a_1 \otimes \cdots \otimes a_{n-1}) := \alpha(a_1 \otimes \cdots \otimes a_{i-1} \otimes \beta \otimes a_i \otimes \cdots \otimes a_{n-1});$$

for any other case, set $\alpha \mathbin{\bar{\circ}_i} \beta$ to be zero. Now define

$$\alpha \mathbin{\bar{\circ}} \beta := \sum_{i=1}^{n}(-1)^{(m-1)(i-1)} \alpha \mathbin{\bar{\circ}_i} \beta$$

and

$$[\alpha, \beta] := \alpha \mathbin{\bar{\circ}} \beta - (-1)^{(n-1)(m-1)} \beta \mathbin{\bar{\circ}} \alpha.$$

Note that $[\alpha, \beta] \in C^{n+m-1}(A)$. The above $[\ ,\ ]$ induces a well-defined Lie bracket in Hochschild cohomology

$$[\ ,\ ] : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m-1}(A)$$

such that $(HH^*(A), \smile, [\ ,\ ])$ is a Gerstenhaber algebra [4].

The complex used to compute the Hochschild homology $HH_*(A)$ is $C_*(A) = A \otimes_{A^e} \mathrm{Bar}_*(A)$. Notice that $C_r(A) = A \otimes_{A^e} A^{\otimes(r+2)} \simeq A^{\otimes(r+1)}$ and the differential $\partial_r : C_r(A) = A^{\otimes(r+1)} \to C_{r-1}(A) = A^{\otimes r}$ sends $a_0 \otimes \cdots \otimes a_r$ to $\sum_{i=0}^{r-1}(-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_r + (-1)^r a_r a_0 \otimes a_1 \otimes \cdots \otimes a_{r-1}$.

There is a Connes' $\mathfrak{B}$-operator in the Hochschild homology theory which is defined as follows. For $a_0 \otimes \cdots \otimes a_r \in C_r(A)$, let $\mathfrak{B}(a_0 \otimes \cdots \otimes a_r) \in C_{r+1}(A)$ be

$$\sum_{i=0}^{r}(-1)^{ir} 1 \otimes a_i \otimes \cdots \otimes a_r \otimes a_0 \otimes \cdots \otimes a_{i-1} + \sum_{i=0}^{r}(-1)^{ir} a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_r \otimes a_0 \otimes \cdots \otimes a_{i-1}.$$

It is easy to check that $\mathfrak{B}$ is a chain map satisfying $\mathfrak{B} \circ \mathfrak{B} = 0$, which induces an operator $\mathfrak{B} : HH_r(A) \to HH_{r+1}(A)$.

Given two algebras A and B, there is a shuffle product $sh : HH_*(A) \otimes HH_*(B) \to HH_*(A \otimes B)$. The definition needs some notation. Let $\mathcal{S}_{s,t}$ denote the set of all (s, t)-shuffles, that is, the set of permutations σ in the symmetric group on $s + t$ letters such that $\sigma(1) < \sigma(2) < \cdots < \sigma(s)$ and $\sigma(s+1) < \sigma(s+2) < \cdots < \sigma(s+t)$. Let $M = \overline{A} \oplus \overline{B}$, then there is a natural map $F: M \to \overline{A \otimes B}$ given by $F(a) = a \otimes 1$ for $a \in A$, and $F(b) = 1 \otimes b$ for $b \in B$. It is easily seen that F is well defined. For each $\sigma \in \mathcal{S}_{s,t}$, we call

$$|\sigma| := \sharp\{(i,j) \mid 1 \leqslant i < j \leqslant s+t, \text{ but } \sigma(i) > \sigma(j)\}$$

the degree of σ, and define

$$F_\sigma : M^{\otimes(s+t)} \longrightarrow (A \otimes B)^{\otimes(s+t)},$$
$$x_1 \otimes x_2 \otimes \cdots \otimes x_{s+t} \mapsto F(x_{\sigma^{-1}(1)}) \otimes F(x_{\sigma^{-1}(2)}) \otimes \cdots \otimes F(x_{\sigma^{-1}(s+t)}).$$

The shuffle product is induced by the map $sh : C_p(A) \otimes C_q(B) \to C_{p+q}(A \otimes B)$ defined as follows:

$$sh\big((a_0 \otimes a_1 \otimes \cdots \otimes a_p) \otimes (b_0 \otimes b_1 \otimes \cdots \otimes b_q)\big) = \sum_{\sigma \in \mathcal{S}_{p,q}} (-1)^{|\sigma|}(a_0 \otimes b_0) F_\sigma(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q).$$

Connes' $\mathfrak{B}$-operator is a derivation for the shuffle product; see [9, Corollary 4.3.4]. That is,

$$\mathfrak{B}\big(sh(x,y)\big) = sh\big(\mathfrak{B}(x), y\big) + (-1)^{|x|} sh\big(x, \mathfrak{B}(y)\big)$$

for homogeneous elements $x \in HH_*(A)$ and $y \in HH_*(B)$.

All the above constructions, the cup product, the Lie bracket, the Connes' $\mathfrak{B}$-operator, carry over to normalized complexes.

We define the tensor product of Gerstenhaber algebras in the following result. It extends slightly a result of Manin, where he introduced the tensor product of Batalin–Vilkovisky algebras; see [11, 9.11.1]. Note that a Batalin–Vilkovisky algebra is a special kind of a Gerstenhaber algebra; see Definition 2.4 for the notion of Batalin–Vilkovisky algebras.

Proposition-Definition 2.2. *Let $(A^\bullet, \smile_A, [\ ,\ ]_A)$ and $(B^\bullet, \smile_B, [\ ,\ ]_B)$ be two Gerstenhaber algebras over k. Then there is a new Gerstenhaber algebra $(L^\bullet, \smile, [\ ,\ ])$ over k given as follows*:

(i) $L^n = \bigoplus_{i+j=n} A^i \otimes B^j$ *as a k-vector space for each* $n \in \mathbb{Z}$;
(ii) $(a \otimes b) \smile (a' \otimes b') = (-1)^{|a'||b|}(a \smile_A a') \otimes (b \smile_B b')$;
(iii) $[a \otimes b, a' \otimes b'] = (-1)^{(|a|+|b|-1)|b'|}[a,a']_A \otimes (b \smile_B b') + (-1)^{|a|(|a'|+|b'|-1)}(a \smile_A a') \otimes [b,b']_B$

where $a, a' \in A^\bullet$ and $b, b' \in B^\bullet$ are homogeneous elements. We call $(L^\bullet, \smile, [\ ,\ ])$ the tensor product *of the two Gerstenhaber algebras $A^\bullet$ and $B^\bullet$, and denote it by $A^\bullet \otimes B^\bullet$.*

Since there is no detailed proof of the corresponding result in [11], we give in detail the hardest part of the proof.

Proof. The hardest part of the proof is to verify that the bracket of $L^\bullet$ satisfies the Jacobi identity, whereas the remaining part is routine. In what follows, we simplify the notation and abuse $[\ ,\ ]$, $[\ ,\ ]_A$ and $[\ ,\ ]_B$ (also $\smile$, $\smile_A$ and $\smile_B$ respectively).

Let

$$S := (-1)^{(|a\otimes b|-1)(|a''\otimes b''|-1)}\big[\big[a \otimes b, a' \otimes b'\big], a'' \otimes b''\big],$$
$$T := (-1)^{(|a'\otimes b'|-1)(|a\otimes b|-1)}\big[\big[a' \otimes b', a'' \otimes b''\big], a \otimes b\big],$$
$$W := (-1)^{(|a''\otimes b''|-1)(|a'\otimes b'|-1)}\big[\big[a'' \otimes b'', a \otimes b\big], a' \otimes b'\big].$$

Using (iii) twice, we have

$$\begin{aligned} S &= (-1)^{(|a|+|b|-1)(|a''|+|b''|-1+|b'|)}(-1)^{(|a|+|b|+|a'|+|b'|-2)|b''|}\big[\big[a,a'\big],a''\big] \otimes \big(\big(b \smile b'\big) \smile b''\big) \\ &\quad + (-1)^{(|a|+|b|-1)(|a''|+|b''|-1+|b'|)}(-1)^{(|a|+|a'|-1)(|a''|+|b''|-1)}\big(\big[a,a'\big] \smile a''\big) \otimes \big[b \smile b', b''\big] \\ &\quad + (-1)^{(|a|+|b|-1)(|a''|+|b''|-1)}(-1)^{|a|(|a'|+|b'|-1)}(-1)^{(|a|+|b|+|a'|+|b'|-2)|b''|}\big[a \smile a', a''\big] \otimes \big(\big[b,b'\big] \smile b''\big) \\ &\quad + (-1)^{(|a|+|b|-1)(|a''|+|b''|-1)}(-1)^{|a|(|a'|+|b'|-1)}(-1)^{(|a|+|a'|)(|a''|+|b''|-1)}\big(\big(a \smile a'\big) \smile a''\big) \otimes \big[\big[b,b'\big],b''\big]. \end{aligned}$$

Write the equation above as $S = S_1 + S_2 + S_3 + S_4$. Similarly, we have the equations $T = T_1 + T_2 + T_3 + T_4$ and $W = W_1 + W_2 + W_3 + W_4$.

Note that

$$S_1 = (-1)^{|a||b'|+|b||b'|+|b||a''|+|a'||b''|+|b'||b''|-|b|-|b'|-|b''|}(-1)^{(|a|-1)(|a''|-1)}\big[\big[a,a'\big],a''\big] \otimes \big(\big(b \smile b'\big) \smile b''\big),$$

while

$$T_1 = (-1)^{|a'||b''|+|b'||b''|+|a||b'|+|b||a''|+|b||b''|-|b|-|b'|-|b''|}(-1)^{(|a'|-1)(|a|-1)}\big[[a',a''],a\big] \otimes \big((b' \smile b'') \smile b\big)$$
$$= (-1)^{|a||b'|+|b||b'|+|b||a''|+|a'||b''|+|b'||b''|-|b|-|b'|-|b''|}(-1)^{(|a'|-1)(|a|-1)}\big[[a',a''],a\big] \otimes \big((b \smile b') \smile b''\big),$$

and

$$W_1 = (-1)^{|a''||b|+|b||b''|+|a'||b''|+|a||b'|+|b||b'|-|b|-|b'|-|b''|}(-1)^{(|a''|-1)(|a'|-1)}\big[[a'',a],a'\big] \otimes \big((b'' \smile b) \smile b'\big)$$
$$= (-1)^{|a||b'|+|b||b'|+|b||a''|+|a'||b''|+|b'||b''|-|b|-|b'|-|b''|}(-1)^{(|a''|-1)(|a'|-1)}\big[[a'',a],a'\big] \otimes \big((b \smile b') \smile b''\big).$$

Then we get $S_1 + T_1 + W_1 = 0$, since $[\ ,\ ]_A$ satisfies the Jacobi identity. In the same way, $S_4 + T_4 + W_4 = 0$ because $[\ ,\ ]_B$ satisfies the Jacobi identity.

For the remaining part, using Poisson rule of $[\ ,\ ]_B$, we have

$$S_2 = (-1)^{|b'||a|-|a'|-|b'|+|a'||a''|+|a'||b''|+|a''||b|+|b||b'|}\big([a,a'] \smile a''\big) \otimes \big((-1)^{|b||b''|-|b|}[b,b''] \smile b' + b \smile [b',b'']\big),$$
$$T_2 = (-1)^{|a||a''|+|b||a''|-|a''|-|b''|+|a||b'|+|a'||b''|+|b'||b''|}\big([a',a''] \smile a\big) \otimes \big((-1)^{|b'||b|-|b'|}[b',b] \smile b'' + b' \smile [b'',b]\big),$$
$$W_2 = (-1)^{|b||a''|-|a|-|b|+|a||a'|+|a||b'|+|a'||b''|+|b||b''|}\big([a'',a] \smile a'\big) \otimes \big((-1)^{|b''||b'|-|b''|}[b'',b'] \smile b + b'' \smile [b,b']\big).$$

On the other hand,

$$\begin{aligned}
&S_3 + T_3 + W_3 \\
&= (-1)^{|b||a''|-|a|-|b|-|b''|+1+|a||a'|+|a||b'|+|a'||b''|+|b'||b''|}\big([a'',a] \smile a'\big) \otimes \big([b,b'] \smile b''\big) \\
&\quad + (-1)^{|a||a''|+|b||a''|-|b|-|a''|-|b''|+1+|a||b'|+|a'||b''|+|b'||b''|}\big([a',a''] \smile a\big) \otimes \big([b,b'] \smile b''\big) \\
&\quad + (-1)^{|b'||a|-|a'|-|b'|-|b|+1+|a'||a''|+|a'||b''|+|a''||b|+|b''||b|}\big([a,a'] \smile a''\big) \otimes \big([b',b''] \smile b\big) \\
&\quad + (-1)^{|a'||a|+|b'||a|-|b'|-|a|-|b|+1+|a'||b''|+|a''||b|+|b''||b|}\big([a'',a] \smile a'\big) \otimes \big([b',b''] \smile b\big) \\
&\quad + (-1)^{|b''||a'|-|a''|-|b''|-|b'|+1+|a''||a|+|a''||b|+|a||b'|+|b||b'|}\big([a',a''] \smile a\big) \otimes \big([b'',b] \smile b'\big) \\
&\quad + (-1)^{|a''||a'|+|b''||a'|-|b''|-|a'|-|b'|+1+|a''||b|+|a||b'|+|b||b'|}\big([a,a'] \smile a''\big) \otimes \big([b'',b] \smile b'\big) \\
&= (-1)^{|b||a''|-|a|-|b|+1+|a||a'|+|a||b'|+|a'||b''|}\big([a'',a] \smile a'\big) \otimes \big((-1)^{|b'||b''|-|b''|}\big([b,b'] \smile b''\big) \\
&\quad + (-1)^{|b||b''|-|b'|}\big([b',b''] \smile b\big) + (-1)^{|a||a''|+|b||a''|-|a''|-|b''|+1+|a||b'|+|a'||b''|}\big([a',a''] \smile a\big) \\
&\quad \otimes \big((-1)^{|b'||b''|-|b|}\big([b,b'] \smile b''\big) + (-1)^{|b'||b|-|b'|}\big([b'',b] \smile b'\big)\big) \\
&\quad + (-1)^{|b'||a|-|a'|-|b'|+1+|a'||a''|+|a'||b''|+|a''||b|}\big([a,a'] \smile a''\big) \\
&\quad \otimes \big((-1)^{|b''||b|-|b|}\big([b',b''] \smile b\big) + (-1)^{|b||b'|-|b''|}\big([b'',b] \smile b'\big)\big) \\
&= (-1)^{|b||a''|-|a|-|b|+|a||a'|+|a||b'|+|a'||b''|+|b||b''|}(-1)^{|b'||b''|-|b''|+1}\big([a'',a] \smile a'\big) \otimes [b'' \smile b, b'] \\
&\quad + (-1)^{|a||a''|+|b||a''|-|a''|-|b''|+|a||b'|+|a'||b''|+|b'||b''|}(-1)^{|b||b'|-|b'|+1}\big([a',a''] \smile a\big) \otimes [b' \smile b'', b] \\
&\quad + (-1)^{|b'||a|-|a'|-|b'|+|a'||a''|+|a'||b''|+|a''||b|+|b||b'|}(-1)^{|b||b''|-|b|+1}\big([a,a'] \smile a''\big) \otimes [b \smile b', b''],
\end{aligned}$$

where the first equality uses the Poisson rule of $[\ ,\ ]_A$, and the last equality uses the Poisson rule of $[\ ,\ ]_B$. Writing the sum above as $U_1 + U_2 + U_3$, and applying the Poisson rule of $[\ ,\ ]_B$ once again, we have

$$S_2 + U_3 = 0, \qquad T_2 + U_2 = 0 \quad \text{and} \quad W_2 + U_1 = 0.$$

From this we have

$$S_2 + T_2 + W_2 + S_3 + T_3 + W_3 = 0,$$

therefore

$$S + T + W = (S_1 + T_1 + W_1) + (S_4 + T_4 + W_4) + (S_2 + T_2 + W_2 + S_3 + T_3 + W_3) = 0 + 0 + 0 = 0. \qquad \square$$

Remark 2.3.

(1) Proposition–Definition 2.2 gives in fact the coproduct in the category of Gerstenhaber algebras.

(2) In Proposition-Definition 2.2, if we define a new bracket $[x,y]^\star=[y,x]$ for $x,y,\in A^\bullet, B^\bullet$ or $L^\bullet$, then one verifies easily that

$$\big[a\otimes b,\ a'\otimes b'\big]^\star=(-1)^{(|a'|-1)|b|}\big[a,\ a'\big]^\star\otimes\big(b\smile b'\big)+(-1)^{|a'|(|b|-1)}\big(a\smile a'\big)\otimes\big[b,b'\big]^\star,$$

which is exactly the definition given by Manin [11]. The difference between our definition and Manin's comes from the fact that we follow the original definitions of the cup product (and the Lie bracket) in [4].

Now we turn to Batalin–Vilkovisky algebras [5,11].

Definition 2.4. A *Batalin–Vilkovisky algebra* (BV algebra for short) is a Gerstenhaber algebra $(A^\bullet,\smile,[\ ,\ ])$ together with an operator $\Delta: A^\bullet\to A^{\bullet-1}$ of degree -1 such that $\Delta\circ\Delta=0$ and

$$[a,b]=-(-1)^{(|a|-1)|b|}\big(\Delta(a\smile b)-\Delta(a)\smile b-(-1)^{|a|}a\smile\Delta(b)\big)$$

for homogeneous elements $a,b\in A^\bullet$.

Remark 2.5. If we define $[a,b]^*=-[b,a]$, then we get

$$[a,b]^*=(-1)^{|a|}\big(\Delta(a\smile b)-\Delta(a)\smile b-(-1)^{|a|}a\smile\Delta(b)\big),$$

which is the equality in the usual definition of a Batalin–Vilkovisky algebra in [5, Proposition 1.2] and [11, §5.1].

Tradler noticed that the Hochschild cohomology algebra of a symmetric algebra is a BV algebra [15], see also [13,3]. For a symmetric algebra A, he showed that the Δ-operator on the Hochschild cohomology corresponds to the Connes' $\mathfrak{B}$-operator on the Hochschild homology via the duality between the Hochschild cohomology and the Hochschild homology.

Recall that a finite dimensional k-algebra A is called symmetric if A is isomorphic to its dual $DA=\mathrm{Hom}_k(A,k)$ as A^e-modules, or equivalently, if there exists a symmetric associative non-degenerate bilinear form $\langle\ ,\ \rangle: A\times A\to k$. This bilinear form induces a duality between the Hochschild cohomology and the homology. In fact,

$$\mathrm{Hom}_k\big(C_*(A),k\big)=\mathrm{Hom}_k\big(A\otimes_{A^e}\mathrm{Bar}_*(A),k\big)\simeq\mathrm{Hom}_{A^e}\big(\mathrm{Bar}_*(A),\mathrm{Hom}_k(A,k)\big)\simeq\mathrm{Hom}_{A^e}\big(\mathrm{Bar}_*(A),A\big)=C^*(A).$$

Via this duality, for $n\geqslant 1$ we obtain an operator $\Delta: HH^n(A)\to HH^{n-1}(A)$ which is the dual of Connes' operator.

We recall the following theorem by Tradler.

Theorem 2.6. *(See [15, Theorem 1].) With the notation above, together with the cup product, the Lie bracket and the Δ-operator defined above, the Hochschild cohomology of A is a BV algebra. More precisely, for $\alpha\in C^n(A)=\mathrm{Hom}_k(A^{\otimes n},A)$, $\Delta(\alpha)\in C^{n-1}(A)=\mathrm{Hom}_k(A^{\otimes(n-1)},A)$ is given by the equation*

$$\big\langle\Delta(\alpha)(a_1\otimes\cdots\otimes a_{n-1}),\ a_n\big\rangle=\sum_{i=1}^{n}(-1)^{i(n-1)}\big\langle\alpha(a_i\otimes\cdots\otimes a_{n-1}\otimes a_n\otimes a_1\otimes\cdots\otimes a_{i-1}),1\big\rangle$$

for $a_1,\dots,a_n\in A$. The same formula holds also for the normalized complex $\overline{C}^*(A)$.

We mention that the Δ-operator depends on the choice of the non-degenerate bilinear form of A. With Theorem 2.6 at hand, in order to obtain the BV algebra structure over the Hochschild cohomology of a symmetric algebra, one needs to know the cup product and the Δ-operator.

Let us recall the tensor product of two BV algebras defined in [11, Proposition in Section 5.8.1].

Definition 2.7. Let $(A^\bullet,\smile_A,[\ ,\ ]_A,\Delta_A)$ and $(B^\bullet,\smile_B,[\ ,\ ]_B,\Delta_B)$ be two BV algebras. Then there is a new BV algebra $(L^\bullet,\smile,[\ ,\ ],\Delta)$, where

(i) $L^n=\bigoplus_{i+j=n}A^i\otimes B^j$ as a k-vector space for $n\in\mathbb{Z}$;
(ii) $(a\otimes b)\smile(a'\otimes b')=(-1)^{|a'||b|}(a\smile_A a')\otimes(b\smile_B b')$;
(iii) $[a\otimes b,a'\otimes b']=(-1)^{(|a|+|b|-1)|b'|}[a,a']_A\otimes(b\smile_B b')+(-1)^{|a|(|a'|+|b'|-1)}(a\smile_A a')\otimes[b,b']_B$;
(iv) $\Delta(a\otimes b)=\Delta_A(a)\otimes b+(-1)^{|a|}a\otimes\Delta_B(b)$

where $a,a'\in A^\bullet$ and $b,b'\in B^\bullet$ are homogeneous elements. We call $(L^\bullet,\smile,[\ ,\ ],\Delta)$ the *tensor product* of the two BV algebras $A^\bullet$ and $B^\bullet$, and denote it by $A^\bullet\otimes B^\bullet$.

3. Tensor product

Let A and B be two k-algebras. In this section, we compare the Gerstenhaber structure of $HH^*(A\otimes B)$ with that of $HH^*(A)\otimes HH^*(B)$. We compare further the BV structure of them when A and B are finite dimensional symmetric algebras.

We shall prove the following result, which is a combination of Lemma 3.1, Theorem 3.3 and Theorem 3.5.

Main Theorem. *Let A and B be two k-algebras such that one of them is finite dimensional. Then there is an isomorphism of Gerstenhaber algebras*

$$HH^*(A\otimes B)\simeq HH^*(A)\otimes HH^*(B).$$

If furthermore, A and B are finite dimensional symmetric algebras, the above isomorphism becomes an isomorphism of BV algebras, once we endow $A\otimes B$ with the non-degenerate bilinear form such that $\langle a\otimes b, a'\otimes b'\rangle=\langle a,a'\rangle\langle b,b'\rangle$ for $a,a'\in A$, $b,b'\in B$.

Before proceeding to the technical proof of this result, let us explain the strategy. In the above isomorphism, the left-handed side is computed by using the normalized bar resolution $\mathbb{B}_*(A\otimes B)$, while the right-handed side can be obtained by using $\mathbb{B}_*(A)\otimes\mathbb{B}_*(B)$. Both of these two resolutions are projective resolutions of $A\otimes B$ as bimodules. So in order to compare the two sides, we need to construct comparison morphisms between these two resolutions. The following map AW_* (resp. EZ_*) is in fact the usual Alexander–Whitney map (resp. the Eilenberg–Zilber or shuffle maps); see [10, X.Theorem 7.4] or [16, Exercise 8.6.5].

Let $\mathbb{B}_*(A)$ and $\mathbb{B}_*(B)$ be the normalized bar resolutions of A and B. Then the tensor product complex

$$\mathbb{B}_*(A)\otimes\mathbb{B}_*(B):\cdots\to\bigoplus_{i=0}^{r}\big(A\otimes\overline{A}^{\otimes(r-i)}\otimes A\otimes B\otimes\overline{B}^{\otimes i}\otimes B\big)\to\cdots\to A\otimes A\otimes B\otimes B(\to A\otimes B)$$

is a projective resolution of $A\otimes B$ as an $(A\otimes B)^e$-module. On the other hand, the normalized bar resolution of $A\otimes B$

$$\mathbb{B}_*(A\otimes B):\cdots\to A\otimes B\otimes\overline{A\otimes B}^{\otimes r}\otimes A\otimes B\to\cdots\to A\otimes B\otimes A\otimes B(\to A\otimes B)$$

is also a projective resolution of $A\otimes B$ as an $(A\otimes B)^e$-module. We are going to construct the maps $AW_*:\mathbb{B}_*(A\otimes B)\to\mathbb{B}_*(A)\otimes\mathbb{B}_*(B)$ and $EZ_*:\mathbb{B}_*(A)\otimes\mathbb{B}_*(B)\to\mathbb{B}_*(A\otimes B)$ which are homotopy equivalences. Notice that a similar formula of AW_* has been considered by May [12].

The map $AW_*:\mathbb{B}(A\otimes B)\to\mathbb{B}(A)\otimes\mathbb{B}(B)$ is defined as follows:

- $AW_0: A\otimes B\otimes A\otimes B\to A\otimes A\otimes B\otimes B$ is defined by twisting the middle two items;
- for $r\geqslant 1$,

$$\begin{aligned}&AW_r(1\otimes 1\otimes a_1\otimes b_1\otimes\cdots\otimes a_r\otimes b_r\otimes 1\otimes 1)\\&\quad=\sum_{t=0}^{r}(-1)^{t(r-t)}a_1a_2\cdots a_t\otimes a_{t+1}\otimes\cdots\otimes a_r\otimes 1\otimes 1\otimes b_1\otimes\cdots\otimes b_t\otimes b_{t+1}\cdots b_r,\end{aligned}$$

  where $a_1,\dots,a_r\in A$ and $b_1,\dots,b_r\in B$, and by convention for $t=0$, $a_1\cdots a_t=1$ and for $t=r$, $b_{t+1}\cdots b_r=1$.

The chain map $EZ_*:\mathbb{B}(A)\otimes\mathbb{B}(B)\to\mathbb{B}(A\otimes B)$ is given as follows:

- $EZ_0: A\otimes A\otimes B\otimes B\to A\otimes B\otimes A\otimes B$ is defined by twisting the middle two items;
- for $r\geqslant 1$, $0\leqslant t\leqslant r$,

$$\begin{aligned}&EZ_r(1\otimes a_1\otimes\cdots\otimes a_{r-t}\otimes 1\otimes 1\otimes b_1\otimes\cdots\otimes b_t\otimes 1)\\&\quad=1\otimes 1\otimes\Bigg(\sum_{\sigma\in\mathcal{S}_{r-t,t}}(-1)^{|\sigma|}F_\sigma(a_1\otimes\cdots\otimes a_{r-t}\otimes b_1\otimes\cdots\otimes b_t)\Bigg)\otimes 1\otimes 1.\end{aligned}$$

We are now in the position to compute the chain maps $AW^*=\mathrm{Hom}_{(A\otimes B)^e}(AW_*,A\otimes B)$ and $EZ^*=\mathrm{Hom}_{(A\otimes B)^e}(EZ_*,A\otimes B)$. Applying the functor $\mathrm{Hom}_{(A\otimes B)^e}(-,A\otimes B)$ to $\mathbb{B}_*(A\otimes B)$ and using the isomorphism

$$\mathrm{Hom}_{(A\otimes B)^e}\big(A\otimes B\otimes\overline{A\otimes B}^{\otimes r}\otimes A\otimes B, A\otimes B\big)\simeq\mathrm{Hom}_k\big(\overline{A\otimes B}^{\otimes r},A\otimes B\big),$$

we obtain the complex $\overline{C}^*(A\otimes B)=\mathrm{Hom}_{(A\otimes B)^e}(\mathbb{B}_*(A\otimes B),A\otimes B)$:

$$A\otimes B\to\mathrm{Hom}_k(\overline{A\otimes B},A\otimes B)\to\cdots\to\mathrm{Hom}_k\big(\overline{A\otimes B}^{\otimes r},A\otimes B\big)\to\mathrm{Hom}_k\big(\overline{A\otimes B}^{\otimes(r+1)},A\otimes B\big)\to\cdots.$$

Similarly, when at least one of two algebras is finite dimensional, we have the following sequence of isomorphisms

$$\begin{aligned}&\mathrm{Hom}_{(A\otimes B)^e}\big(A\otimes \overline{A}^{\otimes(r-t)}\otimes A\otimes B\otimes \overline{B}^{\otimes t}\otimes B, A\otimes B\big)\\ &\quad\simeq \mathrm{Hom}_{A^e}\big(A\otimes \overline{A}^{\otimes(r-t)}\otimes A, A\big)\otimes \mathrm{Hom}_{B^e}\big(B\otimes \overline{B}^{\otimes t}\otimes B, B\big)\\ &\quad\simeq \mathrm{Hom}_k\big(\overline{A}^{\otimes(r-t)}, A\big)\otimes \mathrm{Hom}_k\big(\overline{B}^{\otimes t}, B\big)\end{aligned}$$

which implies that the complex $\mathrm{Hom}_{(A\otimes B)^e}(\mathbb{B}_*(A)\otimes\mathbb{B}_*(B), A\otimes B)$ is isomorphic to

$$A\otimes B\to \big(\mathrm{Hom}_k(\overline{A}, A)\otimes B\big)\oplus\big(A\otimes \mathrm{Hom}_k(\overline{B}, B)\big)\to\cdots\to\bigoplus_{t=0}^{r}\mathrm{Hom}_k\big(\overline{A}^{\otimes(r-t)}, A\big)\otimes \mathrm{Hom}_k\big(\overline{B}^{\otimes t}, B\big)\to\cdots$$

which is exactly the tensor product complex $\bar{C}^*(A)\otimes\bar{C}^*(B)$. From now on, since we suppose that one of the algebras is finite dimensional, we shall identify $\mathrm{Hom}_{(A\otimes B)^e}(\mathbb{B}_*(A)\otimes\mathbb{B}_*(B), A\otimes B)$ with $\bar{C}^*(A)\otimes\bar{C}^*(B)$ without further explanation.

Accordingly, the map

$$AW^r=\mathrm{Hom}_{(A\otimes B)^e}(AW_r, A\otimes B):\bigoplus_{t=0}^{r}\mathrm{Hom}_k\big(\overline{A}^{\otimes(r-t)}, A\big)\otimes \mathrm{Hom}_k\big(\overline{B}^{\otimes t}, B\big)\longrightarrow \mathrm{Hom}_k\big(\overline{A\otimes B}^{\otimes r}, A\otimes B\big)$$

is given as follows:

- $AW^0=\mathrm{Hom}_{(A\otimes B)^e}(AW_0, A\otimes B)=\mathrm{id}$;
- for $r\geqslant 1$, $0\leqslant t\leqslant r$, and $\alpha\in\mathrm{Hom}_k(\overline{A}^{\otimes(r-t)}, A)$, $\beta\in\mathrm{Hom}_k(\overline{B}^{\otimes t}, B)$,

$$\begin{aligned}&AW^r(\alpha\otimes\beta)\big((a_1\otimes b_1)\otimes\cdots\otimes(a_r\otimes b_r)\big)\\ &\quad=(-1)^{t(r-t)}\big(a_1\cdots a_t\cdot\alpha(a_{t+1}\otimes\cdots\otimes a_r)\big)\otimes\big(\beta(b_1\otimes\cdots\otimes b_t)\cdot b_{t+1}\cdots b_r\big).\end{aligned}$$

Before defining EZ^r, let us recall an isomorphism

$$\begin{aligned}&\vartheta:\mathrm{Hom}_k\big(\overline{A}^{\otimes(r-t)}, A\big)\otimes\mathrm{Hom}_k\big(\overline{B}^{\otimes t}, B\big)\longrightarrow\mathrm{Hom}_k\big(\overline{A}^{\otimes(r-t)}\otimes\overline{B}^{\otimes t}, A\otimes B\big)\\ &\alpha\otimes\beta\mapsto\vartheta(\alpha\otimes\beta),\end{aligned}\tag{3.1}$$

where $\vartheta(\alpha\otimes\beta)$ sends $a_1\otimes\cdots\otimes a_{r-t}\otimes b_1\otimes\cdots\otimes b_t$ to $\alpha(a_1\otimes\cdots\otimes a_{r-t})\otimes\beta(b_1\otimes\cdots\otimes b_t)$. Then

$$EZ^r=\mathrm{Hom}_{(A\otimes B)^e}(EZ_r, A\otimes B):\mathrm{Hom}_k\big(\overline{A\otimes B}^{\otimes r}, A\otimes B\big)\longrightarrow\bigoplus_{t=0}^{r}\mathrm{Hom}_k\big(\overline{A}^{\otimes(r-t)}, A\big)\otimes\mathrm{Hom}_k\big(\overline{B}^{\otimes t}, B\big)$$

is given by

- $EZ^0=\mathrm{Hom}_{(A\otimes B)^e}(EZ_0, A\otimes B)=\mathrm{id}_{A\otimes B}$;
- for $r\geqslant 1$ and $\varphi\in\mathrm{Hom}_k(\overline{A\otimes B}^{\otimes r}, A\otimes B)$, let $EZ^r(\varphi)=(\vartheta^{-1}(\xi_0), \vartheta^{-1}(\xi_1), \ldots, \vartheta^{-1}(\xi_r))$, where $\xi_t\in\mathrm{Hom}_k(\overline{A}^{\otimes(r-t)}\otimes\overline{B}^{\otimes t}, A\otimes B)$ sends $a_1\otimes\cdots\otimes a_{r-t}\otimes b_1\otimes\cdots\otimes b_t$ to $\varphi(\sum_{\sigma\in\mathcal{S}_{r-t,t}}(-1)^{|\sigma|}F_\sigma(a_1\otimes\cdots\otimes a_{r-t}\otimes b_1\otimes\cdots\otimes b_t))$.

The following result is well known. However, we could not find a proof in the literature, so we supply a proof. We are very grateful to the referee for suggesting to us the following simple conceptual proof, while our proof (see Remark 3.2) is much more complicated using the maps AW^* and EZ^*.

Lemma 3.1. *Let A and B be two k-algebras such that one of them is finite dimensional. Then there is an isomorphism of graded algebras*

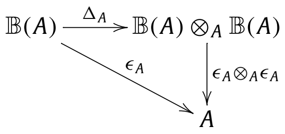
$$HH^*\big(EZ^*\big):HH^*(A\otimes B)\simeq HH^*(A)\otimes HH^*(B):HH^*\big(AW^*\big).$$

Proof. The isomorphism $HH^*(A)\otimes HH^*(B)\simeq HH^*(A\otimes B)$ is given by $HH^*(AW^*)$. So given bimodule homomorphisms $\alpha:\mathbb{B}(A)\to A$ and $\beta:\mathbb{B}(B)\to B$, write $\alpha\times\beta=(\alpha\otimes\beta)\circ AW_*:\mathbb{B}(A\otimes B)\to A\otimes B$. What we need to prove is that

$$(\alpha\smile\beta)\times\big(\alpha'\smile\beta'\big)=(-1)^{|\beta||\alpha'|}\big(\alpha\times\alpha'\big)\smile\big(\beta\times\beta'\big)$$

for $\alpha, \alpha':\mathbb{B}(A)\to A$ and $\beta, \beta':\mathbb{B}(B)\to B$.

Since $\epsilon_A:\mathbb{B}(A)\to A$ and $\epsilon_A\otimes_A\epsilon_A:\mathbb{B}(A)\otimes_A\mathbb{B}(A)\to A$ are projective resolutions of A as bimodules, there exists a comparison morphism $\Delta_A:\mathbb{B}(A)\to\mathbb{B}(A)\otimes_A\mathbb{B}_*(A)$ lifting the identity of A, which is unique up to homotopy of A-bimodules. Hence there is a commutative triangle:

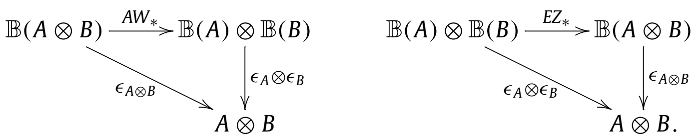


Let $\alpha : \mathbb{B}(A) \to A$ and $\beta : \mathbb{B}(A) \to A$ be two bimodule maps. By definition, the cup product of α and β is $\alpha \smile \beta = \Delta_A(\alpha \otimes_A \beta)$. If α and β are cocycles, then $\alpha \smile \beta$ is a cocycle and its cohomological class is independent of the choice of Δ_A. A possible choice of Δ_A is given by (for $a_0, \dots, a_n \in A$)

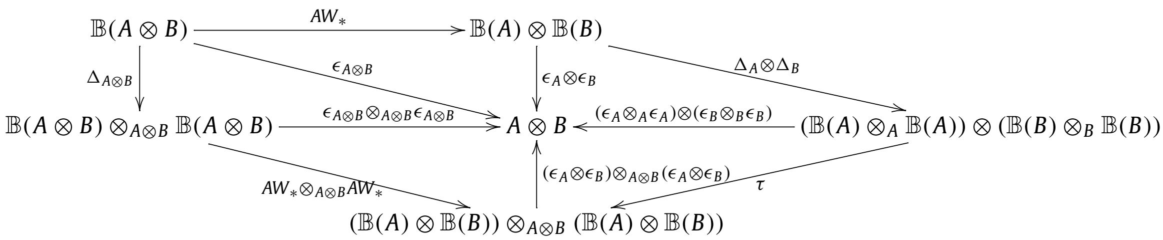
$$\Delta_A(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = \sum_{p=0}^{n} (a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1) \otimes_A (1 \otimes a_{p+1} \otimes \cdots \otimes a_n \otimes a_{n+1}).$$

With this map, we recover the usual definition of cup product.

The Alexander–Whitney map AW_* and the Eilenberg–Zilber map EZ_* fit also into the commutative triangles:



Therefore, combining several commutative triangles as above, we obtain a diagram with all triangles commutative:



where τ is the middle four interchange isomorphism sending $x_1 \otimes x_2 \otimes y_1 \otimes y_2$ to $(-1)^{|x_2||y_1|} x_1 \otimes y_1 \otimes x_2 \otimes y_2$. Since both $\tau \circ (\Delta_A \otimes \Delta_B) \circ AW_*$ and $(AW_* \otimes_{A\otimes B} AW_*) \circ \Delta_{A\otimes B}$ lift the identity of $A \otimes B$, they are homotopy equivalent. The outer pentagon of the above diagram is hence commutative up to homotopy. By composing with $(\alpha \otimes \beta) \otimes_{A\otimes B} (\alpha' \otimes \beta')$, we deduce that

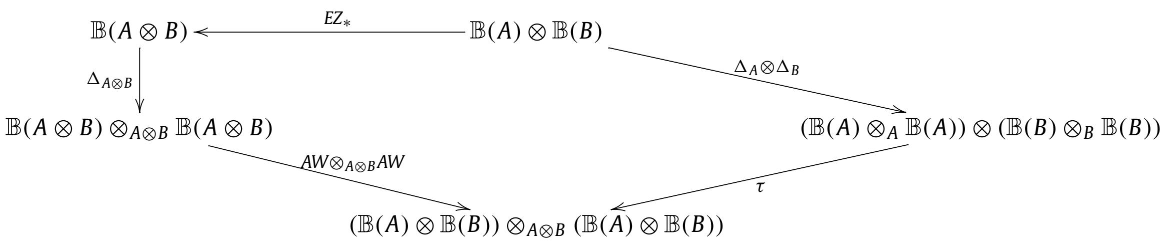
$$\begin{aligned}(-1)^{|\beta||\alpha'|}(\alpha \times \alpha') \smile (\beta \times \beta') &= (\alpha \otimes \beta) \otimes_{A\otimes B} (\alpha' \otimes \beta') \circ \tau \circ (\Delta_A \otimes \Delta_B) \circ AW_* \\ &= (\alpha \otimes \beta) \otimes_{A\otimes B} (\alpha' \otimes \beta') \circ (AW_* \otimes_{A\otimes B} AW_*) \circ \Delta_{A\otimes B} \\ &= (\alpha \smile \beta) \times (\alpha' \smile \beta').\end{aligned}$$

We have proved that

$$H^*(AW^*) : HH^*(A) \otimes HH^*(B) \to HH^*(A \otimes B)$$

is an isomorphism of graded algebras. $\square$

Remark 3.2. The outer pentagon of the diagram in the proof of Lemma 3.1 is only commutative up to homotopy and it is NOT strictly commutative, as one may readily verify (even with the specified choice of Δ_A, etc.). However, if we replace AW_* by EZ_* in the first row of the diagram, then a tedious computation shows that the outer pentagon is commutative:



.

This enables us to obtain a formula on the cochain level. Let $\alpha \in \mathrm{Hom}_k(\overline{A}^{\otimes n}, A)$, $\beta \in \mathrm{Hom}_k(\overline{B}^{\otimes m}, B)$, $\alpha' \in \mathrm{Hom}_k(\overline{A}^{\otimes s}, A)$ and $\beta' \in \mathrm{Hom}_k(\overline{B}^{\otimes t}, B)$ with $n, m, s, t \in \mathbb{Z}_{\geqslant 0}$. We obtain the equality

$$EZ^{n+m+s+t}\big(AW^{n+m}(\alpha\otimes\beta)\smile AW^{s+t}\big(\alpha'\otimes\beta'\big)\big)=(-1)^{ms}\big(\alpha\smile\alpha'\big)\otimes\big(\beta\smile\beta'\big) \tag{3.2}$$

on the cochain level, which implies the isomorphism on the cohomological level.

We remark also that $AW_*\circ EZ_*=Id$, but $EZ_*\circ AW_*\neq Id$, as $\mathbb{B}(A)\otimes\mathbb{B}(B)$ is much smaller than $\mathbb{B}(A\otimes B)$. This also explain the outer pentagon of the diagram in the proof of Lemma 3.1 is only commutative up to homotopy, since the above pentagon in this remark is strictly commutative.

We investigate further the Gerstenhaber algebra structure over the Hochschild cohomology of the tensor product of two k-algebras. We have reviewed the tensor product of two Gerstenhaber algebras in Proposition-Definition 2.2.

Theorem 3.3. *Let A and B be two k-algebras such that one of them is finite dimensional. Then there is an isomorphism of Gerstenhaber algebras*

$$HH^*(A\otimes B)\simeq HH^*(A)\otimes HH^*(B).$$

Proof. It is sufficient to prove that the isomorphism in Lemma 3.1 preserves Lie bracket. In fact, we show a little bit more. To be precise, given $\alpha\in \mathrm{Hom}_k(\overline{A}^{\otimes n},A)$, $\beta\in\mathrm{Hom}_k(\overline{B}^{\otimes m},B)$, $\alpha'\in\mathrm{Hom}_k(\overline{A}^{\otimes s},A)$ and $\beta'\in\mathrm{Hom}_k(\overline{B}^{\otimes t},B)$, we have the following equation on the cochain level

$$\begin{aligned}&EZ^{n+m+s+t-1}\big(\big[AW^{n+m}(\alpha\otimes\beta),AW^{s+t}\big(\alpha'\otimes\beta'\big)\big]\big)\\&\quad=(-1)^{t(n+m-1)}\big[\alpha,\alpha'\big]\otimes\big(\beta\smile\beta'\big)+(-1)^{n(s+t-1)}\big(\alpha\smile\alpha'\big)\otimes\big[\beta,\beta'\big]-(-1)^{n(s+t)}\delta\big(\big(\alpha'\,\bar\circ\,\alpha\big)\otimes\big(\beta\,\bar\circ\,\beta'\big)\big)\\&\qquad-(-1)^{(n-1)(s+t-1)}\big(\alpha'\,\bar\circ\,\alpha\big)\otimes\big(\delta(\beta)\bar\circ\beta'-(-1)^t\beta\,\bar\circ\,\delta\big(\beta'\big)\big)-(-1)^{n(s+t-1)}\big(\delta\big(\alpha'\big)\,\bar\circ\,\alpha-(-1)^n\alpha'\,\bar\circ\,\delta(\alpha)\big)\\&\qquad\otimes\big(\beta\,\bar\circ\,\beta'\big).\end{aligned}\tag{3.3}$$

Then we deduce the desired formula on the cohomological level, since α,β,α' and β' lie in the corresponding cocycle, and $\delta((\alpha'\,\bar\circ\,\alpha)\otimes(\beta\,\bar\circ\,\beta'))$ lies in the coboundary.

We only show the case $n,m,s,t\geqslant 1$, since the cases when some of them are zero are easier.

Recall that

$$\begin{aligned}\big[AW^{n+m}(\alpha\otimes\beta),AW^{s+t}\big(\alpha'\otimes\beta'\big)\big]&=AW^{n+m}(\alpha\otimes\beta)\,\bar\circ\,AW^{s+t}\big(\alpha'\otimes\beta'\big)\\&\quad-(-1)^{(n+m-1)(s+t-1)}AW^{s+t}\big(\alpha'\otimes\beta'\big)\,\bar\circ\,AW^{n+m}(\alpha\otimes\beta),\end{aligned}$$

where

$$AW^{n+m}(\alpha\otimes\beta)\,\bar\circ\,AW^{s+t}\big(\alpha'\otimes\beta'\big)=\sum_{i=1}^{n+m}(-1)^{(i-1)(s+t-1)}\big(AW^{n+m}(\alpha\otimes\beta)\,\bar\circ_i\,AW^{s+t}\big(\alpha'\otimes\beta'\big)\big).$$

Now let

$$EZ^{n+m+s+t-1}\big(AW^{n+m}(\alpha\otimes\beta)\,\bar\circ_i\,AW^{s+t}\big(\alpha'\otimes\beta'\big)\big)=\big(\vartheta^{-1}\big(\xi_0^i\big),\dots,\vartheta^{-1}\big(\xi_{n+m+s+t-1}^i\big)\big)$$

with $\xi_j^i\in\mathrm{Hom}_k(\overline{A}^{\otimes(n+m+s+t-1-j)}\otimes\overline{B}^{\otimes j},A\otimes B)$. Then for $a_1,\dots,a_{n+m+s+t-1-j}\in A$ and $b_1,\dots,b_j\in B$,

$$\begin{aligned}&\xi_j^i(a_1\otimes\cdots\otimes a_{n+m+s+t-1-j}\otimes b_1\otimes\cdots\otimes b_j)\\&\quad=\sum_{\sigma\in\mathcal{S}_{n+m+s+t-1-j,j}}(-1)^{|\sigma|}\big(AW^{n+m}(\alpha\otimes\beta)\,\bar\circ_i\,AW^{s+t}\big(\alpha'\otimes\beta'\big)\big)\big(F_\sigma(a_1\otimes\cdots\otimes a_{n+m+s+t-1-j}\otimes b_1\otimes\cdots\otimes b_j)\big).\end{aligned}$$

Suppose that

$$F_\sigma(a_1\otimes\cdots\otimes a_{n+m+s+t-1-j}\otimes b_1\otimes\cdots\otimes b_j)=\big(x_1^\sigma\otimes y_1^\sigma\big)\otimes\cdots\otimes\big(x_{n+m+s+t-1}^\sigma\otimes y_{n+m+s+t-1}^\sigma\big),$$

with $x_1^\sigma,\dots,x_{n+m+s+t-1}^\sigma\in A$ and $y_1^\sigma,\dots,y_{n+m+s+t-1}^\sigma\in B$. We have two cases.

Case 1: If $m+1\leqslant i\leqslant n+m$, then

$$\begin{aligned}&\big(AW^{n+m}(\alpha\otimes\beta)\,\bar\circ_i\,AW^{s+t}\big(\alpha'\otimes\beta'\big)\big)\big(\big(x_1^\sigma\otimes y_1^\sigma\big)\otimes\cdots\otimes\big(x_{n+m+s+t-1}^\sigma\otimes y_{n+m+s+t-1}^\sigma\big)\big)\\&\quad=AW^{n+m}(\alpha\otimes\beta)\big(\big(x_1^\sigma\otimes y_1^\sigma\big)\otimes\cdots\otimes\big(x_{i-1}^\sigma\otimes y_{i-1}^\sigma\big)\otimes AW^{s+t}\big(\alpha'\otimes\beta'\big)\big(\big(x_i^\sigma\otimes y_i^\sigma\big)\otimes\cdots\\&\qquad\otimes\big(x_{i+s+t-1}^\sigma\otimes y_{i+s+t-1}^\sigma\big)\big)\otimes\big(x_{i+s+t}^\sigma\otimes y_{i+s+t}^\sigma\big)\otimes\cdots\otimes\big(x_{n+m+s+t-1}^\sigma\otimes y_{n+m+s+t-1}^\sigma\big)\big)\\&\quad=(-1)^{nm+st}\big(x_1^\sigma\cdots x_m^\sigma\cdot\alpha\big(x_{m+1}^\sigma\otimes\cdots\otimes x_{i-1}^\sigma\otimes\big(x_i^\sigma\cdots x_{i+t-1}^\sigma\cdot\alpha'\big(x_{i+t}^\sigma\otimes\cdots\otimes x_{i+t+s-1}^\sigma\big)\big)\otimes x_{i+t+s}^\sigma\otimes\cdots\\&\qquad\otimes x_{n+m+s+t-1}^\sigma\big)\big)\otimes\big(\beta\big(y_1^\sigma\otimes\cdots\otimes y_m^\sigma\big)\cdot y_{m+1}^\sigma\cdots y_{i-1}^\sigma\cdot\beta'\big(y_i^\sigma\otimes\cdots\otimes y_{i+t-1}^\sigma\big)\cdot y_{i+t}^\sigma\cdots y_{n+m+s+t-1}^\sigma\big).\end{aligned}$$

Observe that the only non-zero case is when $j = m + t$ and $\sigma \in \mathcal{S}_{n+s-1,m+t}$ is the following permutation:

$$\begin{pmatrix} 1,\dots,i-m-1, & i-m,\dots,n+s-1, & n+s,\dots,n+s+m-1, & n+s+m,\dots,n+m+s+t-1 \\ m+1,\dots,i-1, & i+t,\dots,n+m+s+t-1, & 1,\dots,m, & i,\dots,i+t-1 \end{pmatrix},$$

hence

$$EZ^{n+m+s+t-1}\big(AW^{n+m}(\alpha\otimes\beta)\,\bar{\circ}_i\,AW^{s+t}(\alpha'\otimes\beta')\big) = \big(0,\dots,0,\vartheta^{-1}(\xi^i_{m+t}),0,\dots,0\big).$$

We identify $(0,\dots,0,\vartheta^{-1}(\xi^i_{m+t}),0,\dots,0)$ with $\vartheta^{-1}(\xi^i_{m+t})$, and obtain that

$$\begin{aligned}
&\xi^i_{m+t}(a_1\otimes\cdots\otimes a_{n+s-1}\otimes b_1\otimes\cdots\otimes b_{m+t}) \\
&\quad= (-1)^{m(n+s-1)+t(n+m+s-i)}\big(AW^{n+m}(\alpha\otimes\beta)\,\bar{\circ}_i\,AW^{s+t}(\alpha'\otimes\beta')\big)\big((1\otimes b_1)\otimes\cdots\otimes(1\otimes b_m)\otimes(a_1\otimes 1)\otimes\cdots \\
&\qquad\otimes(a_{i-m-1}\otimes 1)\otimes(1\otimes b_{m+1})\otimes\cdots\otimes(1\otimes b_{m+t})\otimes(a_{i-m}\otimes 1)\otimes\cdots\otimes(a_{n+s-1}\otimes 1)\big) \\
&\quad= (-1)^{m(n+s-1)+t(n+m+s-i)}(-1)^{nm+st}\alpha\big(a_1\otimes\cdots\otimes a_{i-m-1}\otimes\alpha'(a_{i-m}\otimes\cdots\otimes a_{i-m+s-1})\otimes a_{i-m+s}\otimes\cdots \\
&\qquad\otimes a_{n+s-1}\big)\otimes\beta(b_1\otimes\cdots\otimes b_m)\beta'(b_{m+1}\otimes\cdots\otimes b_{m+t}) \\
&\quad= \vartheta\big((-1)^{m(s-1)+t(n+m-i)}\big((\alpha\,\bar{\circ}_{i-m}\,\alpha')\otimes(\beta\smile\beta')\big)\big)(a_1\otimes\cdots\otimes a_{n+s-1}\otimes b_1\otimes\cdots\otimes b_{m+t}).
\end{aligned}$$

Therefore

$$\vartheta^{-1}\big(\xi^i_{m+t}\big) = (-1)^{m(s-1)+t(n+m-i)}\big((\alpha\,\overline{\circ_{i-m}}\,\alpha')\otimes(\beta\smile\beta')\big).$$

Case 2: Similarly, for $1\leqslant i\leqslant m$, the only non-zero case is when $j = m+t-1$ and $\sigma\in\mathcal{S}_{n+s,m+t-1}$ is the permutation as follows:

$$\begin{pmatrix} 1,\dots,s, & s+1,\dots,n+s, & n+s+1,\dots,i+n+s+t-1, & i+n+s+t,\dots,n+m+s+t-1 \\ i+t,\dots,i+s+t-1, & m+s+t,\dots,n+m+s+t-1, & 1,\dots,i+t-1, & i+s+t,\dots,m+s+t-1 \end{pmatrix}.$$

We have

$$EZ^{n+m+s+t-1}\big(AW^{n+m}(\alpha\otimes\beta)\,\bar{\circ}_i\,AW^{s+t}(\alpha'\otimes\beta')\big) = \big(0,\dots,0,\vartheta^{-1}(\xi^i_{m+t-1}),0,\dots,0\big)$$

and

$$\vartheta^{-1}\big(\xi^i_{m+t-1}\big) = (-1)^{s(i-1)+n(t-1)}\big(\alpha'\smile\alpha\big)\otimes\big(\beta\,\bar{\circ}_i\,\beta'\big).$$

Now it follows that

$$\begin{aligned}
&EZ^{n+m+s+t-1}\big(AW^{n+m}(\alpha\otimes\beta)\,\bar{\circ}\,AW^{s+t}(\alpha'\otimes\beta')\big) \\
&\quad= \sum_{i=1}^{n+m}(-1)^{(i-1)(s+t-1)}EZ^{n+m+s+t-1}\big(AW^{n+m}(\alpha\otimes\beta)\,\bar{\circ}_i\,AW^{s+t}(\alpha'\otimes\beta')\big) \\
&\quad= \sum_{i=m+1}^{n+m}(-1)^{(i-1)(s+t-1)}\vartheta^{-1}\big(\xi^i_{m+t}\big) + \sum_{i=1}^{m}(-1)^{(i-1)(s+t-1)}\vartheta^{-1}\big(\xi^i_{m+t-1}\big) \\
&\quad= \sum_{i=m+1}^{n+m}(-1)^{t(n+m-1)+(i-m-1)(s-1)}\big(\alpha\,\bar{\circ}_{i-m}\,\alpha'\big)\otimes\big(\beta\smile\beta'\big) + \sum_{i=1}^{m}(-1)^{n(t-1)+(i-1)(t-1)}\big(\alpha'\smile\alpha\big)\otimes\big(\beta\,\bar{\circ}_i\,\beta'\big) \\
&\quad= (-1)^{t(n+m-1)}\big(\alpha\,\bar{\circ}\,\alpha'\big)\otimes\big(\beta\smile\beta'\big) + (-1)^{n(t-1)}\big(\alpha'\smile\alpha\big)\otimes\big(\beta\,\bar{\circ}\,\beta'\big).
\end{aligned}$$

Exchanging α with α' and n with s, and in the meanwhile, exchanging β with β' and m with t, we obtain another equation

$$\begin{aligned}
&EZ^{n+m+s+t-1}\big(AW^{s+t}(\alpha'\otimes\beta')\,\bar{\circ}\,AW^{n+m}(\alpha\otimes\beta)\big) \\
&\quad= (-1)^{m(s+t-1)}\big(\alpha'\,\bar{\circ}\,\alpha\big)\otimes\big(\beta'\smile\beta\big) + (-1)^{s(m-1)}\big(\alpha\smile\alpha'\big)\otimes\big(\beta'\,\bar{\circ}\,\beta\big).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&EZ^{n+m+s+t-1}\left(\left[AW^{n+m}(\alpha\otimes\beta), AW^{s+t}\left(\alpha'\otimes\beta'\right)\right]\right)\\
&\quad=(-1)^{t(n+m-1)}\left(\alpha\,\bar{\circ}\,\alpha'\right)\otimes\left(\beta\smile\beta'\right)+(-1)^{n(t-1)}\left(\alpha'\smile\alpha\right)\otimes\left(\beta\,\bar{\circ}\,\beta'\right)-(-1)^{(n-1)(s+t-1)}\left(\alpha'\,\bar{\circ}\,\alpha\right)\otimes\left(\beta'\smile\beta\right)\\
&\qquad-(-1)^{n(s+t-1)+(m-1)(t-1)}\left(\alpha\smile\alpha'\right)\otimes\left(\beta'\,\bar{\circ}\,\beta\right)\\
&\quad=(-1)^{t(n+m-1)}\left[\alpha,\alpha'\right]\otimes\left(\beta\smile\beta'\right)+(-1)^{n(s+t-1)}\left(\alpha\smile\alpha'\right)\otimes\left[\beta,\beta'\right]\\
&\qquad-(-1)^{(n-1)(s+t-1)}\left(\alpha'\,\bar{\circ}\,\alpha\right)\otimes\left(\beta'\smile\beta-(-1)^{mt}\beta\smile\beta'\right)-(-1)^{n(s+t-1)}\left(\alpha\smile\alpha'-(-1)^{ns}\alpha'\smile\alpha\right)\otimes\left(\beta\,\bar{\circ}\,\beta'\right).
\end{aligned}$$

To finish the proof, we apply [4, Theorem 3] to obtain

$$(-1)^n\left(\alpha\smile\alpha'-(-1)^{ns}\alpha'\smile\alpha\right)=\delta\left(\alpha'\,\bar{\circ}\,\alpha\right)-(-1)^{n-1}\delta\left(\alpha'\right)\bar{\circ}\,\alpha-\alpha'\,\bar{\circ}\,\delta(\alpha)$$

and

$$(-1)^t\left(\beta'\smile\beta-(-1)^{mt}\beta\smile\beta'\right)=\delta\left(\beta\,\bar{\circ}\,\beta'\right)-(-1)^{t-1}\delta(\beta)\,\bar{\circ}\,\beta'+\beta\,\bar{\circ}\,\delta\left(\beta'\right).$$

Here δ is the differential of the complexes $\bar{C}^*(A)$ and $\bar{C}^*(B)$. Denote by δ also the differential of the complex $\bar{C}^*(A)\otimes\bar{C}^*(B)$, we have

$$\delta\left(\left(\alpha'\,\bar{\circ}\,\alpha\right)\otimes\left(\beta\,\bar{\circ}\,\beta'\right)\right)=\delta\left(\alpha'\,\bar{\circ}\,\alpha\right)\otimes\left(\beta\bar{\circ}\beta'\right)+(-1)^{n+s-1}\left(\alpha'\,\bar{\circ}\,\alpha\right)\otimes\delta\left(\beta\,\bar{\circ}\,\beta'\right).$$

From these, we finally deduce Eq. (3.3). □

Let us include an immediate consequence of Theorem 3.3 which generalizes slightly [1, Theorem 7.1]. Recall that $HH^0(A)=Z(A)$ the center of A, and $HH^1(A)\simeq Der(A)/InnDer(A)$, where $Der(A)$ is the space of derivations of A and $InnDer(A)$ is the subspace of inner derivations of A. We have

Corollary 3.4. *Let A and B be two k-algebras such that one of them is finite dimensional. Then there is an isomorphism of Lie algebras*

$$\left(\frac{Der(A)}{InnDer(A)}\otimes Z(B)\right)\times\left(Z(A)\otimes\frac{Der(B)}{InnDer(B)}\right)\simeq\frac{Der(A\otimes B)}{InnDer(A\otimes B)}.$$

In the remaining part of this section, we assume that A and B are finite dimensional symmetric k-algebras with bilinear forms $\langle\ ,\ \rangle_A$ and $\langle\ ,\ \rangle_B$ respectively. Then both $HH^*(A)$ and $HH^*(B)$ have induced BV algebra structures as in Theorem 2.6. The tensor product $A\otimes B$ is a symmetric algebra; indeed, its symmetric bilinear form might chose to be as follows

$$\begin{aligned}
&\langle\ ,\ \rangle: A\otimes B\times A\otimes B\longrightarrow k,\\
&\left(a\otimes b, a'\otimes b'\right)\mapsto\langle a,a\rangle_A\left\langle b,b'\right\rangle_B.
\end{aligned}$$

We shall investigate the BV structure over $HH^*(A\otimes B)$ induced by this bilinear form; see Theorem 2.6. In Definition 2.7, we have reviewed the tensor product of two BV algebras. We have the following isomorphism.

Theorem 3.5. *Let A and B be two finite dimensional symmetric algebras. Then there is an isomorphism of BV algebras*

$$HH^*(A\otimes B)\simeq HH^*(A)\otimes HH^*(B).$$

Proof. Since the cup product and the Δ-operator can determine the Lie bracket, it follows from Lemma 3.1 that we only need to prove that the isomorphism above preserves the Δ-operator. By Definition 2.7, it suffices to show that Δ-operator is a derivation under the identification given by the isomorphism $HH^*(AW^*): HH^*(A\otimes B)\xrightarrow{\simeq}HH^*(A)\otimes HH^*(B)$.

For a symmetric algebra A, for any $n\geqslant 0$ there is an isomorphism between $HH^n(A)$ and $\mathrm{Hom}_k(HH_n(A),k)$ induced by the following canonical isomorphisms

$$\theta_A:\bar{C}^*(A)\simeq\mathrm{Hom}_{A^e}\left(\mathbb{B}_n(A),A\right)\simeq\mathrm{Hom}_{A^e}\left(\mathbb{B}_n(A),D(A)\right)\simeq\mathrm{Hom}_k\left(A\otimes_{A^e}\mathbb{B}_n(A),k\right)\simeq\mathrm{Hom}_k\left(\bar{C}_*(A),k\right).$$

This induces a non-degenerate pairing $\langle\ ,\ \rangle:\bar{C}^*(A)\times\bar{C}_*(A)\to k$ inducing $\langle\ ,\ \rangle: HH^*(A)\times HH_*(A)\to k$.

Via this duality, even on the cochain level, EZ^* is dual to the shuffle product, that is, we have a commutative diagram

$$\begin{array}{ccc}
\bar{C}^*(A)\otimes\bar{C}^*(B) & \xleftarrow{\ EZ^*\ } & \bar{C}^*(A\otimes B)\\
\downarrow{\scriptstyle\theta_A\otimes\theta_B} & & \downarrow{\scriptstyle\theta_{A\otimes B}}\\
\mathrm{Hom}_k(\bar{C}_*(A),k)\otimes\mathrm{Hom}_k(\bar{C}_*(B),k) & & \mathrm{Hom}_k(\bar{C}_*(A\otimes B),k)\\
& \searrow{\scriptstyle\psi,\ \simeq} & \downarrow{\scriptstyle\mathrm{Hom}_k(sh,k)}\\
& & \mathrm{Hom}_k(\bar{C}_*(A)\otimes\bar{C}_*(B),k).
\end{array}$$

In fact, let $f \in \bar{C}^n(A \otimes B) = \mathrm{Hom}_k(\overline{A \otimes B}^{\otimes n}, A \otimes B)$. Write $EZ^*(f) = \sum_{i+j=n} g_i \otimes h_j \in \bigoplus_{i+j=n} \bar{C}^i(A) \otimes \bar{C}^j(B)$. Then for $p+q=n$, $\vartheta(f_p \otimes g_q) : \bar{A}^{\otimes p} \otimes \bar{B}^{\otimes q} \to A \otimes B$ sends $a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q$ to

$$\sum_{\sigma \in \mathcal{S}_{p,q}} (-1)^{|\sigma|} f \circ F_\sigma(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q),$$

where ϑ is defined in formula (3.1). The map $\psi \circ (\theta_A \otimes \theta_B) \circ EZ^*(f)$ sends $a_0 \otimes \cdots \otimes a_p \otimes b_0 \otimes \cdots \otimes b_q$ to

$$\begin{aligned}
&\langle a_0, f_p(a_1 \otimes \cdots \otimes a_p)\rangle_A \langle b_0, g_q(b_1 \otimes \cdots \otimes b_q)\rangle_B \\
&\quad = \langle a_0 \otimes b_0,\ f_p(a_1 \otimes \cdots \otimes a_p) \otimes g_q(b_1 \otimes \cdots \otimes b_q)\rangle_{A\otimes B} \\
&\quad = \sum_{\sigma \in \mathcal{S}_{p,q}} (-1)^{|\sigma|} \langle a_0 \otimes b_0, f \circ F_\sigma(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q)\rangle_{A\otimes B}.
\end{aligned}$$

On the other hand, on sees easily that the map $\mathrm{Hom}_k(sh, k) \circ \theta_{A\otimes B}(f)$ sends $a_0 \otimes \cdots \otimes a_p \otimes b_0 \otimes \cdots \otimes b_q$ to

$$\sum_{\sigma \in \mathcal{S}_{p,q}} (-1)^{|\sigma|} \langle a_0 \otimes b_0, f \circ F_\sigma(a_1 \otimes \cdots \otimes a_p \otimes b_1 \otimes \cdots \otimes b_q)\rangle_{A\otimes B}.$$

The above diagram is thus commutative.

Since Connes' $\mathfrak{B}$-operator is a derivation for the shuffle product [9, Corollary 4.3.4], its dual: the Δ-operator is also a derivation via the isomorphism $HH^*(AW^*) : HH^*(A) \otimes HH^*(B) \simeq HH^*(A \otimes B)$. This completes the proof. □

Remark 3.6. Using a tedious computation similar to the proof of Theorem 3.3, we could obtain an equality on the cochain level. Let $\alpha \in \mathrm{Hom}_k(\bar{A}^{\otimes(r-t)}, A)$ and $\beta \in \mathrm{Hom}_k(\bar{B}^{\otimes t}, B)$, with $r \geqslant 1$, $0 \leqslant t \leqslant r$. Denote by Δ^A, Δ^B and $\Delta^{A\otimes B}$ the Δ-operator of the complexes $\bar{C}^*(A)$, $\bar{C}^*(B)$ and $\bar{C}^*(A \otimes B)$ respectively. We obtain that

$$EZ^{r-1} \Delta_r^{A\otimes B} AW^r(\alpha \otimes \beta) = \Delta_{r-t}^A(\alpha) \otimes \beta + (-1)^{r-t} \alpha \otimes \Delta_t^B(\beta) \tag{3.4}$$

on the cochain level, which deduces the isomorphism on the cohomological level.

4. Application: Hochschild cohomology of the group algebra of a finite abelian group

In this section, we indicate how to compute the BV structure over the Hochschild cohomology of the group algebra of a finite abelian group.

Let k be a field of characteristic $p > 0$ and let G be a finite abelian group such that p divides the order of G. Now G can be decomposed as follows:

$$G = C_{p^{n_1}} \times C_{p^{n_2}} \times \cdots \times C_{p^{n_r}} \times H$$

with $r \geqslant 0$, $1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_r$, where C_{p^s} denotes the cyclic group of order p^s and p does not divide the order of H. Now

$$kG \simeq kC_{p^{n_1}} \otimes kC_{p^{n_2}} \otimes \cdots \otimes kC_{p^{n_r}} \otimes kH$$

and thus

$$HH^*(kG) \simeq HH^*(kC_{p^{n_1}}) \otimes HH^*(kC_{p^{n_2}}) \otimes \cdots \otimes HH^*(kC_{p^{n_r}}) \otimes HH^*(kH)$$

which is an isomorphism of BV algebras by Theorem 3.5. We point out that the Hochschild cohomology ring of a finite abelian group is studied in [2] and [6].

As kH is semi-simple by Maschke's theorem, $HH^*(kH) = HH^0(kH) = kH$, so we only need to consider abelian p-groups. By Theorem 3.5, in order to compute the BV structure over $HH^*(kG)$, one needs to compute the BV structure over $HH^*(kC_{p^s})$, but this is well-known, as $kC_{p^s} \simeq k[X]/(X^{p^s})$.

Lemma 4.1. *(See [17, Theorems 4.7 and 4.8].) Let k be a field of characteristic $p > 0$ and $A = k[X]/(X^{p^s})$ with $s \geqslant 1$.*

(1) *If $p \neq 2$, then as a BV algebra, $HH^*(A) = k[x, y, z]/(x^{p^s}, y^2)$ with $|x| = 0$, $|y| = 1$ and $|z| = 2$ and the Δ-operator is given by $\Delta(x^r y^\epsilon z^s) = \epsilon l x^{l-1} z^k$ for $0 \leqslant r \leqslant p^s - 1$, $\epsilon \in \{0, 1\}$ and $s \geqslant 0$.*
(2) *If $p = 2$, then as a BV algebra, $HH^*(A) = k[x, y, z]/(x^{2^s}, y^2 - 2^{s-1}x^{2^s-2}z)$ with $|x| = 0$, $|y| = 1$ and $|z| = 2$ and the Δ-operator is given by $\Delta(x^r y^\epsilon z^s) = \epsilon l x^{l-1} z^k$ for $0 \leqslant r \leqslant p^s - 1$, $\epsilon \in \{0, 1\}$ and $s \geqslant 0$.*

Let us isolate the case where the group is C_p, the cyclic group of order p for later use.

Corollary 4.2. *(See [17].) Let k be a field of characteristic $p > 0$.*

(1) *If $p \neq 2$, then as a BV algebra, $HH^*(kC_p) = k[x, y, z]/(x^p, y^2)$ with $|x| = 0$, $|y| = 1$ and $|z| = 2$ and the Δ-operator over $HH^*(kC_p)$ is given by $\Delta(x^r y^\epsilon z^s) = \epsilon r x^{r-1} z^s$ for $r \in \mathbb{Z}/p$, $\epsilon \in \mathbb{Z}/2$ and $s \geqslant 0$. As a consequence, the Gerstenhaber Lie bracket is generated (using the Poisson rule) by*

$$[x, y] = 1, \qquad [y, z] = 0 = [z, y].$$

More generally, for $r, r' \in \mathbb{Z}/p$, $\epsilon, \epsilon' \in \mathbb{Z}/2$ and $s, s' \geqslant 0$, we have

$$\left[x^r y^\epsilon z^s, x^{r'} y^{\epsilon'} z^{s'}\right] = (-1)^{1+(\epsilon-1)\epsilon'}\left((\epsilon+\epsilon')(r+r')x^{r+r'-1} - \epsilon r x^{r+r'-1} y^{\epsilon'} - (-1)^\epsilon \epsilon' r' x^{r+r'-1} y^\epsilon\right) z^{s+s'}.$$

(2) *If $p = 2$, then as a BV algebra, $HH^*(kC_2) = k[x, y]/(x^2)$ with $|x| = 0$ and $|y| = 1$. The Δ-operator over $HH^*(kC_2)$ is given by $\Delta(x^r y^s) = \epsilon r x^{r-1} y^{2[\frac{s}{2}]}$ for $r \in \mathbb{Z}/2$ and $s \geqslant 0$, where $[x]$ denotes the biggest integer not bigger than x and $\epsilon = s - 2[\frac{s}{2}]$. As a consequence, the Gerstenhaber Lie bracket is generated (using the Poisson rule) by $[x, y] = 1$. More generally, for $r, r' \in \mathbb{Z}/p$ and $s, s' \geqslant 0$, set $\epsilon = s - 2[\frac{s}{2}]$ and $\epsilon' = s' - 2[\frac{s'}{2}]$, then we have*

$$\left[x^r y^s, x^{r'} y^{s'}\right] = \left((\epsilon+\epsilon')(r+r')x^{r+r'-1} - \epsilon r x^{r+r'-1} y^{\epsilon'} - \epsilon' r' x^{r+r'-1} y^\epsilon\right) y^{2[\frac{s}{2}]+2[\frac{s'}{2}]}.$$

We refrain from giving here a rather complicated formula for the BV structure over $HH^*(kG)$ for a finite abelian group G. On the contrary, we consider as examples elementary abelian p-groups. In this case, we give the precise formula for the Δ-operator and the Gerstenhaber Lie algebra structure can be obtained accordingly.

Theorem 4.3. *Let k be a field of characteristic $p > 0$ and $G = C_p^n$ be the elementary abelian group of rank $n \geqslant 1$.*

(1) *If $p \neq 2$, then*

$$HH^*\left(kC_p^n\right) \simeq k[x_1, \ldots, x_n, z_1, \ldots, z_n]/\left(x_1^p, \ldots, x_n^p\right) \otimes \Lambda(y_1, \ldots, y_n)$$

with $|x_i| = 0$, $|y_i| = 1$, $|z_i| = 2$ for any $1 \leqslant i \leqslant n$ and where $\Delta(y_1, \ldots, y_n)$ is the exterior algebra with n generators. The Δ-operator on the right-hand side is given by the following formula: for $r_1, \ldots, r_n, s_1, \ldots, s_n \geqslant 0$ and $\epsilon_1, \ldots, \epsilon_n \in \{0, 1\}$,

$$\begin{aligned}
&\Delta\left(x_1^{r_1} y_1^{\epsilon_1} z_1^{s_1} x_2^{r_2} y_2^{\epsilon_2} z_2^{s_2} \cdots x_n^{r_n} y_n^{\epsilon_n} z_n^{s_n}\right)\\
&\quad = \epsilon_1 r_1 x_1^{r_1-1} z_1^{s_1} x_2^{r_2} y_2^{\epsilon_2} z_2^{s_2} \cdots x_n^{r_n} y_n^{\epsilon_n} z_n^{s_n}\\
&\qquad + (-1)^{\epsilon_1} \epsilon_2 r_2 x_1^{r_1} y_1^{\epsilon_1} z_1^{s_1} x_2^{r_2-1} z_2^{s_2} \cdots x_n^{r_n} y_n^{\epsilon_n} z_n^{s_n}\\
&\qquad + \cdots\\
&\qquad + (-1)^{\epsilon_1+\cdots+\epsilon_{n-1}} \epsilon_n r_n x_1^{r_1} y_1^{\epsilon_1} z_1^{s_1} x_2^{r_2} y_2^{\epsilon_2} z_2^{s_2} \cdots x_n^{r_n-1} z_n^{s_n}.
\end{aligned}$$

The Gerstenhaber Lie bracket is generated (using the Poisson rule) by

$$[x_i, x_j] = 0, \qquad [y_i, y_j] = 0, \qquad [x_i, y_j] = \delta_{ij}$$

and z_i are central for $1 \leqslant i, j \leqslant n$.

(2) *If $p = 2$, then*

$$HH^*\left(kC_2^n\right) \simeq k[x_1, \ldots, x_n, y_1, \ldots, y_n]/\left(x_1^2, \ldots, x_n^2\right)$$

with $|x_i| = 0$, $|y_i| = 1$ for any $1 \leqslant i \leqslant n$. The Δ-operator on the right-hand side is given by the following formula: for $r_1, \ldots, r_n, s_1, \ldots, s_n \geqslant 0$, set $\epsilon_i = s_i - 2[\frac{s_i}{2}]$ for each $1 \leqslant i \leqslant n$, where $[x]$ denotes the biggest integer not bigger than x, and we have

$$\begin{aligned}
&\Delta\left(x_1^{r_1} y_1^{s_1} x_2^{r_2} y_2^{s_2} \cdots x_n^{r_n} y_n^{s_n}\right)\\
&\quad = \epsilon_1 r_1 x_1^{r_1-1} y_1^{2[\frac{s_1}{2}]} x_2^{r_2} y_2^{s_2} \cdots x_n^{r_n} y_n^{s_n}\\
&\qquad + \epsilon_2 r_2 x_1^{r_1} y_1^{s_1} x_2^{r_2-1} y_2^{2[\frac{s_2}{2}]} \cdots x_n^{r_n} y_n^{s_n}\\
&\qquad + \cdots\\
&\qquad + \epsilon_n r_n x_1^{r_1} y_1^{s_1} x_2^{r_2} y_2^{s_2} \cdots x_n^{r_n-1} y_n^{2[\frac{s_n}{2}]}.
\end{aligned}$$

The Gerstenhaber Lie bracket is generated (using the Poisson rule) by

$$[x_i, x_j] = 0, \qquad [y_i, y_j] = 0, \qquad [x_i, y_j] = \delta_{ij}.$$

Proof. This follows from Corollary 4.2 and Theorem 3.5. The formula for the Δ-operator uses the formula

$$\Delta(a \otimes b) = \Delta(a) \otimes b + (-1)^{|a|} a \otimes \Delta(b).$$

The Gerstenhaber Lie bracket is computed using the formula

$$[a, b] = -(-1)^{(|a|-1)|b|}\big(\Delta(a \smile b) - \Delta(a) \smile b - (-1)^{|a|} a \smile \Delta(b)\big). \qquad \square$$

Remark 4.4. In the paper [14], Sélène Sanchez-Flores computed the Lie bracket over the Hochschild cohomology of a cyclic group, using a method different from ours. For a cyclic group, it is not difficult to see that we obtain the same result, but our method can deal with any abelian group instead of cyclic ones. The paper [17] also considered cyclic group case using the BV formalism.

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