L1-Regression based Subdivision Schemes for Noisy Data

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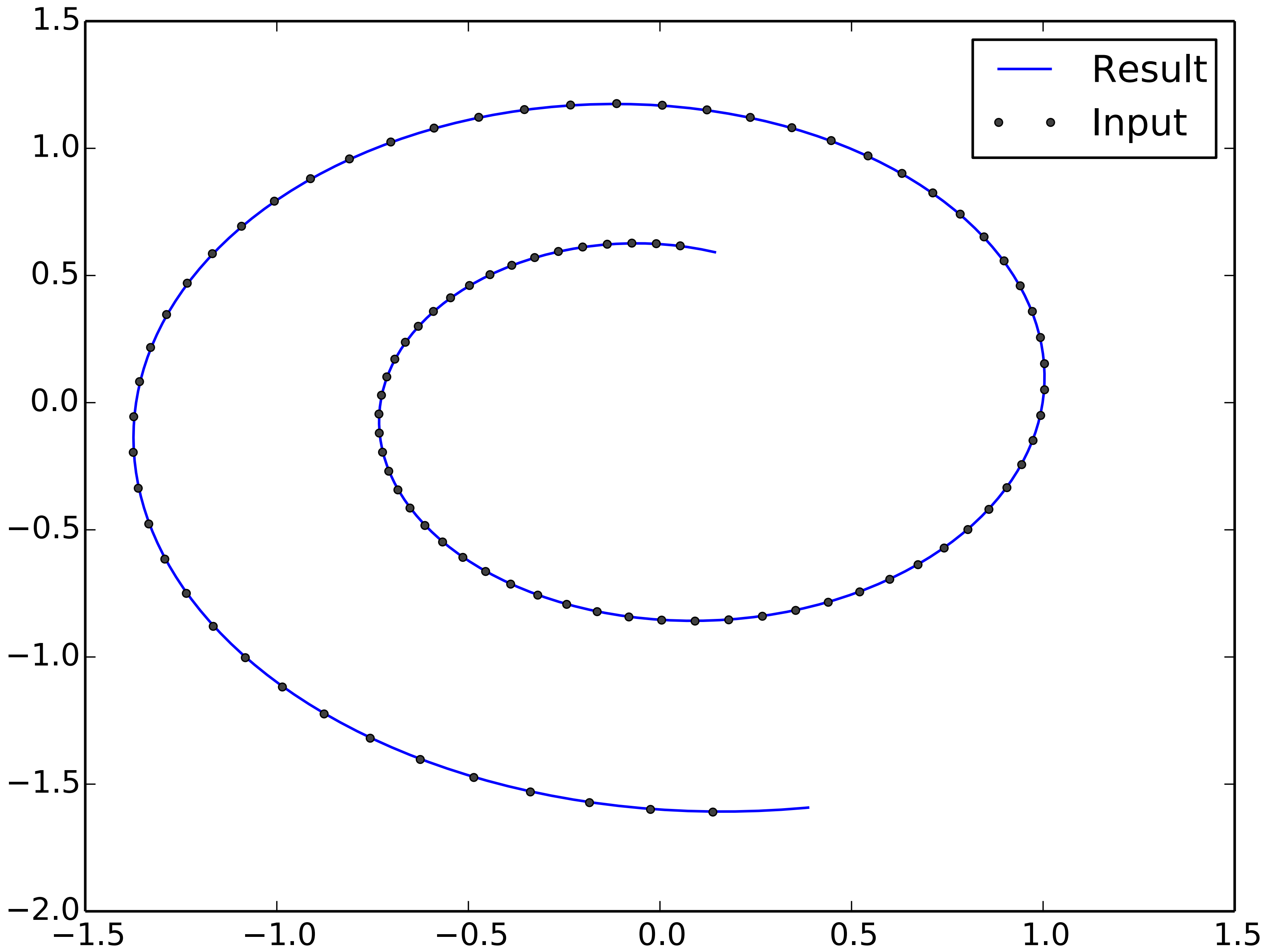
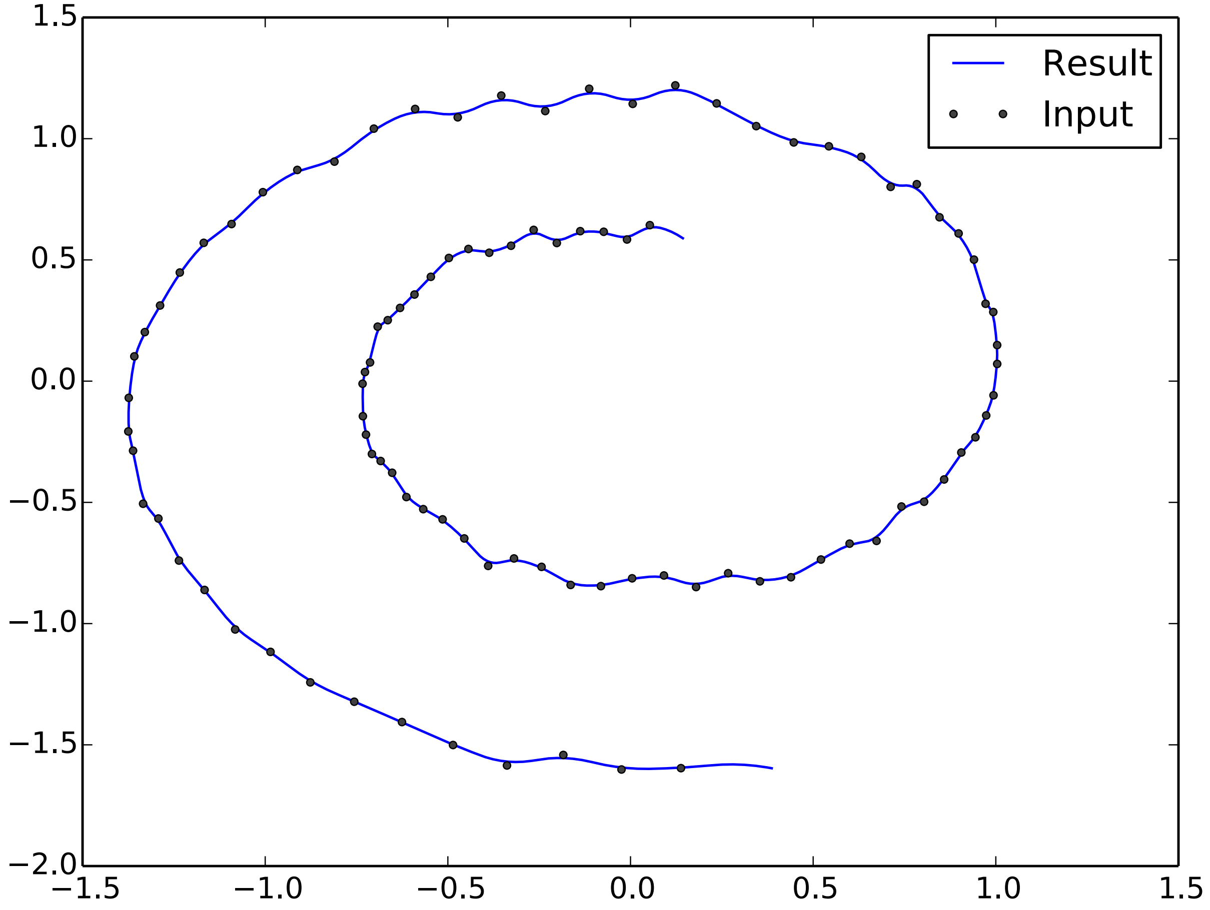
Subdivision

- Subdivision for curve/surface fitting: simple, easy to implement, arbitrary topology, local support
- Only clean input data is considered in classical subdivision schemes

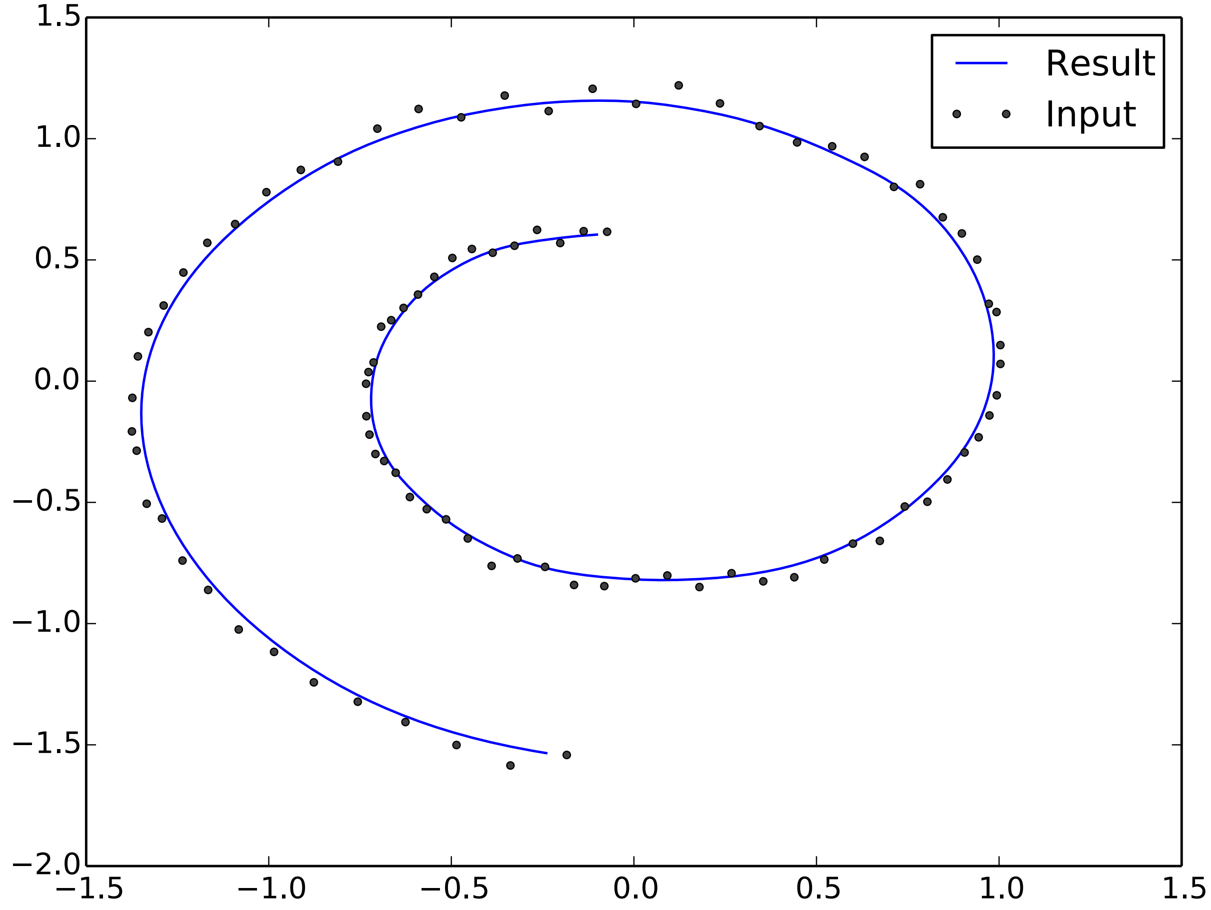


Classical Chaikin Scheme

Data with Gaussian Noise



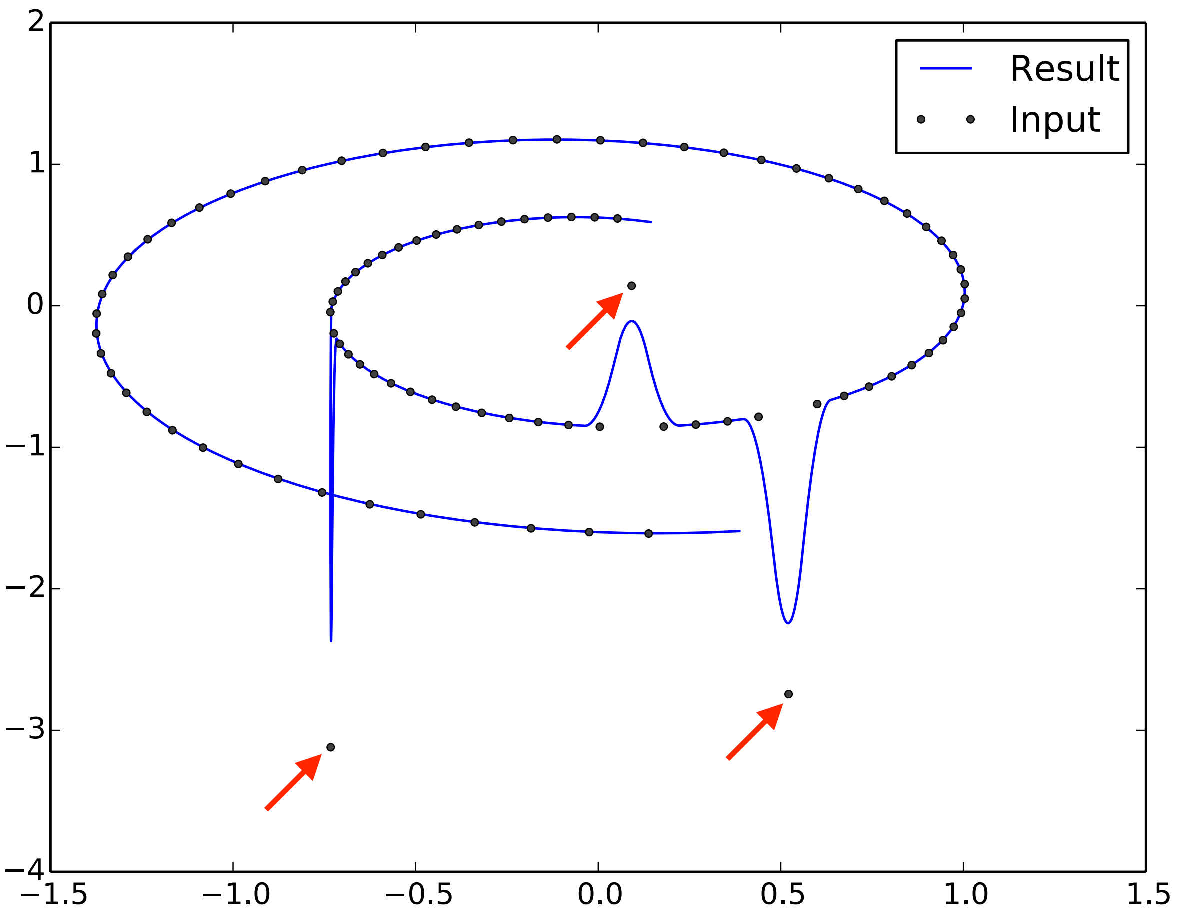
Classical Scheme



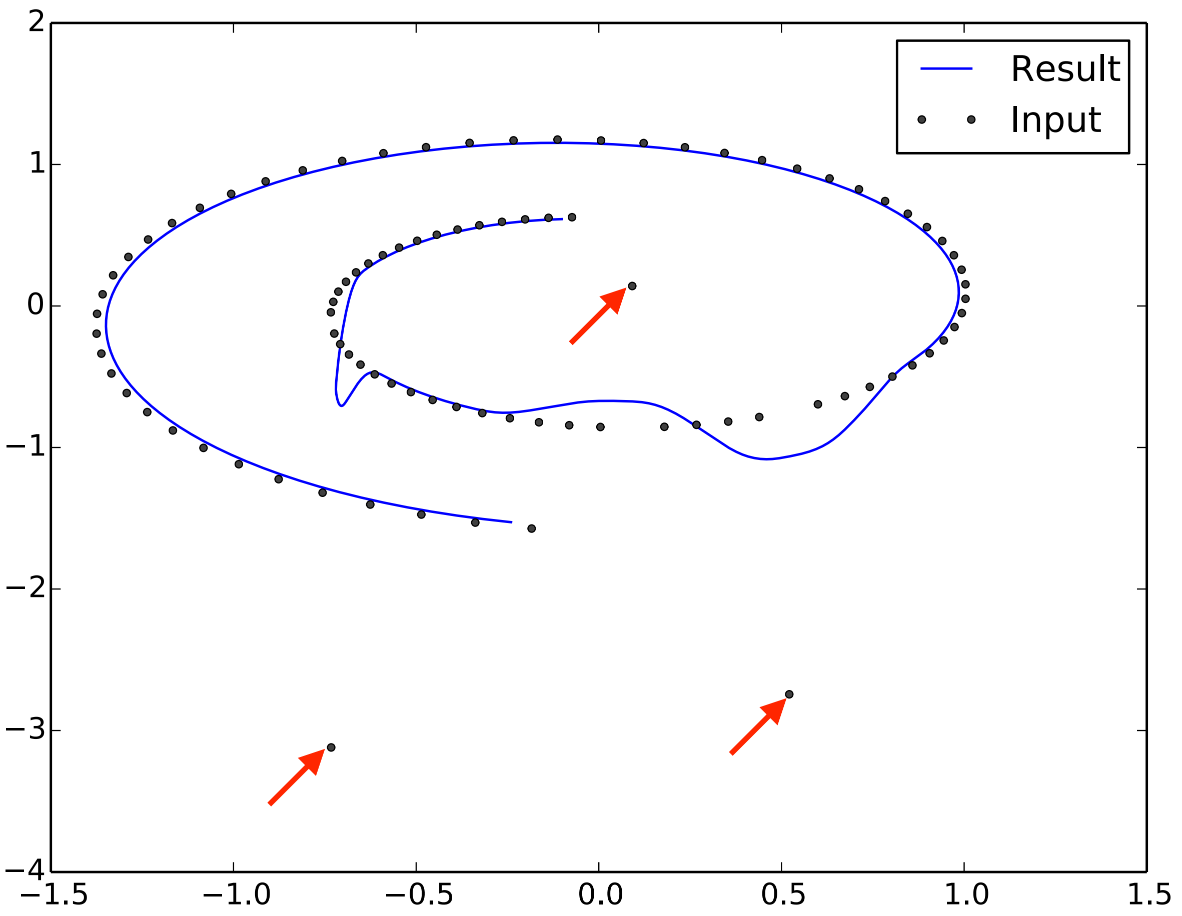
LS Scheme

[Dyn, et.al. Univariate subdivision schemes for noisy data, 2013]

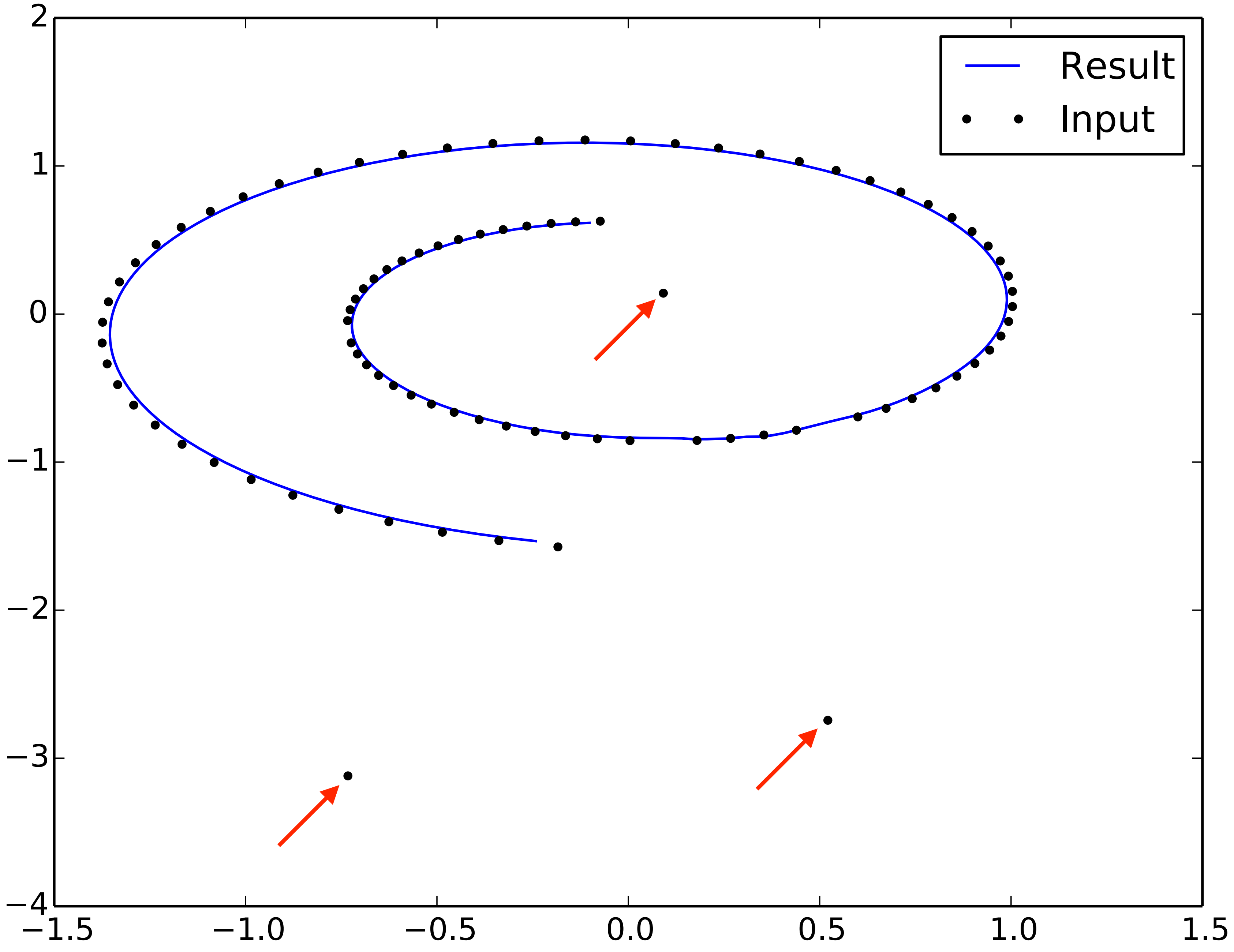
Data with Outliers



Classical Scheme



LS Scheme



Proposed L1 Scheme

Image Denoising: Gaussian

Restore

$$\min_u \sum_i \sum_j \|\nabla u(i,j)\| + \frac{\lambda}{2}\|u - f\|_2^2$$

Image Denoising: Outliers

Restore

$$\min_u \sum_i \sum_j \|\nabla u(i,j)\| + \frac{\lambda}{2}\|u - f\|_1$$

Extend this idea to subdivision

Least Square Subdivision

Given data $y_1, \cdots, y_m$ at nodes $x_1, \cdots, x_m$

Find a linear polynomial minimizing the squared residual

$$\min_{\beta_1, \beta_2 \in R} \sum_{i=1}^{m} (y_i - (\beta_1 + \beta_2 x_i))^2$$

[Dyn, et.al. Univariate subdivision schemes for noisy data, 2013]

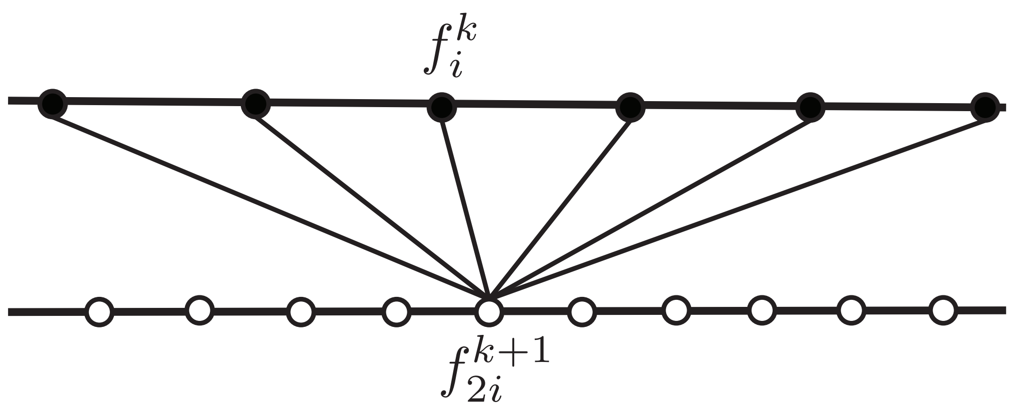
Proposed L1 Model

LS Scheme: $$\min_{\beta_1,\beta_2 \in R} \sum_{i=1}^{m} (y_i - (\beta_1 + \beta_2 x_i))^2$$

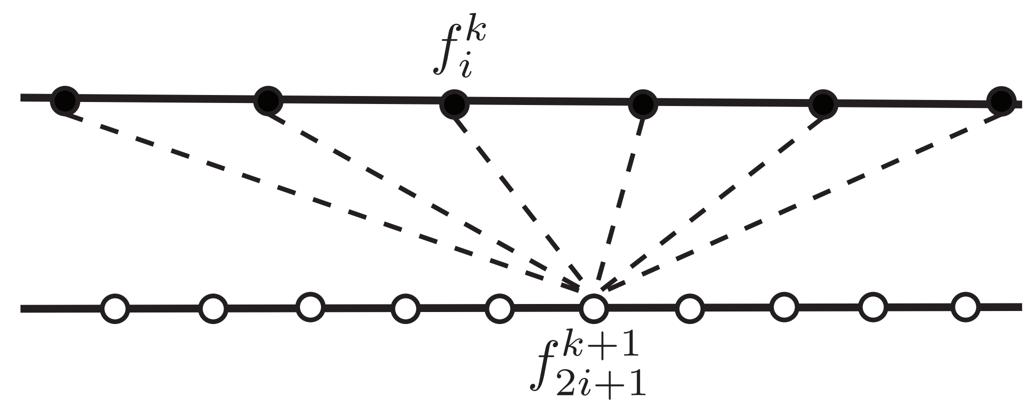
Proposed L1 Scheme: $$\min_{\beta_1,\beta_2 \in R} \sum_{i=1}^{m} |y_i - (\beta_1 + \beta_2 x_i)|$$

Refinement Rule

Even:

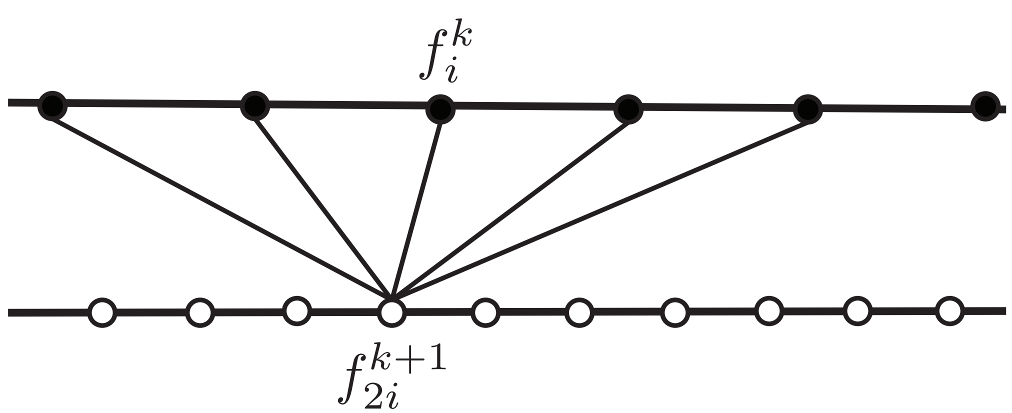


$1/4$

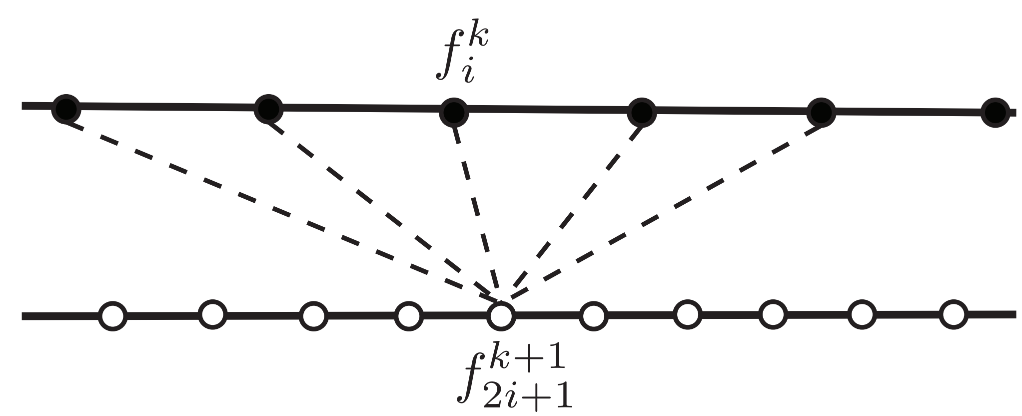


$3/4$

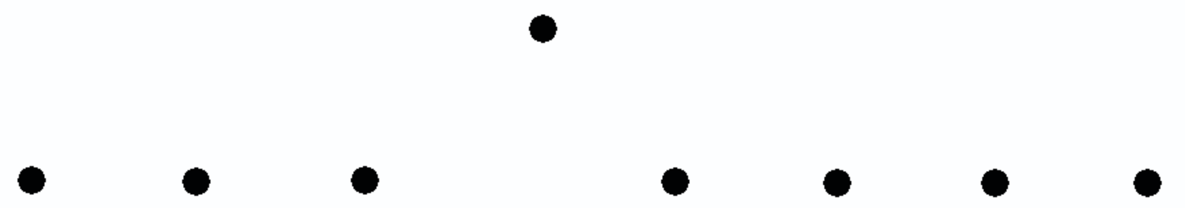
Odd:

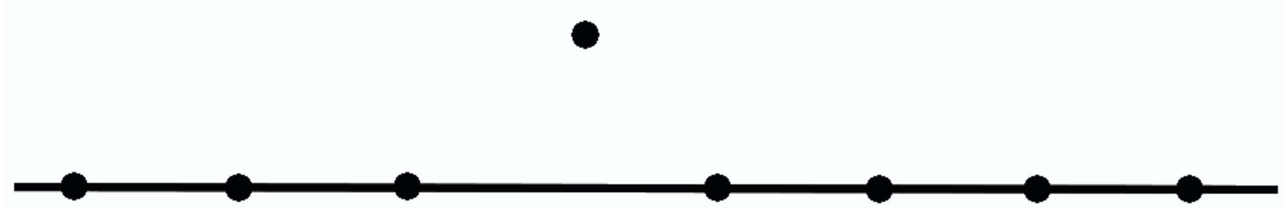


$-1/4$



$1/4$

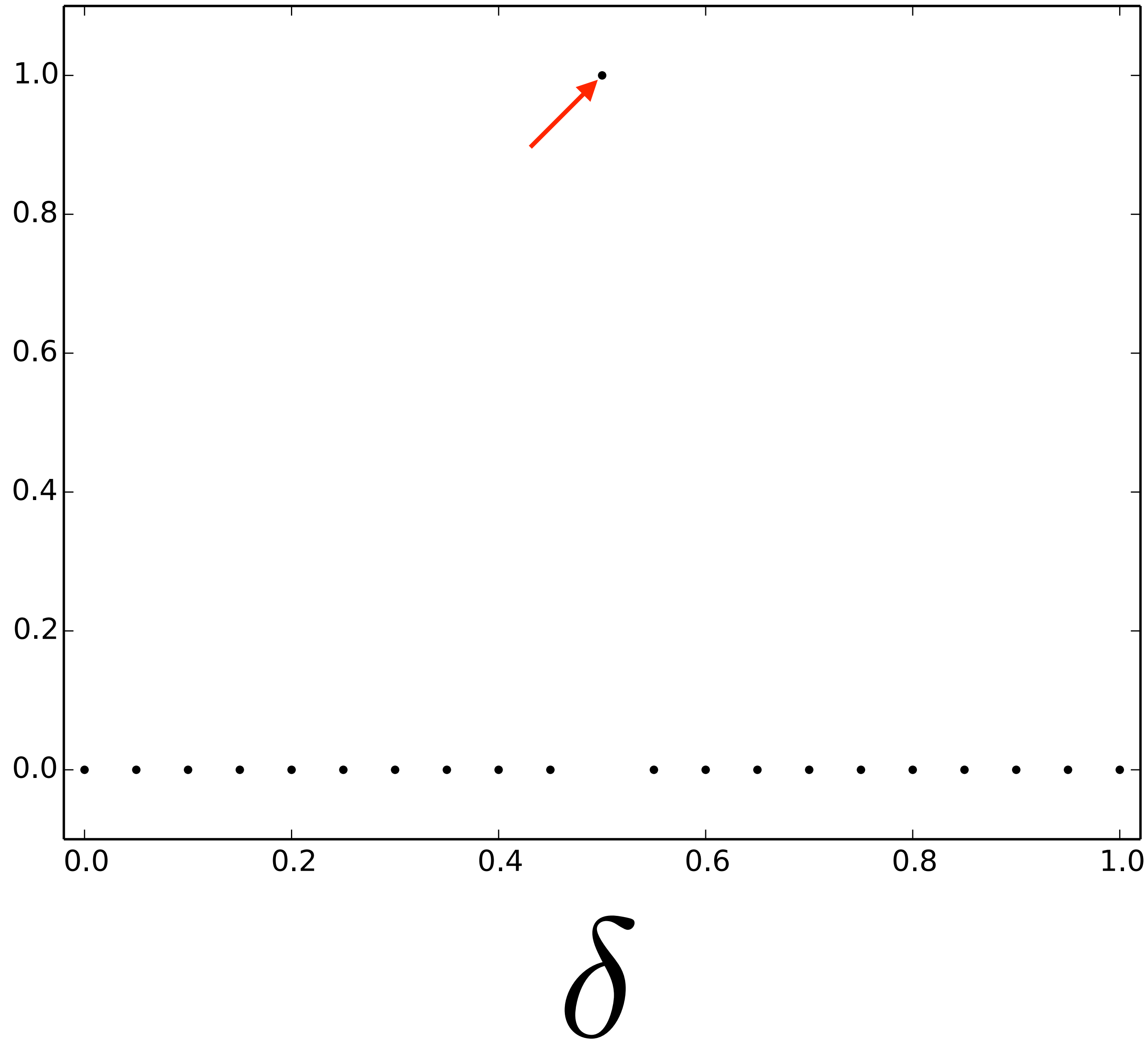




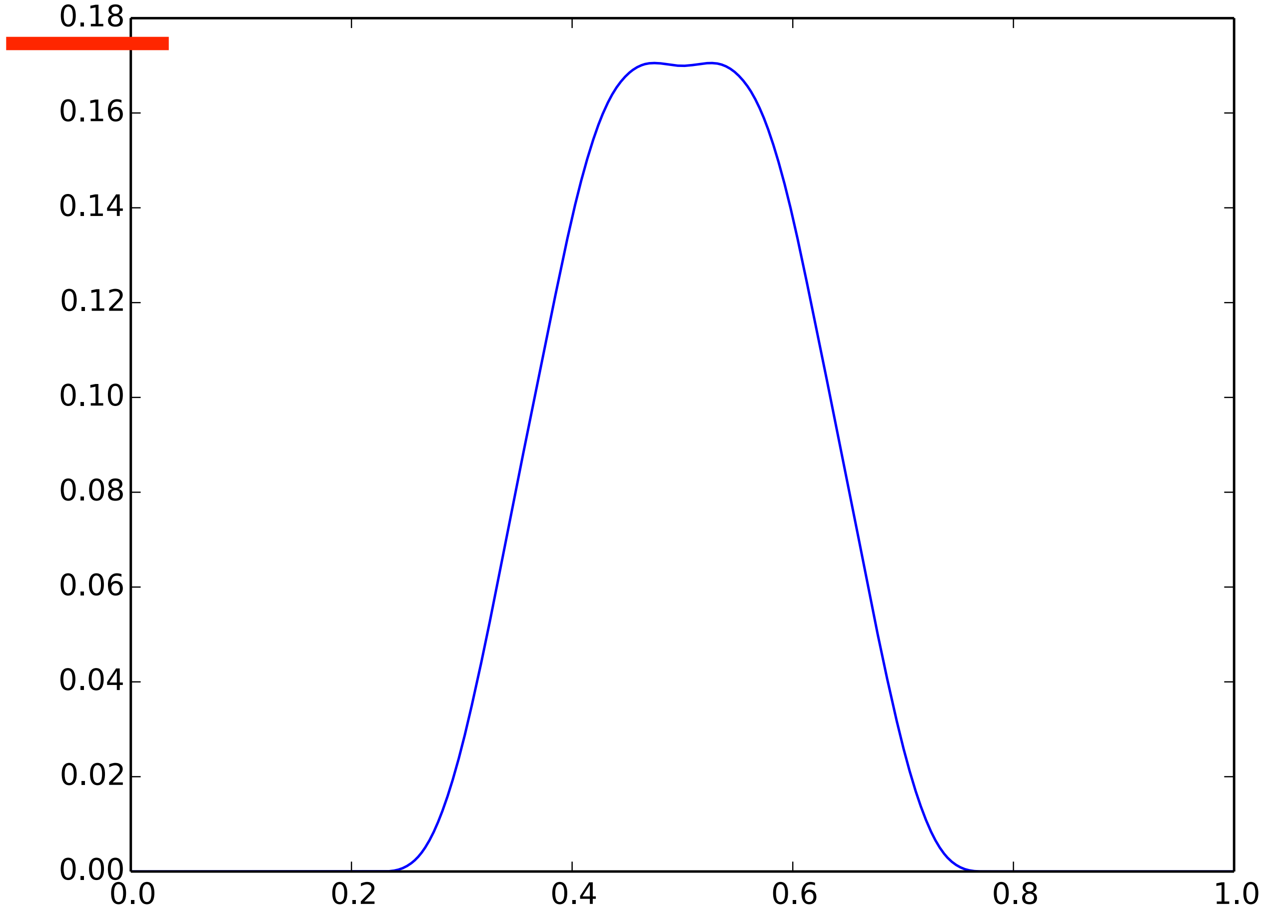
Even:

Odd:

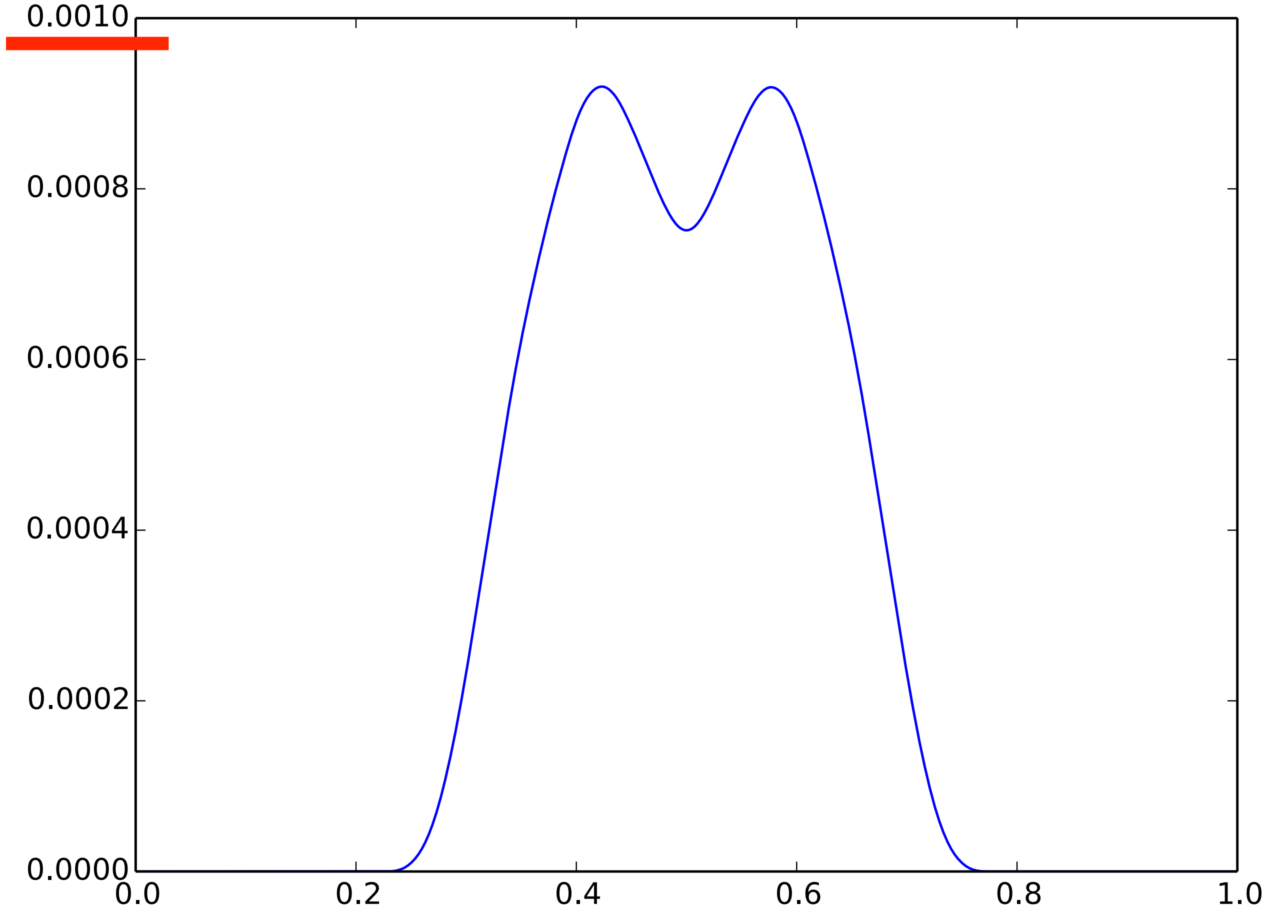
Given Data



$$\phi_m = S_m^\infty \delta$$



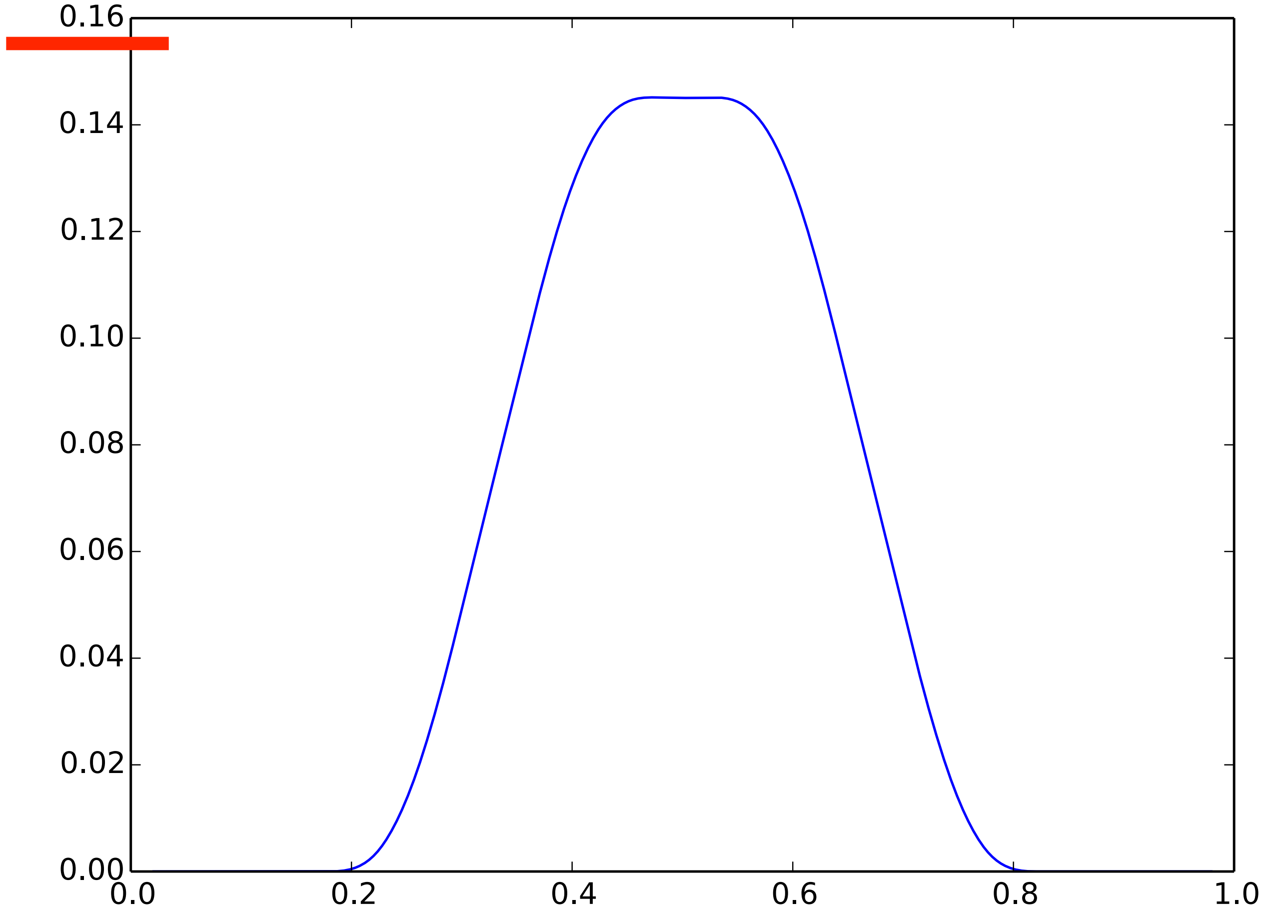
LS Scheme



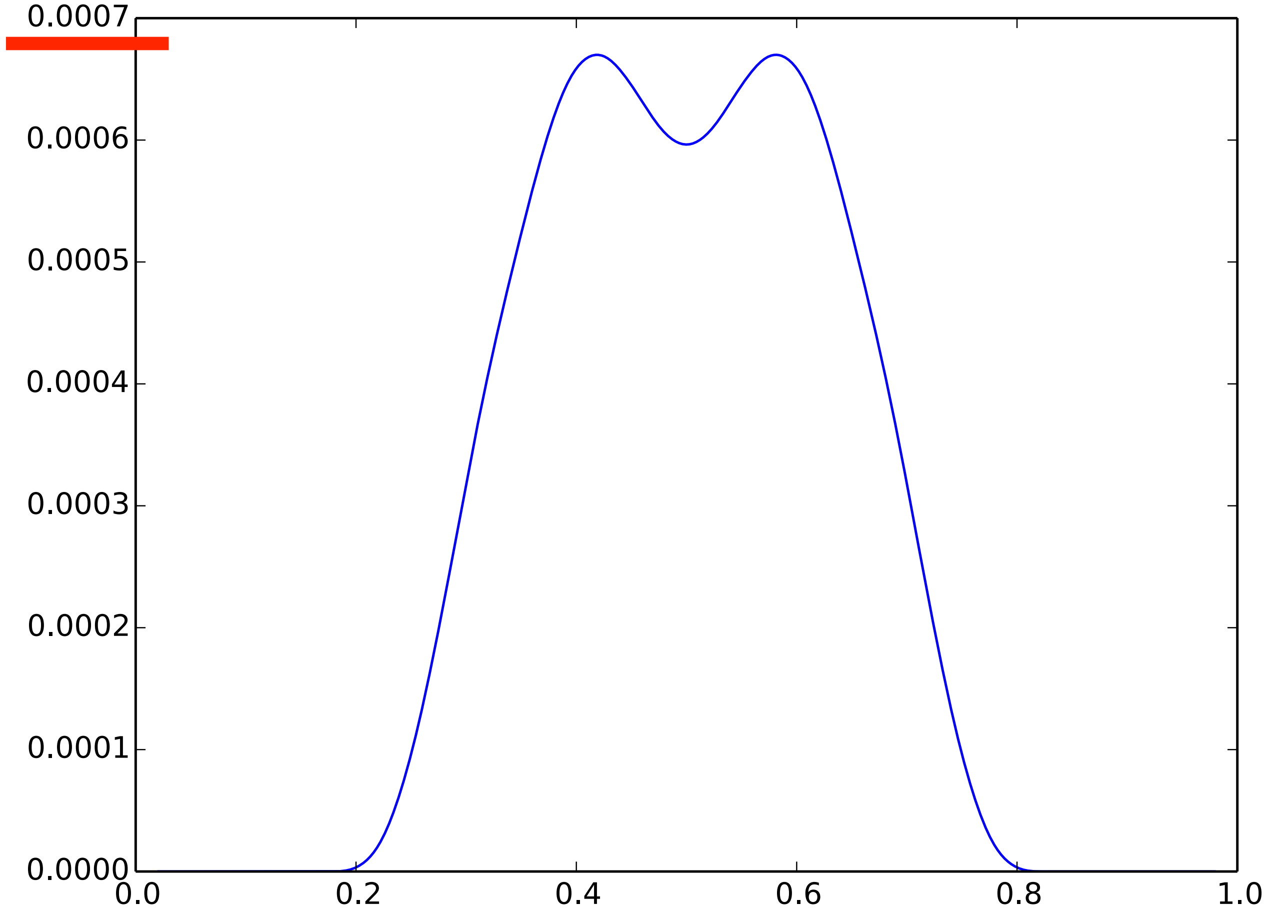
L1 Scheme

m = 6

$$\phi_m = S_m^{\infty}\delta$$



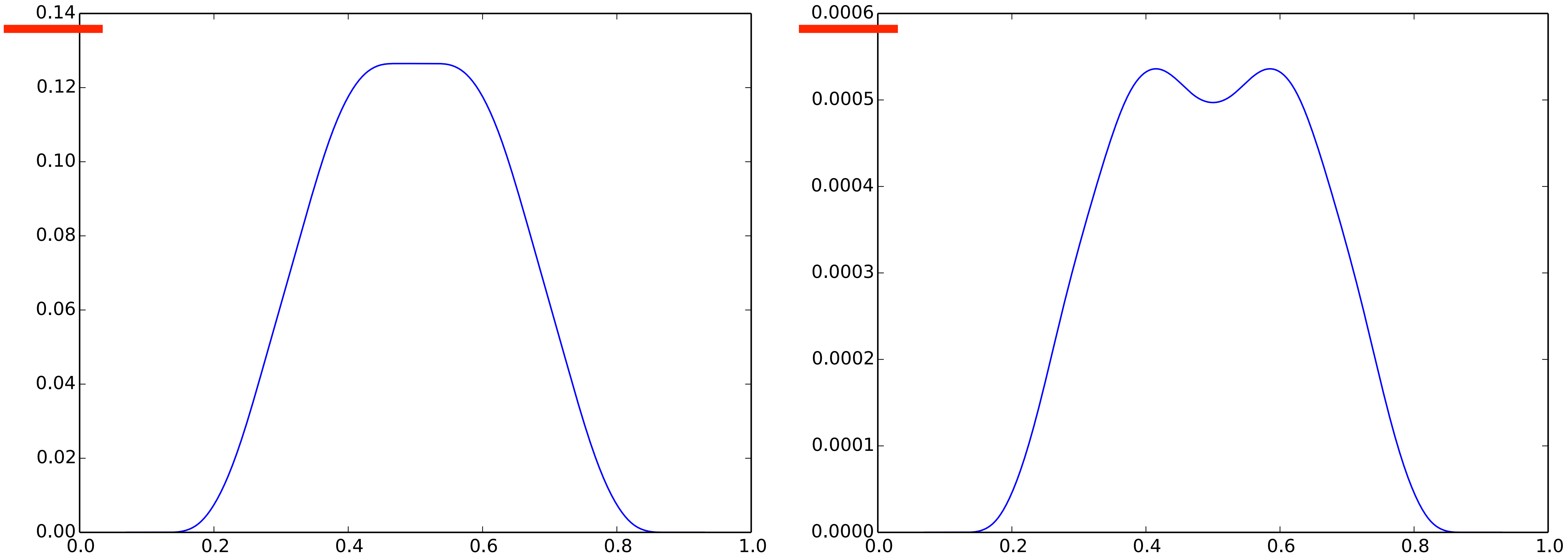
LS Scheme



L1 Scheme

m = 7

$$\phi_m = S_m^{\infty}\delta$$



LS Scheme

L1 Scheme

m = 8

Optimization Algorithm

Convex, but non-differentiable!

$$\min_{\beta_1,\beta_2\in R}\sum_{r=-n+1}^{n}|f_r-\beta_1-\beta_2 r| = \frac{1}{|f_r-\beta_1-\beta_2 r|}(f_r-\beta_1-\beta_2 r)^2$$

$$\beta_{1,\delta}^{k+1},\beta_{2,\delta}^{k+1} = \arg\min_{\beta_1,\beta_2\in R}\sum_{r=-n+1}^{n} w_r^k (f_r-\beta_1-\beta_2 r)^2$$

$$k=k+1$$

$$w_r^k = 1/\left[(f_r-\beta_{1,\delta}^k-\beta_{2,\delta}^k r)^2+\delta\right]^{1/2}$$

Iterative Reweighted Least Square (IRLS) formulation

Closed Form Expression

$$f_{2i}^{(k+1)} = \sum_{r=-n+1}^{n} \left\{ 1 - \left(\tau_{2,i}^{(m)} - \frac{1}{4}\tau_{1,i}^{(m)} \right) \left(\frac{r\tau_{1,i}^{(m)} - \tau_{2,i}^{(m)}}{\tau_{1,i}^{(m)}\tau_{3,i}^{(m)} - \left(\tau_{2,i}^{(m)}\right)^2} \right) \right\} \frac{w_{i+r}^{(m)} f_{i+r}^{(k)}}{\tau_{1,i}^{(m)}}$$

$$f_{2i+1}^{(k+1)} = \sum_{r=-n+1}^{n} \left\{ 1 - \left(\tau_{2,i}^{(m)} - \frac{3}{4}\tau_{1,i}^{(m)} \right) \left(\frac{r\tau_{1,i}^{(m)} - \tau_{2,i}^{(m)}}{\tau_{1,i}^{(m)}\tau_{3,i}^{(m)} - \left(\tau_{2,i}^{(m)}\right)^2} \right) \right\} \frac{w_{i+r}^{(m)} f_{i+r}^{(k)}}{\tau_{1,i}^{(m)}}$$

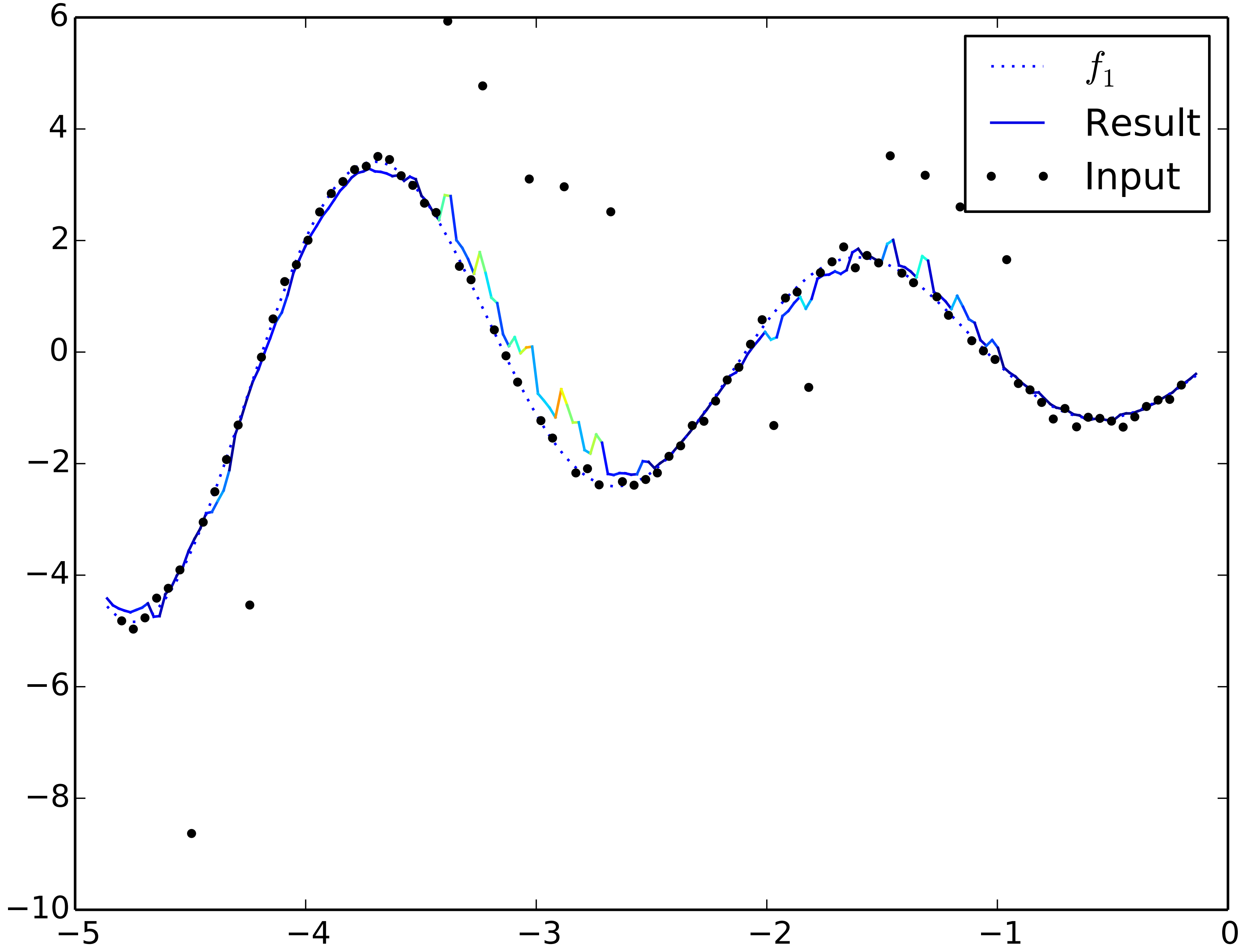
Recap of L1 Scheme

Assign weights to local points

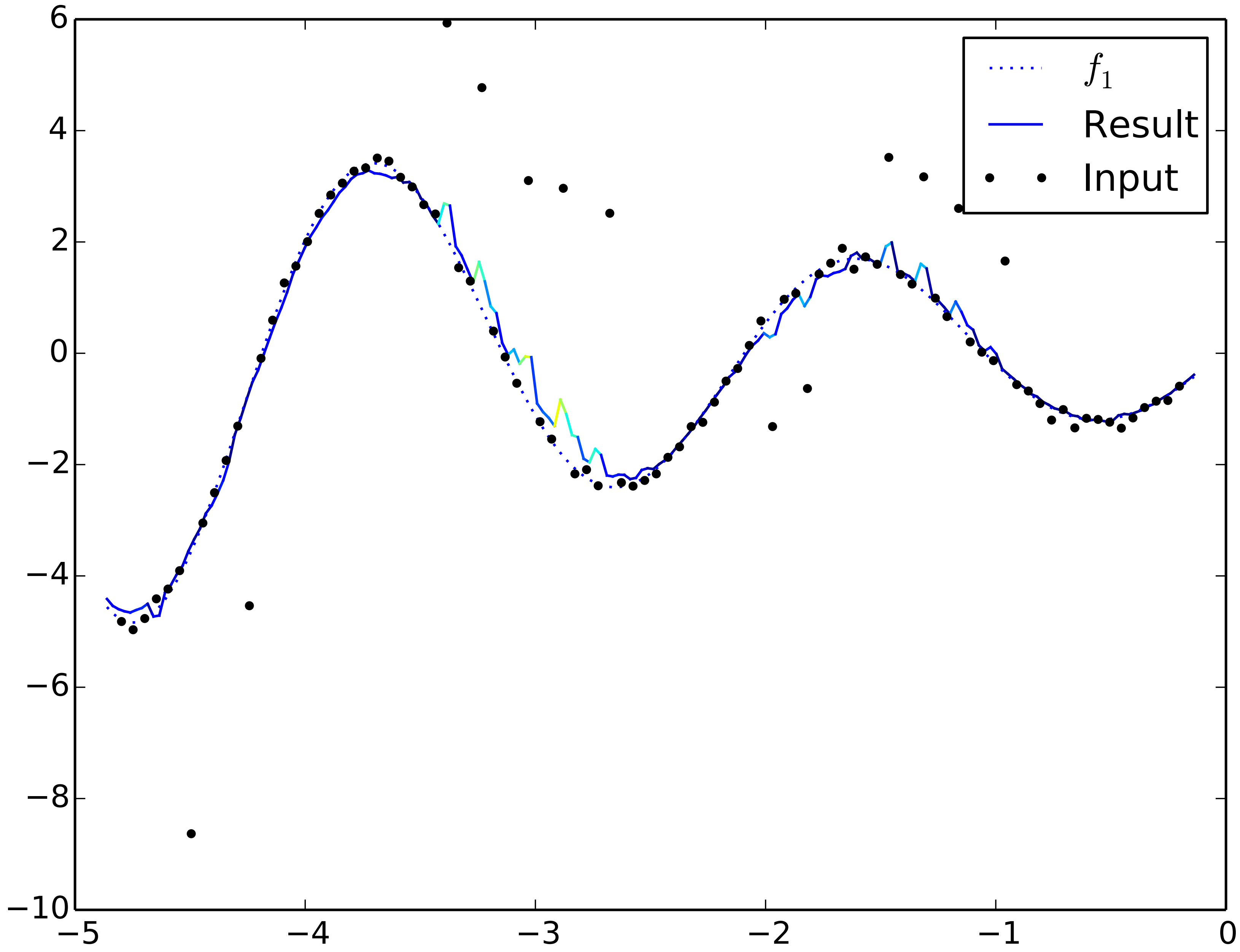
Adjust the vertex positions

- Iterative formulation
- Closed form expression for each iteration

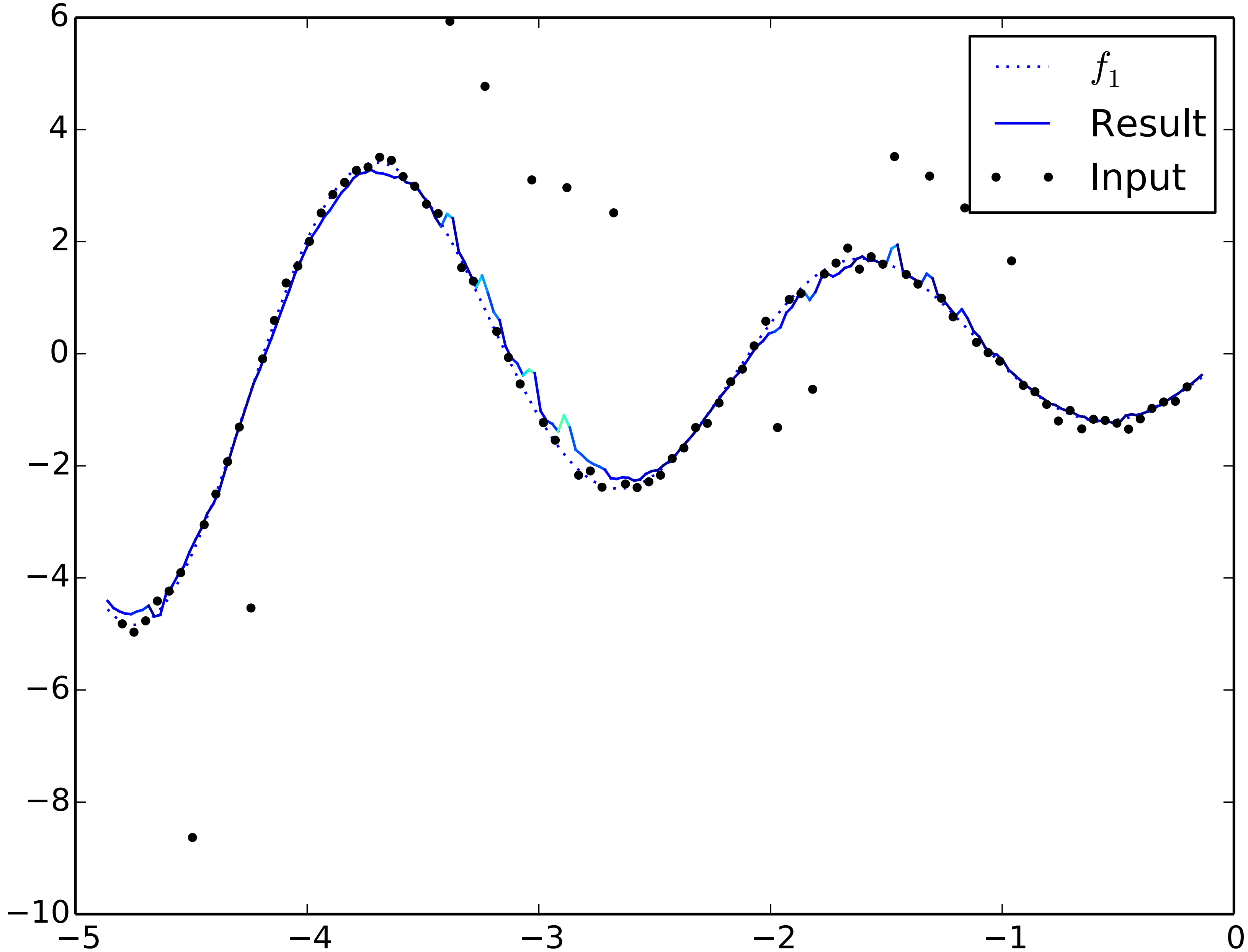
Iterative Results of IRLS



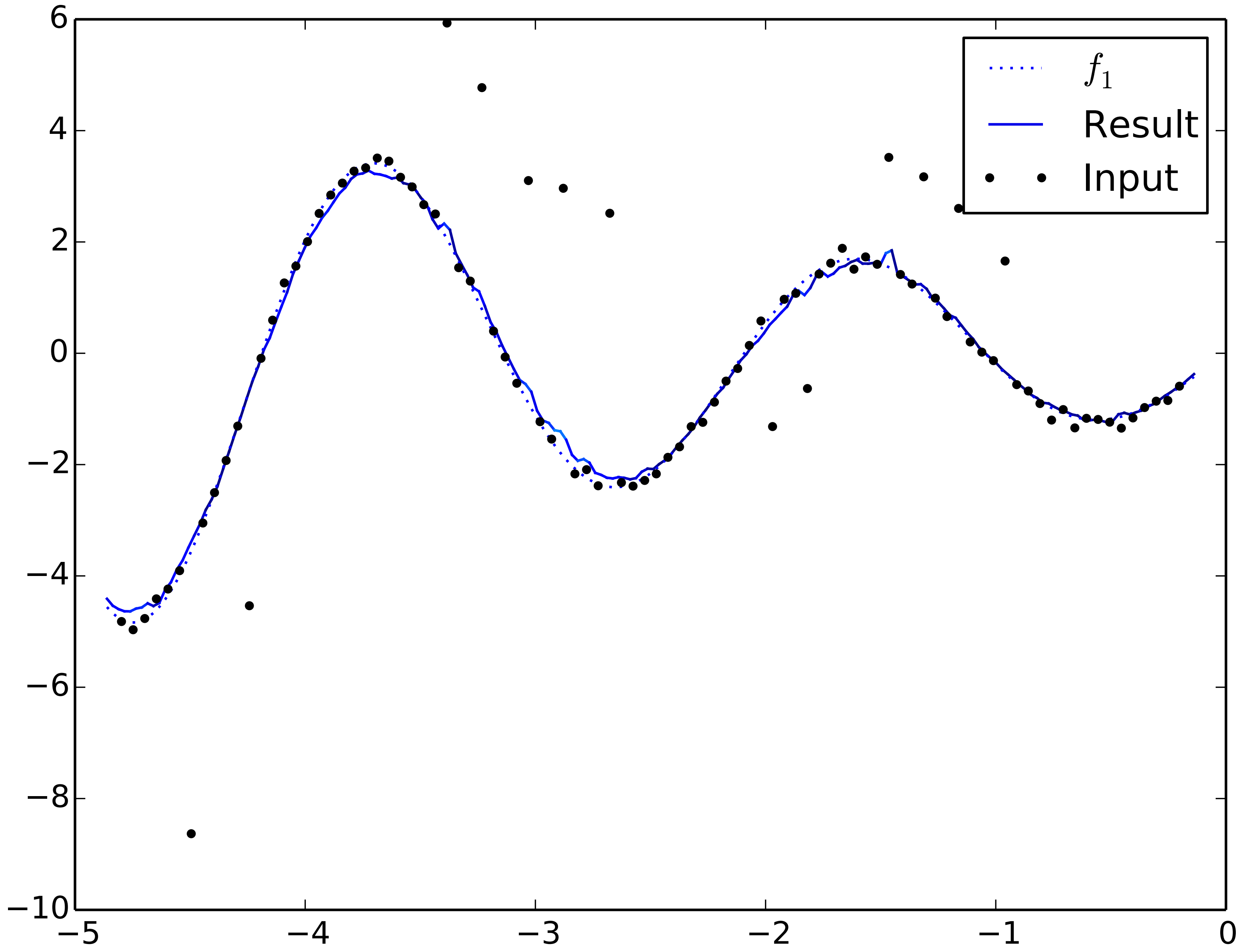
K = 1, Level = 1



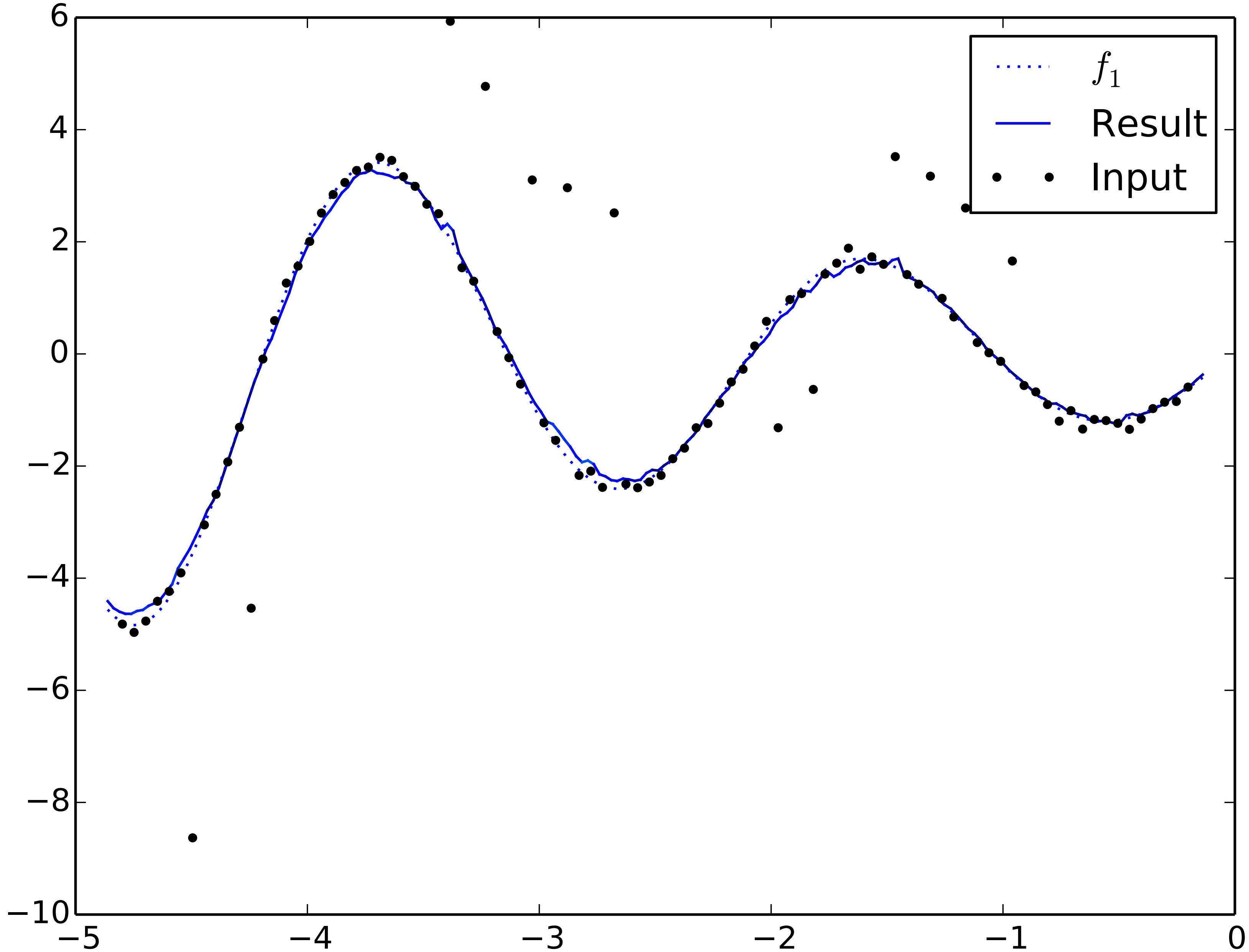
K = 2, Level = 1



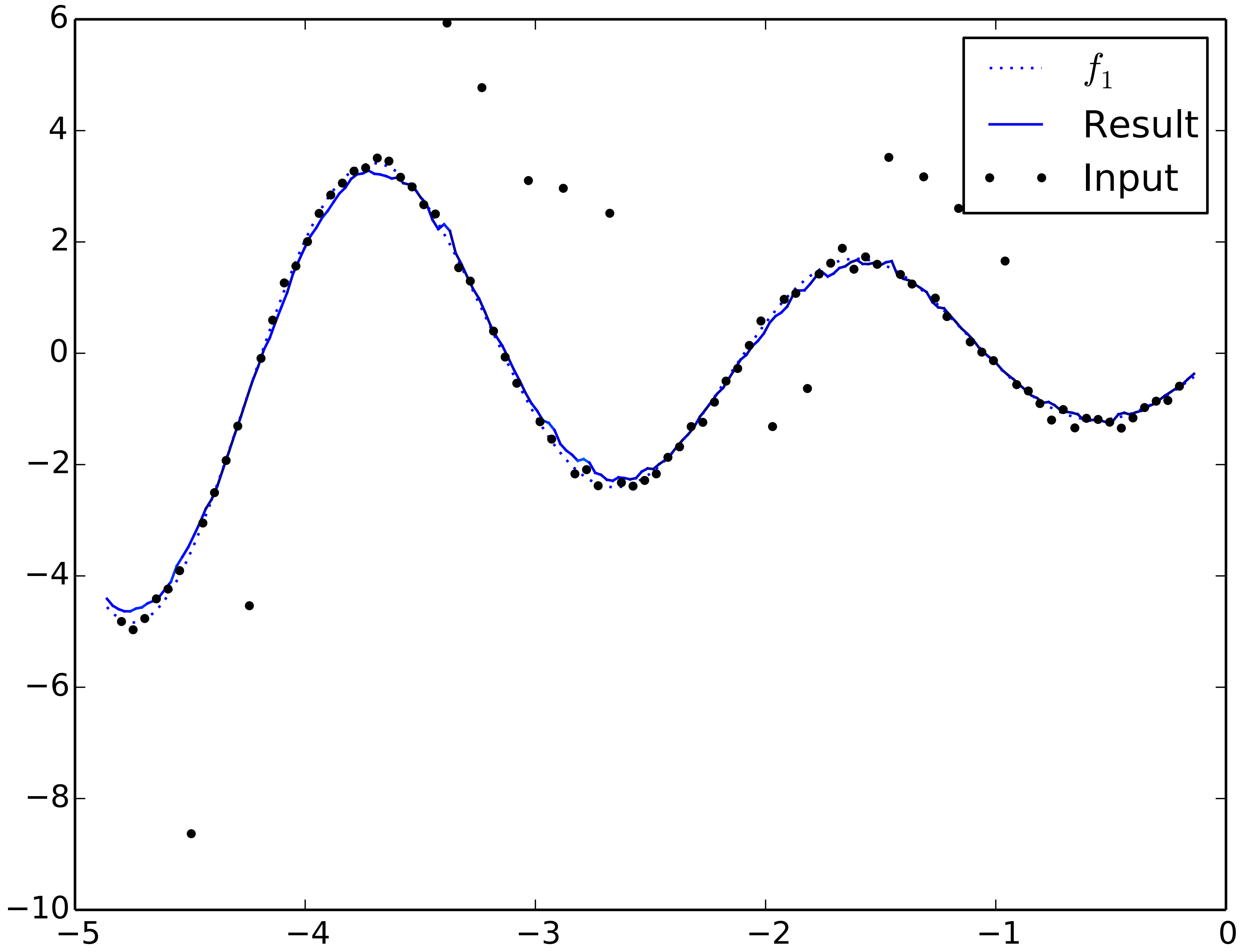
K = 4, Level = 1



K = 8, Level = 1

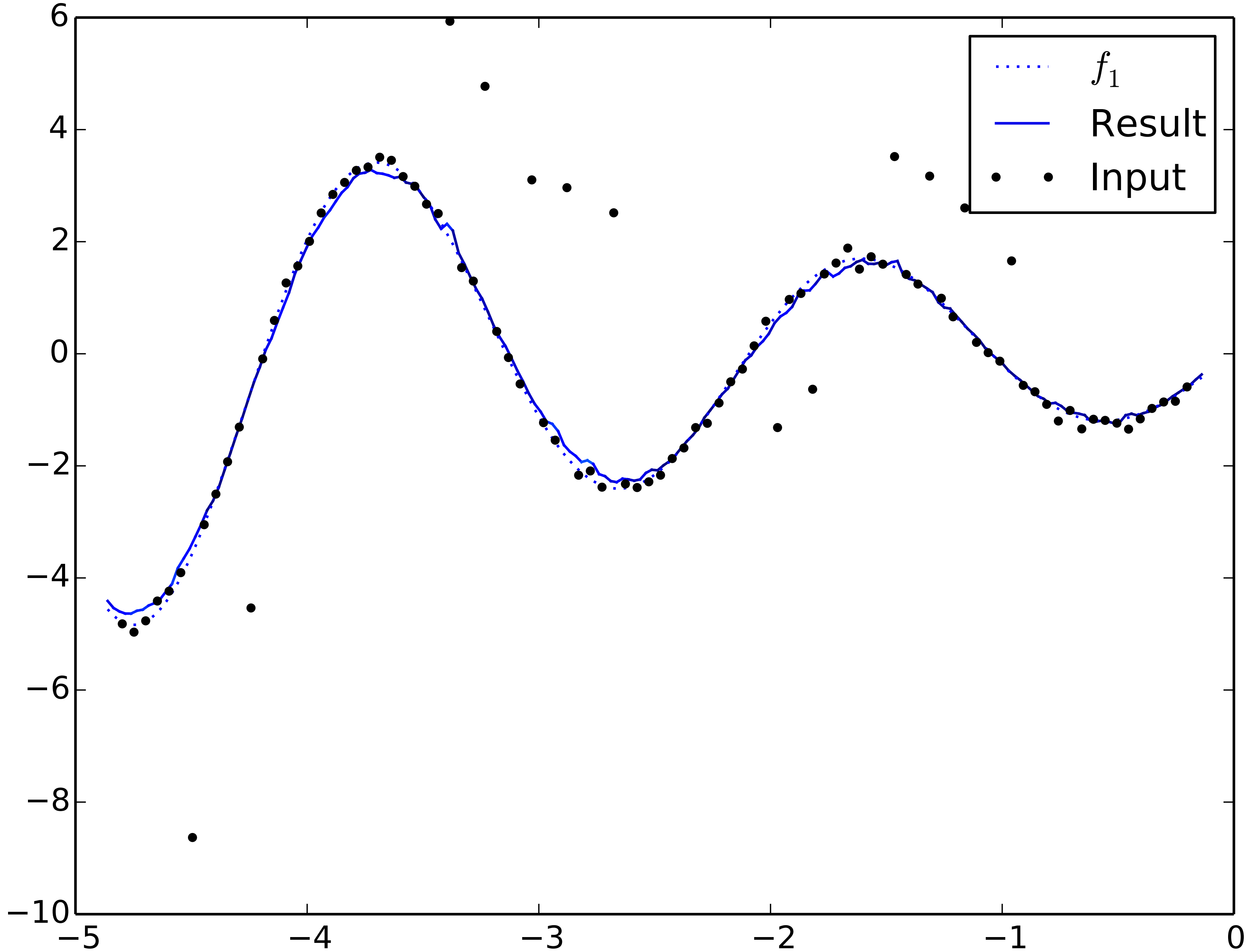


K = 16, Level = 1

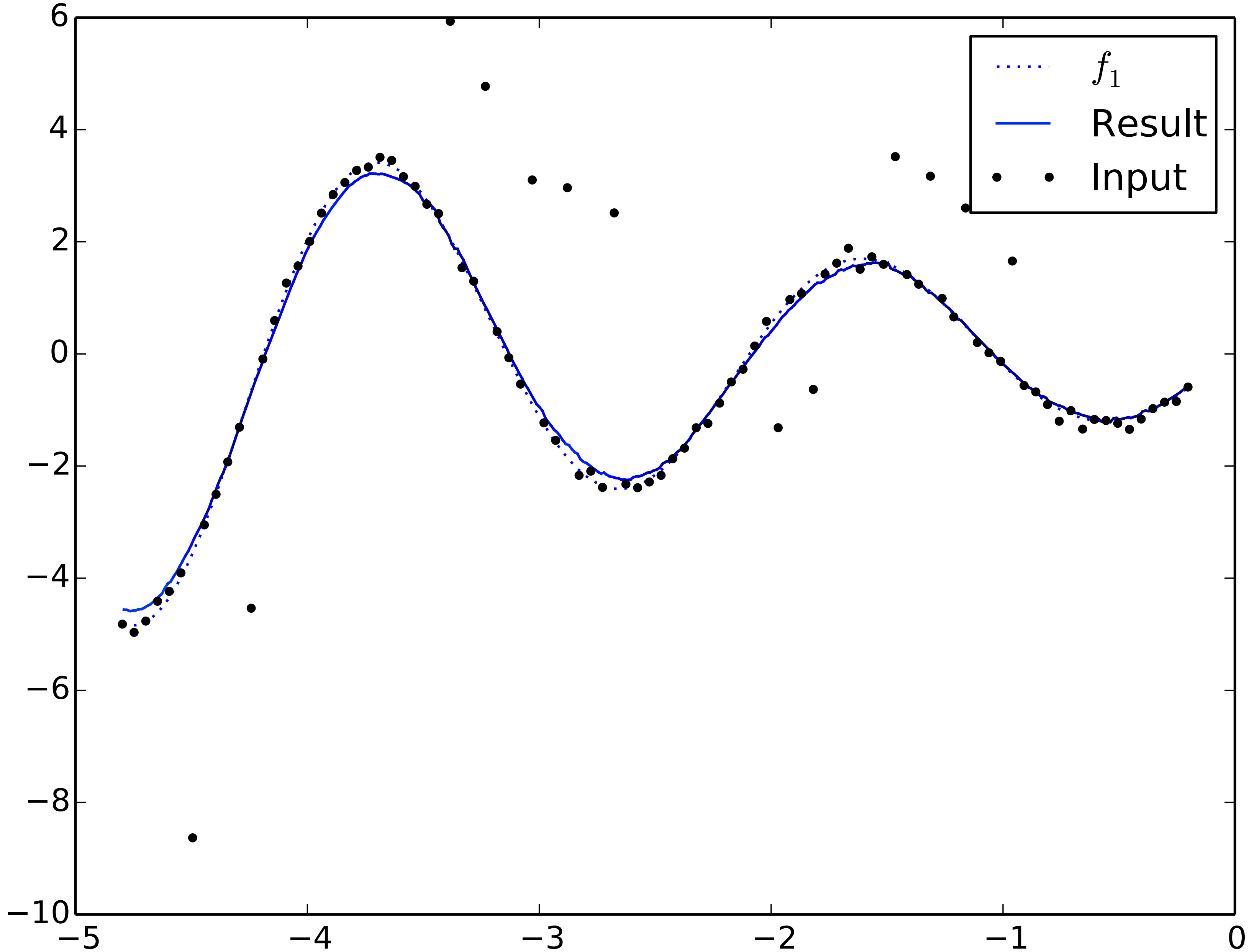


K = 500, Level = 1

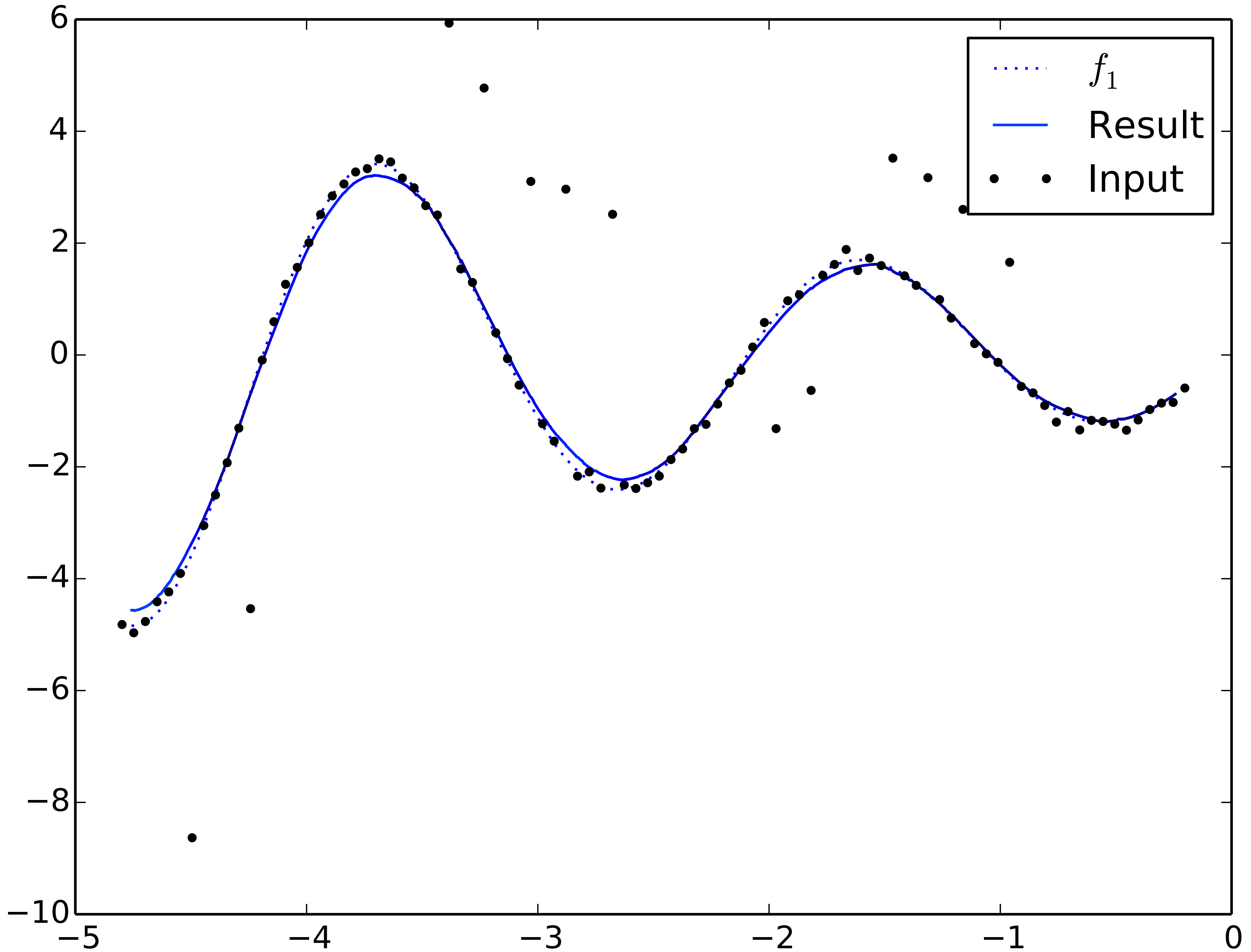
Iterative Results of Subdivision



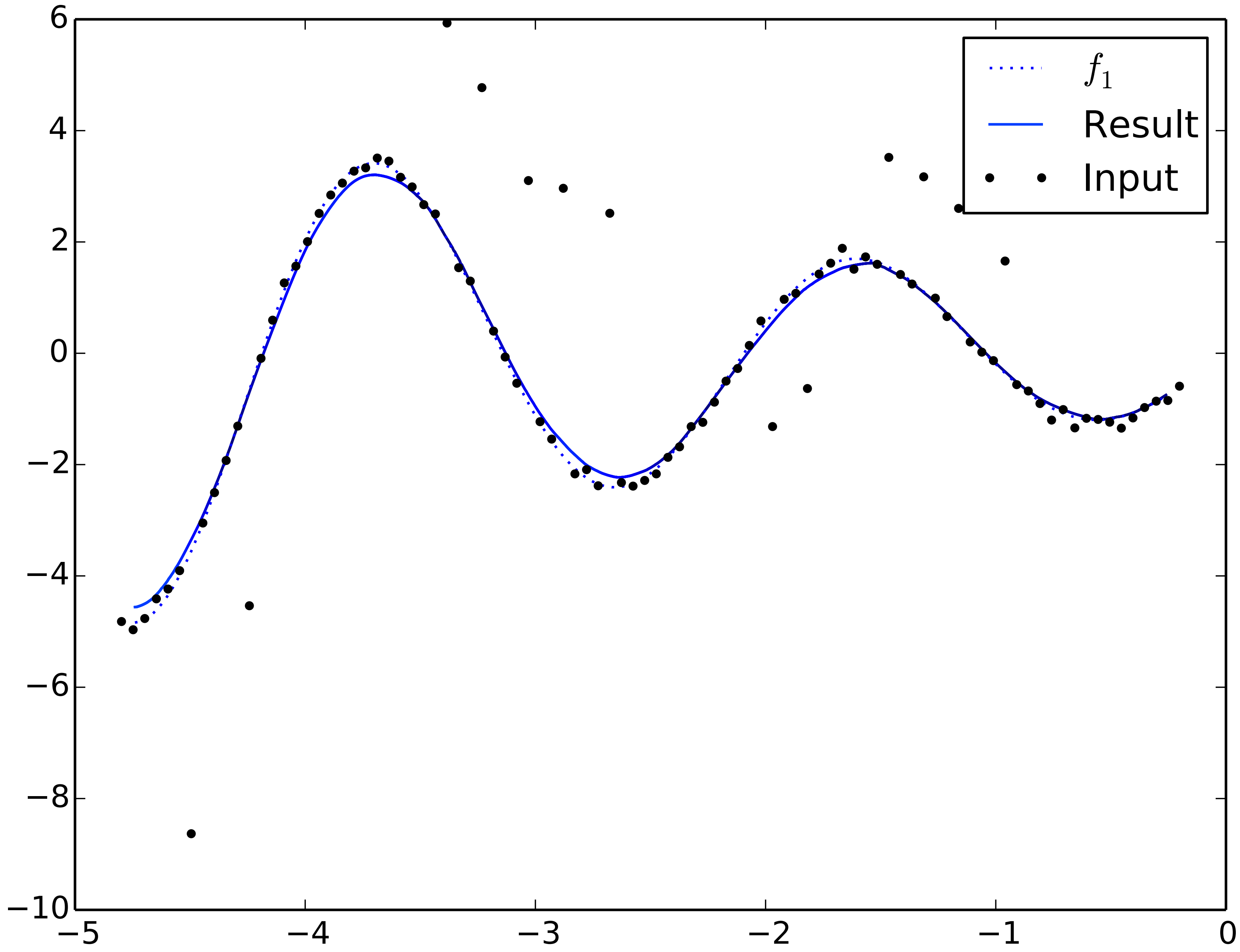
Level = 1



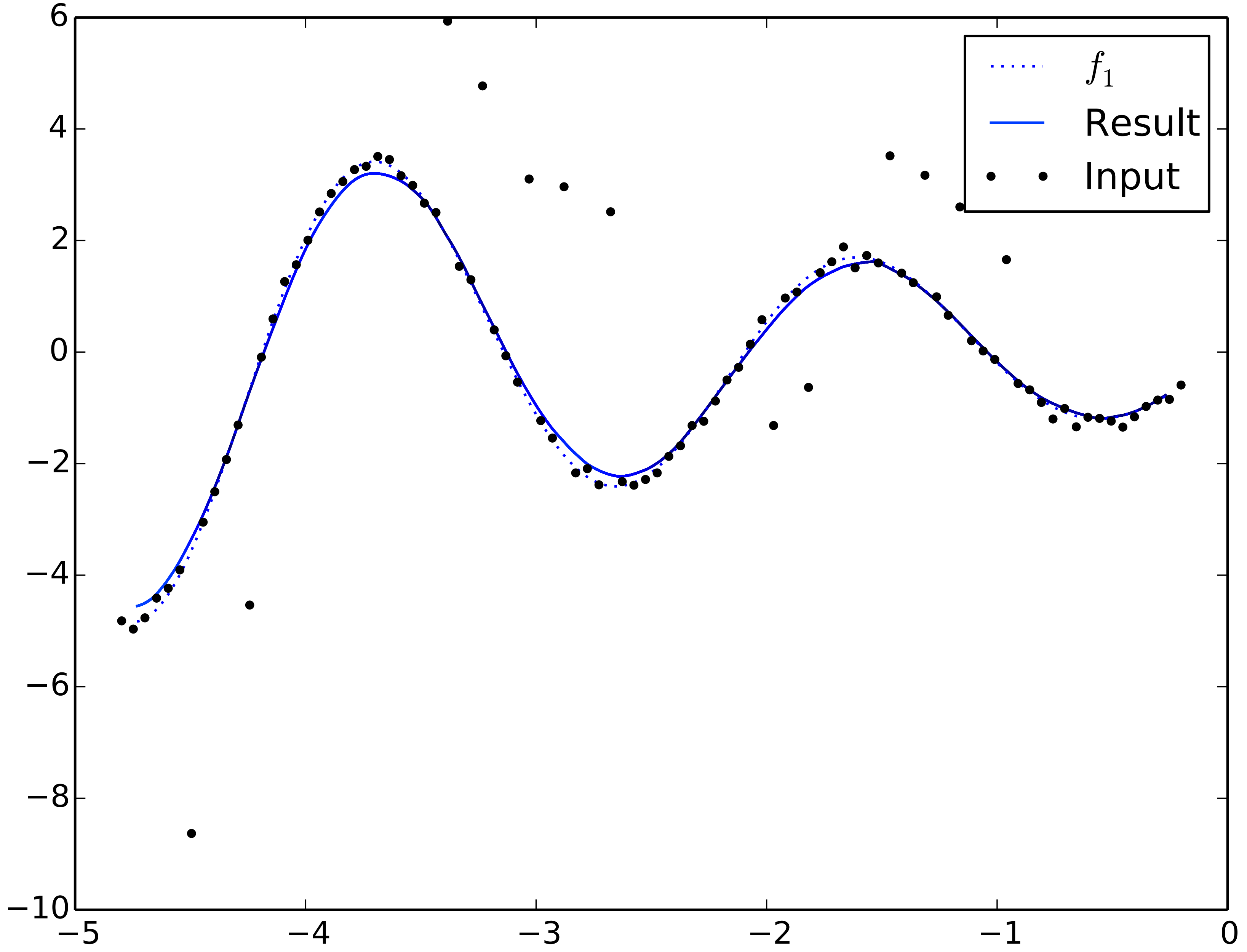
Level = 2



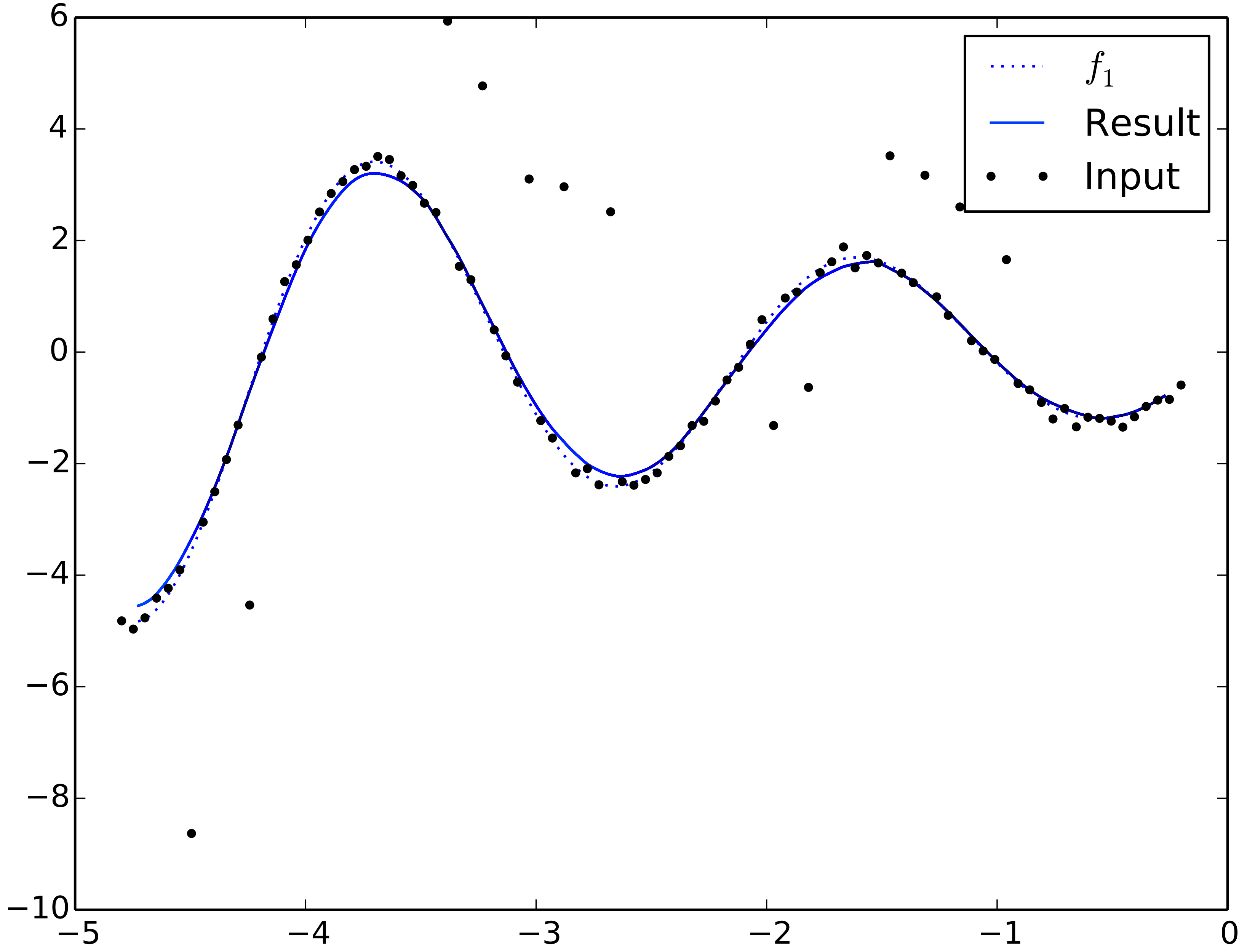
Level = 3



Level = 4



Level = 5



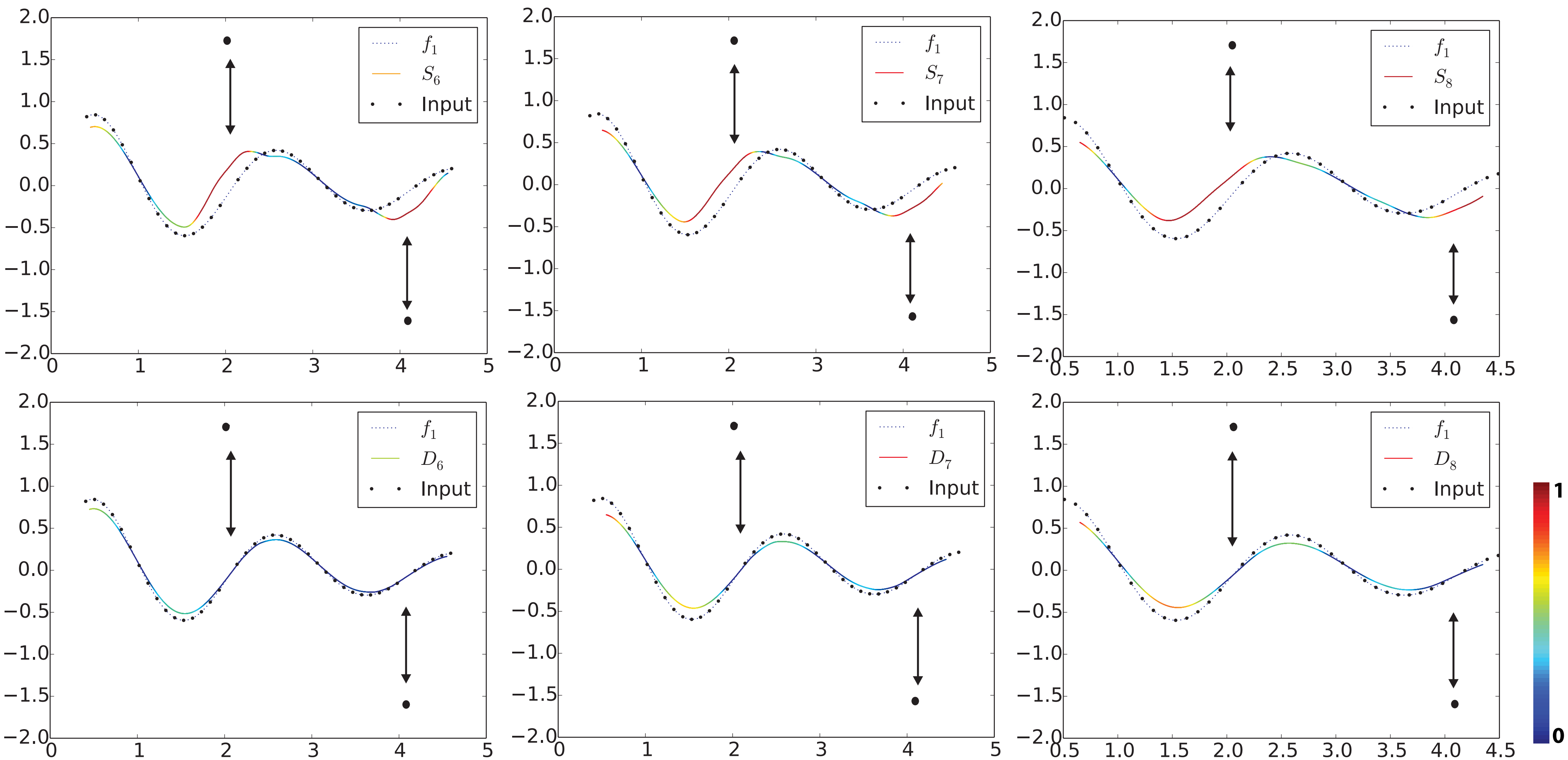
Level = 6

More Results

Data with Outliers

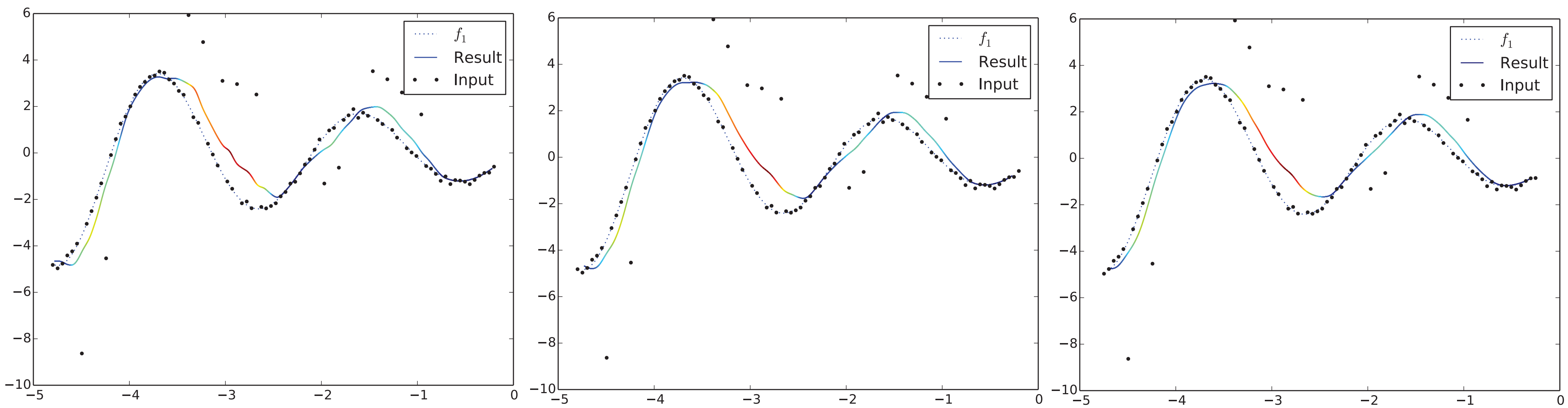
LS:

L1:

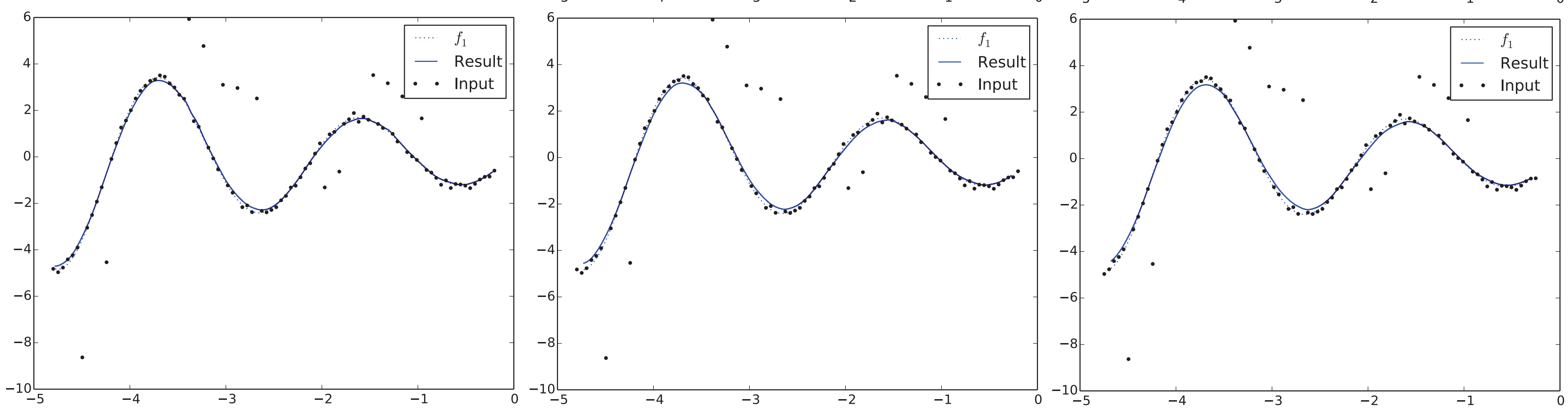


Data with Noise and Outliers

LS:



L1:



Surface Fitting

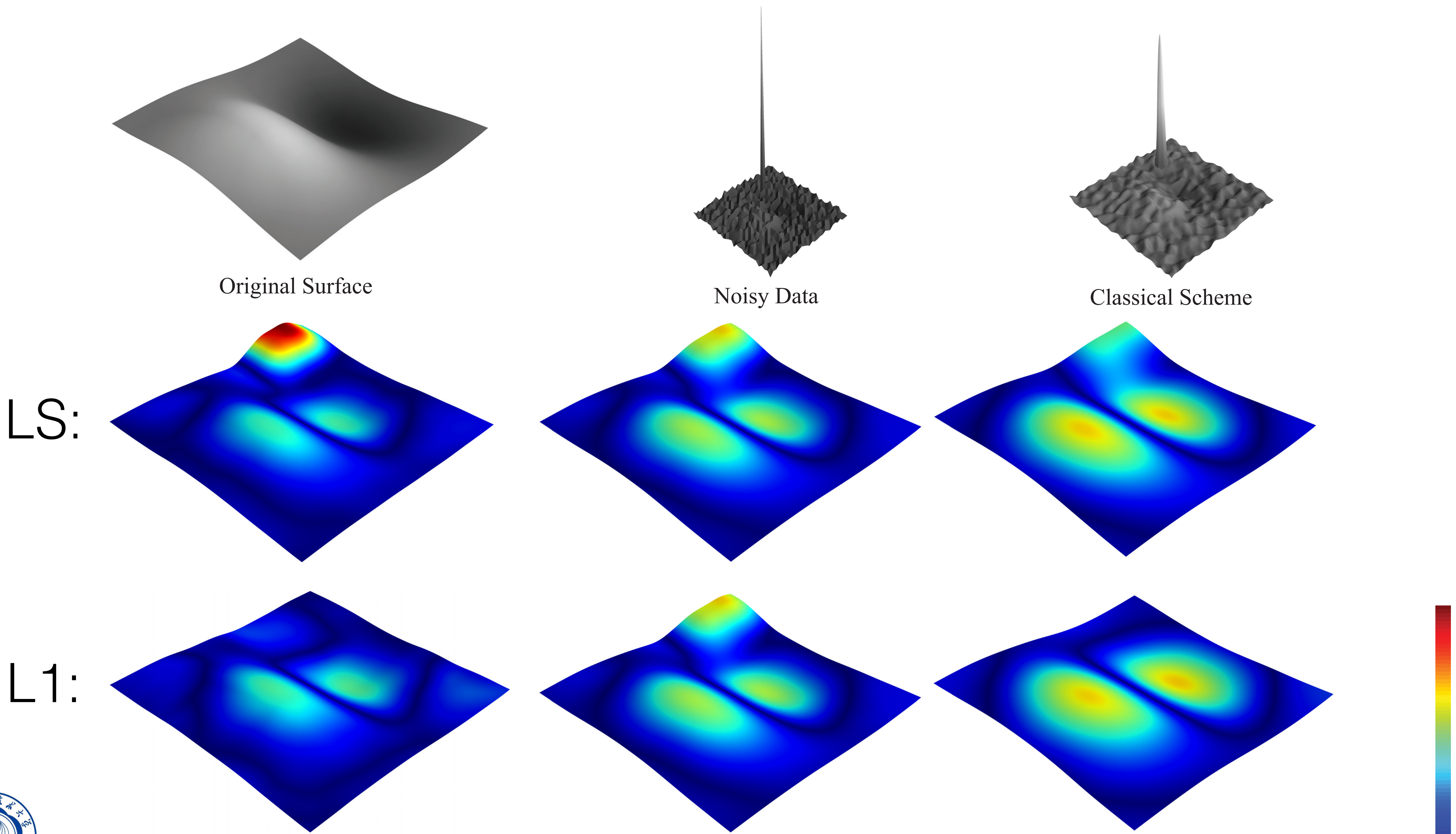
$$\beta_1, \beta_2, \beta_3 = \arg \min_{\beta_1, \beta_2, \beta_3 \in R} \sum_{r=-n+1}^{n} \sum_{s=-n+1}^{n} |f_{r,s} - \beta_1 - r\beta_2 - s\beta_3|$$

$$\Updownarrow$$

$$\beta_{1,\delta}^{k+1}, \beta_{2,\delta}^{k+1}, \beta_{3,\delta}^{l+1} = \arg \min_{\beta_1, \beta_2, \beta_3 \in R} \sum_{r=-n+1}^{n} \sum_{s=-n+1}^{n} w_{r,s}^{k} \left(f_{r,s} - \beta_1 - r\beta_2 - s\beta_3\right)^2$$

$$w_{r,s}^{k} = 1 / \left[\left(f_{r,s} - \beta_{1,\delta}^{k} - r\beta_{2,\delta}^{k} - s\beta_{3,\delta}^{k}\right)^2 + \delta\right]^{1/2}$$

Surface Example



Conclusion

	Computation	Noise Type
Classical Scheme	Closed Form	
LS Scheme	Closed Form	Gaussian
L1 Scheme	Closed Form Iterative	Gaussian Impulsive Outliers

Thank you!