Homework 3:

Problem 3.1: (Calderón-Zygmund decomposition on L^q) Fix a function $f \in L^q(\mathbb{R}^n)$ for some $1 \leq q < \infty$ and let $\alpha > 0$. Then there exist functions g and b on $\mathbb{R}^n$ such that

(1) $f = g + b$.
(2) $\|g\|_{L^q} \leq \|f\|_{L^q}$ and $\|g\|_{L^\infty} \leq 2^{\frac{n}{q}}\alpha$.
(3) $b = \sum_j b_j$, where each b_j is supported in a cube Q_j. Furthermore, the cubes Q_k and Q_j have disjoint interiors when $j \neq k$.
(4) $\|b_j\|_{L^q}^q \leq 2^{n+q}\alpha^q|Q_j|$.
(5) $\int_{Q_j} b_j(x)\,\mathrm{d}x = 0$.
(6) $\sum_j |Q_j| \leq \alpha^{-q}\|f\|_{L^q}^q$.
(7) $\|b\|_{L^q} \leq 2^{\frac{n+q}{q}}\|f\|_{L^q}$ and $\|b\|_{L^1} \leq 2\alpha^{1-q}\|f\|_{L^q}^q$.