Homework 3, Spring 2023:

Problem 3.1:
For $\phi \in \mathscr{S}$, $H\phi \in L^1$ if and only if $\int \phi = 0$.

Problem 3.2:
Check that the Hilbert transform on the line with kernel x^{-1} is a singular integral kernel.

Problem 3.3: (Calderón-Zygmund decomposition on L^q)
Fix a function $f \in L^q(\mathbb{R}^n)$ for some $1 \le q < \infty$ and let $\alpha > 0$. Then there exist functions g and b on $\mathbb{R}^n$ such that

(1) $f = g + b$.
(2) $\|g\|_{L^q} \le \|f\|_{L^q}$ and $\|g\|_{L^\infty} \le 2^{\frac{n}{q}}\alpha$.
(3) $b = \sum_j b_j$, where each b_j is supported in a cube Q_j. Furthermore, the cubes Q_k and Q_j have disjoint interiors when $j \ne k$.
(4) $\|b_j\|_{L^q}^q \le 2^{n+q}\alpha^q |Q_j|$.
(5) $\int_{Q_j} b_j(x)\,\mathrm{d}x = 0$.
(6) $\sum_j |Q_j| \le \alpha^{-q}\|f\|_{L^q}^q$.
(7) $\|b\|_{L^q} \le 2^{\frac{n+q}{q}}\|f\|_{L^q}$ and $\|b\|_{L^1} \le 2\alpha^{1-q}\|f\|_{L^q}^q$.