

Homework 8, 2023 spring:

Problem 8.1: Fix a nonzero Schwartz function h on the line whose Fourier transform is supported in the interval $[-1/8, 1/8]$. For $\{a_j\}$ a sequence of numbers, set

$$f(x) = \sum_{j=1}^{\infty} a_j e^{2\pi i 2^j x} h(x).$$

Prove that for all $1 < p < \infty$ there exists a constant C_p such that

$$\|f\|_{L^p(\mathbb{R})} \leq C_p \left(\sum_j |a_j|^2 \right)^{1/2} \|h\|_{L^p}.$$

Problem 8.2: (The exercise on P.27 Chapter 4, Wolff) Using translation and multiplication by characters, construct a sequence of Schwartz functions $\{\phi_n\}$ so that

- (1) Each $\{\phi_n\}$ has the same L^p norm.
- (2) Each $\widehat{\phi_n}$ has the same $L^{p'}$ norm.
- (3) The supports of the $\widehat{\phi_n}$ are disjoint.
- (4) The supports of the $\{\phi_n\}$ are “essentially disjoint” meaning that

$$\left\| \sum_{n=1}^N \phi_n \right\|_p^p \approx \sum_{n=1}^N \|\phi_n\|_p^p \approx N$$

uniformly in N .