Homework 9, 2023 spring:

Problem 9.1: Suppose that ϕ is a real C^∞ phase function satisfying the non-degeneracy condition

$$\det\left(\frac{\partial^2\phi}{\partial x_j \partial y_k}\right) \neq 0$$

on the support of $a(x,y) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. Then for $\lambda > 0$,

$$\left\| \int_{\mathbb{R}^n} e^{i\lambda\phi(x,y)} a(x,y) f(y) \, \mathrm{d}y \right\|_{L^2(\mathbb{R}^n)} \leq C\lambda^{-n/2} \|f\|_{L^2(\mathbb{R}^n)}.$$

Problem 9.2: Prove the identity

$$\int_0^\infty e^{i\lambda x^k} e^{-x^k} x^l \, \mathrm{d}x = c_{k,l}(1 - i\lambda)^{-(l+1)/k}$$

for any integers $k \geq 2$ and $l \geq 0$, where $c_{k,l}$'s are constants.

Problem 9.3: Suppose that $\phi : \mathbb{R} \to \mathbb{R}$ satisfies that

$$\phi(x_0) = \phi'(x_0) = \phi''(x_0) = 0,$$

while $\phi'''(x_0) \neq 0$. If $\psi \in C_c^\infty(\mathbb{R})$ is supported in a sufficiently small neighborhood of x_0, prove that

$$\int_{\mathbb{R}} e^{i\lambda\phi(x)} \psi(x) \, \mathrm{d}x = \lambda^{-1/3} \sum_{j=0}^{N} a_j \lambda^{-j/3} + O(\lambda^{-(N+2)/3})$$

for all $\lambda > 1$ and nonegative integer N.