PHYS5251P: Exercise 3, Spring 2024, USTC
'Introduction to Quantum Information'

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1. (1) What conditions should a good entanglement measures meet?

   (2) Describe the definition of distillable entanglement and entanglement cost and their relationship.

   (3) Write down the monogamy of entanglement and describe its physical meanings.

2. (1) Calculate the amount of entanglement of the state $\rho = \lambda |\phi^+\rangle \langle\phi^+| + (1-\lambda)|\psi^+\rangle\langle\psi^+|, (0 \leq \lambda \leq 1)$ with negativity measure, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle, |\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

   (2) Derive the value scope for λ when the state ρ is entangled using negativity measure.

3. Consider the density matrix $\rho_w = r|\phi^+\rangle\langle\phi^+| + \frac{1-r}{4}I_4$, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is Bell state and $0 \leq r \leq 1$. Calculate the concurrence of ρ_w.

4. For the 2-qubit state $\rho = p|\Psi^-\rangle\langle\Psi^-| + (1-p)\frac{\mathbb{I}}{4}$, where $0 \leq p \leq 1$, $|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$, calculate the EOF (Entanglement of Formation) of ρ.

5. Calculate the von Neumann entropy of the following density matrix,

   (1) $\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, (2) $\rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, (3) $\rho_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, (4) $\rho_4 = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

6. (1) Prove the subadditivity of the von Neumann entropy

$$|S(A) - S(B)| \leq S(A, B) \leq S(A) + S(B).$$

(2) Prove the concavity of the von Neumann entropy

$$S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i).$$

(3) Suppose ABC is a composite quantum system. Prove that

$$S(A|B, C) \leq S(A|B).$$

7. Suppose $\{P_i\}$ is a complete set of orthogonal projectors and ρ is a density operator. Prove that the entropy of the state $\rho' \equiv \sum_i P_i \rho P_i$ of the system after the measurement is at least as great as the orignal entropy, $S(\rho') \geq S(\rho)$, with equality if and only if $\rho = \rho'$.

8. Let ρ be a single qubit density matrix, and recall the Bloch sphere representation

$$\rho = \frac{I + \boldsymbol{r} \cdot \boldsymbol{\sigma}}{2},$$

where $\boldsymbol{r}$ is a real three-dimensional vector and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Express the von Neumann entropy of ρ in terms of $\boldsymbol{r}$ and the binary entropy function $H(p) = -p \log p - (1-p) \log(1-p)$.

9. (1) For the singlet state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

prove that the correlation function $E(A_i, B_j)_{quantum} = \langle\psi^-| \, A_i \otimes B_j \, |\psi^-\rangle \equiv \langle\psi^-| \, (\vec{a}_i \cdot \vec{\sigma}) \otimes (\vec{b}_j \cdot \vec{\sigma}) \, |\psi^-\rangle$ is

$$E(A_i, B_j)_{quantum} = -\vec{a}_i \cdot \vec{b}_j.$$

(2) Prove the CHSH inequality

$$|E(A_1, B_1) + E(A_1, B_2) + E(A_2, B_1) - E(A_2, B_2)| \leq 2,$$

in which $E(A_i, B_j)$ is the expectation value of the correlation experiment A_i, B_j.

(3) What's the maximal violation of the CHSH inequality allowed by quantum mechanics? Give the corresponding quantum state and specify the measurement operators.

10. (Tsirelson's inequality) Suppose $Q = \vec{q} \cdot \vec{\sigma}$, $R = \vec{r} \cdot \vec{\sigma}$, $S = \vec{s} \cdot \vec{\sigma}$, $T = \vec{t} \cdot \vec{\sigma}$, where $\vec{q}$, $\vec{r}$, $\vec{s}$ and $\vec{t}$ are real unit vectors in three dimensions and $\vec{\sigma} = (\sigma_x \ \sigma_y \ \sigma_z)$. Show that

$$(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 = 4I + [Q, R] \otimes [S, T].$$

Use this result to prove that

$$\langle Q \otimes S \rangle + \langle R \otimes S \rangle + \langle R \otimes T \rangle - \langle Q \otimes T \rangle \leq 2\sqrt{2}.$$

11. Consider the CHSH game with the following choices for Alice's and Bob's observables:

$$\begin{aligned}
&P : \sigma_z^A \\
&Q : \cos(\frac{\pi}{4})\sigma_z^A + \sin(\frac{\pi}{4})\sigma_x^A \\
&R : \sigma_z^B \\
&S : \cos(\frac{\pi}{4})\sigma_z^B - \sin(\frac{\pi}{4})\sigma_x^B,
\end{aligned}$$

where $\sigma_z^A = \sigma_z^A \otimes I^B$ is an observable on Alice's qubit only, and so on. Let the two-qubit state shared by Alice and Bob be an imperfect entangled state:

$$\rho = p\frac{I}{4} + (1-p)|\psi^-\rangle\langle\psi^-|,$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Calculate the CHSH quantity:

$$|CHSH| = |E(P, R) + E(Q, R) + E(P, S) - E(Q, S)|$$

for this state, as a function of p. For what values of p is the CHSH inequality violated?

12. Consider the CHSH game in which the referee chooses questions $r, s \in \{0,1\}$ uniformly, and Alice and Bob must each answer a single bit: a for Alice, b for Bob, in which $a, b \in \{0,1\}$. They win if $a \oplus b = r \wedge s$ and lose otherwise.

    (1) Give the maximum probability of winning with the classical strategy.

    (2) Suppose Alice and Bob share a maximum quantum entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, please derive the maximum probability of winning and give the corresponding quantum strategy.

13. Consider the GHZ game in which the referee chooses questions $rst \in \{000, 011, 101, 110\}$ uniformly, and Alice, Bob and Charles must each answer a single bit: a for Alice, b for Bob, c for Charles, in which $a, b, c \in \{0,1\}$. They win if $a \oplus b \oplus c = r \vee s \vee t$ and lose otherwise. Suppose Alice, Bob and Charles share a GHZ state $|\psi\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle)$, give a quantum strategy that maximize probability of winning.

14. Two players, Alice and Bob, are required to independently fill a 3×3 magic square. As shown in Fig. 1, the referee randomly sends two queries $x, y \in \{0,1,2\}$ to Alice and Bob, respectively. Here, x labels rows and y labels columns. Alice and Bob are required to reply with three numbers with specific conditions. Denote Alice's answers in a row as $[a_0^x, a_1^x, a_2^x]$ and Bob's answers in a column as $[b_0^y, b_1^y, b_2^y]$, where $a_i^x, b_j^y \in \{-1, +1\}$ for any $i, j \in \{0,1,2\}$. Alice's answers must satisfy $\prod_i a_i^x = +1$, while Bob's should satisfy $\prod_j b_j^y = -1$ for any x and y. During the game, Alice and Bob are forbidden to communicate with each other. They win the game if the overlapped entry of Alice's row and Bob's column is always the same, i.e.,$a_y^x = b_x^y$ for each x and y.

    (1) Give the maximum probability of winning with the classical strategy.

(2) Suppose Alice and Bob share a maximum quantum entangled state $|\phi\rangle_{A_1A_2B_1B_2} = |\psi\rangle_{A_1B_1} \otimes |\psi\rangle_{A_2B_2}$ with $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, and Alice has systems A_1A_2 and Bob has B_1B_2. Please derive the maximum probability of winning and give the corresponding quantum strategy.

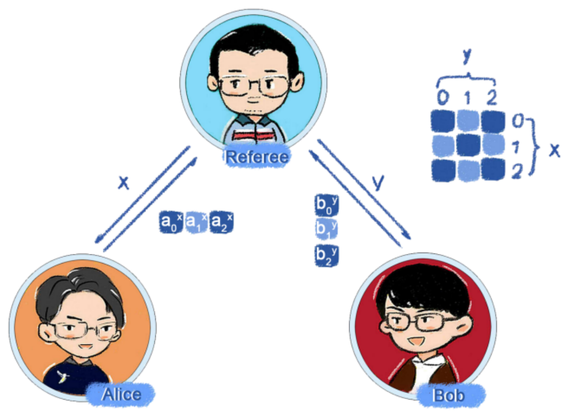

FIG. 1. The Mermin-Peres magic square game.