PHYS5251P: Exercise 4, Spring 2023, USTC
'Introduction to Quantum Information'

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1. Consider two qubits A and B, and an arbitrary unknown quantum state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$. Can we use the operator $\Omega_{AB} = I_A \otimes |0\rangle_{BB}\langle 0|$ to remove the copy state of system B, i.e. $|\Psi\rangle_A |\Psi\rangle_B \longrightarrow |\Psi\rangle_A |0\rangle_B$?

2. Derive the Bell's theorem without inequalities from the GHZ state
$$|\psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle).$$

3. Consider a 2-qubit quantum state $\rho_{AB} = \frac{1}{8}I + \frac{1}{2}|\psi^-\rangle\langle\psi^-|$, where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.

   (1) Give the spectral decomposition of ρ_{AB}.

   (2) Suppose one measures $\vec{n}\cdot\vec{\sigma}_A$ and measures $\vec{m}\cdot\vec{\sigma}_B$ with $\vec{n}\cdot\vec{m} = \cos\theta$, calculate the probability that both outcomes are $+1$.

   (3) Use the realignment criterion to find out whether ρ_{AB} is entangled or not.

4. Suppose Alice and Bob share the two-qubit state $|\psi_{AB}\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$. Recall that a quantum measurement is specified by a set of operators $\{M_0, M_1\}$ such that $\sum_k M_k^\dagger M_k = I$.

   (1) Suppose the quantum measurement $\{M_0, M_1\}$ acting on $|\psi_{AB}\rangle$ output the outcome corresponding to $M_0 = a|0\rangle\langle 0| + b|1\rangle\langle 1|$ with probability $1/4$, give the values of a and b such that M_0 produces the post-measurement state $(|00\rangle + |11\rangle)/\sqrt{2}$.

(2) Give an operator M_1 such that $M_0^\dagger M_0 + M_1^\dagger M_1 = I$ with M_0 being the result of the above question. With what probability does the outcome corresponding to M_1 occur, acting on $|\psi_{AB}\rangle$, and what is the post-measurement state?

5. (1) Prove that $S(\rho) \le \log D$, where D is the number of the non-zero eigenvalues of the density operator ρ.

   (2) For two density operators ρ_1 and ρ_2, prove the inequality

$$\mathrm{tr}(\rho_1 \log \rho_1 + \rho_2 \log \rho_2) \ge \mathrm{tr}(\rho_1 \log \rho_2 + \rho_2 \log \rho_1).$$

6. Quantum teleportation is a process by which quantum information can be transmitted from one location to another, with the help of quantum entanglement.

   (1) Suppose the initials states are $|\psi\rangle_1 = \alpha |0\rangle_1 + \beta |1\rangle_1, |\psi^-\rangle_{23} = \frac{1}{\sqrt{2}}(|0\rangle_2 |1\rangle_3 - |1\rangle_2 |0\rangle_3)$. Show that particle 3 can be projected onto the same state as particle 1 by some local operators after Bell state measurement on particle 1 and 2.

   (2) Suppose the initials states are $|\psi\rangle_1 = \alpha |0\rangle_1 + \beta |1\rangle_1, |GHZ\rangle_{234} = \frac{1}{\sqrt{2}}(|0\rangle_2 |0\rangle_3 |0\rangle_4 + |1\rangle_2 |1\rangle_3 |1\rangle_4)$. Show that particle 4 can be projected onto the same state as particle 1 by some local operators after Bell state measurement on particle 1 and 2 and local X measurement on particle 3.

   (3) Explain why we can't use quantum teleportation to achieve superluminal communication.

7. Consider the following 4-qubit state:

$$|\theta\rangle_{1234} = |\psi\rangle_1 \otimes \frac{1}{\sqrt{2}}(I_2 \otimes U)\left[(|00\rangle_{23} + |11\rangle_{23}) \otimes |\phi\rangle_4\right],$$

   where $|\psi\rangle_1 = a|0\rangle_1 + b|1\rangle_1$ is an arbitrary qubit state and $|\phi\rangle_4$ is another one, U is a two-qubit unitary operator acting on qubit 3 and 4. After Bell state measurement on qubit 1 and 2 of $|\theta\rangle_{1234}$, how can we turn the state of qubit 3 and 4 to be $U(|\psi\rangle \otimes |\phi\rangle)$?

8. Consider a noisy entangled pair with density matrix

$$\rho_\epsilon = (1-\epsilon)\,|\psi^-\rangle\langle\psi^-| + \epsilon\frac{I}{4},$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. The fidelity between two density matrices σ and τ is defined as $F(\sigma,\tau) = (\mathrm{Tr}\sqrt{\sqrt{\sigma}\tau\sqrt{\sigma}})^2 = F(\tau,\sigma)$.

(1) Calculate the fidelity between ρ_ϵ and $|\psi^-\rangle\langle\psi^-|$. (Recall that for pure state σ, we have $\sqrt{\sigma} = \sigma = \sigma^2$.)

(2) Give the lower bound of fidelity $F(\rho_\epsilon, |\psi^-\rangle\langle\psi^-|)$ when the state ρ_ϵ is entangled using PPT criterion.

(3) Suppose one uses ρ_ϵ instead of perfect EPR states to teleport a pure qubit state ρ. Find the fidelity between the teleported qubit state and ρ.

9. Suppose three EPR sources produce three pairs of entangled photons, pair 1-2, 3-4 and 5-6. The initial states are $|\phi^+\rangle_{12} = \frac{|00\rangle_{12}+|11\rangle_{12}}{\sqrt{2}}$, $|\phi^+\rangle_{34} = \frac{|00\rangle_{34}+|11\rangle_{34}}{\sqrt{2}}$, $|\phi^+\rangle_{56} = \frac{|00\rangle_{56}+|11\rangle_{56}}{\sqrt{2}}$. Then photons 2, 4, and 6 are projected to GHZ-state $\frac{|000\rangle+|111\rangle}{\sqrt{2}}$. What is the state of the photons 1, 3 and 5?

10. In quantum information theory, dense coding is a technique used to send two bits of classical information using only one qubit. Suppose that Alice and Bob share an EPR pair

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B),$$

Show the detailed protocol to realize the dense coding.

11. The polarization dependent beam splitter (PDBS), which has transmission rate T_H for horizontal polarization mode and transmission rate T_V for vertical polarization mode, can be used to construct controlled phase gate. In figure (a), a PDBS performs the following transformation on input single photon with path mode a:

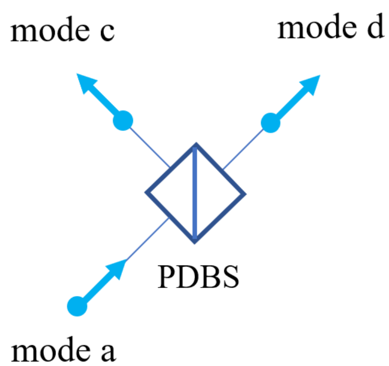
$$\alpha|H_a\rangle + \beta|V_a\rangle \longrightarrow \alpha(\sqrt{T_H}|H_d\rangle + i\sqrt{1-T_H}|H_c\rangle) + \beta(\sqrt{T_V}|V_d\rangle + i\sqrt{1-T_V}|V_c\rangle),$$

where $|H_d\rangle$ denotes horizontal polarized photon on path mode d and similar for the other terms. In figure (b), two input modes a and b are overlapped at PDBS_0, with a $\mathrm{PDBS}_{1/2}$ on each of its output mode.

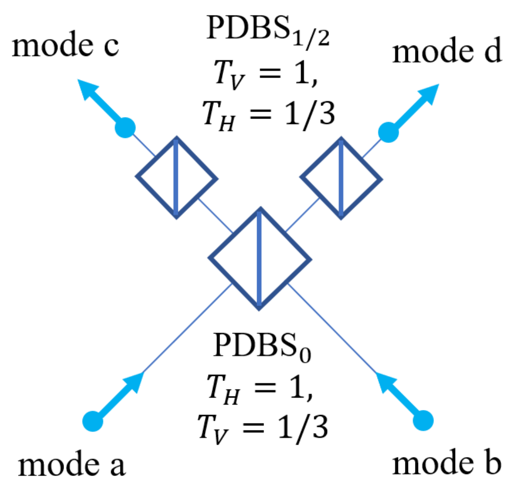
(1) Show that, conditioned on the coincidence detection of output modes c and d, the setup in figure (b) can implement the controlled phase gate perfectly,

$$c_{HH}\left|H_a H_b\right\rangle + c_{HV}\left|H_a V_b\right\rangle + c_{VH}\left|V_a H_b\right\rangle + c_{VV}\left|V_a V_b\right\rangle$$
$$\longrightarrow c_{HH}\left|H_d H_c\right\rangle + c_{HV}\left|H_d V_c\right\rangle + c_{VH}\left|V_d H_c\right\rangle - c_{VV}\left|V_d V_c\right\rangle .$$

(2) Calculate the probability to obtain a coincidence in the outputs.



(a) PDBS



(b) Setup for the controlled phase gate