PHYS5251P: Exercise 5, Spring 2024, USTC
'Introduction to Quantum Information'

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1. Let $\vec{n}$ be a normalized real vector in three dimensions and let θ be real. Prove that the equality
$$f(\theta\vec{n}\cdot\vec{\sigma}) = \frac{f(\theta)+f(-\theta)}{2}I + \frac{f(\theta)-f(-\theta)}{2}\vec{n}\cdot\vec{\sigma}$$
holds for any function $f(\cdot)$.

2. Consider a particle with initial state $|0\rangle$. We perform N sequential measurements $\sigma_k \equiv \vec{n}_k \cdot \vec{\sigma}$ with $\vec{n}_k = \left(\sin\left(\frac{k\pi}{2N}\right), 0, \cos\left(\frac{k\pi}{2N}\right)\right)$ and $k = 1, 2, \cdots, N$. What's the probability that all outcomes are $+1$? What if $N \to \infty$?

3. (a) Give stabilizer generator sets for the following states.

   (1) $(|0\rangle + \mathrm{i}\,|1\rangle)/\sqrt{2}$ (2) $|1\rangle$

   (3) $(|00\rangle + |11\rangle)/\sqrt{2}$ (4) $(|00\rangle - |11\rangle)/\sqrt{2}$

   (5) $(|01\rangle + |10\rangle)/\sqrt{2}$ (6) $(|01\rangle - |10\rangle)/\sqrt{2}$

   (7) $(|000\rangle + |111\rangle)/\sqrt{2}$ (8) $(|{+}0{+}\rangle + |{-}1{-}\rangle)/\sqrt{2}$,

   where $|\pm\rangle$ denote the eigenkets of Pauli X.

   (b) Give stabilizer generator sets for the following vector spaces, specified by the basis sets given.

   (1) $\{|001\rangle, |110\rangle\}$

   (2) $\left\{(|00\rangle + |11\rangle)/\sqrt{2}, (|01\rangle + |10\rangle)/\sqrt{2}\right\}$.

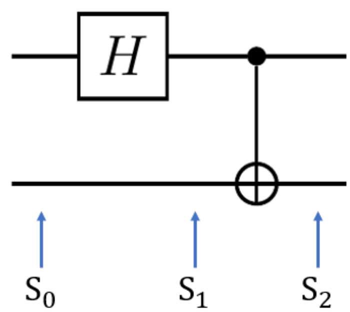
4. (1) For the 4-qubit state $|\psi\rangle = (|0011\rangle + |1100\rangle)/\sqrt{2}$, please write down its stabilizer generator set.

   (2) For 4-qubit cluster state $|\psi\rangle = (|+\rangle|0\rangle|+\rangle|0\rangle + |+\rangle|0\rangle|-\rangle|1\rangle + |-\rangle|1\rangle|-\rangle|0\rangle + |-\rangle|1\rangle|+\rangle|1\rangle)/2$, please write down its stabilizer generator set.

5. (a) Denote the controlled-NOT gate as U, calculate the following and express your results with only Pauli operators. The subscripts denote the labels of qubits.

   (1) $U(X_1I_2)U^\dagger$ (2) $U(Z_1I_2)U^\dagger$ (3) $U(I_1X_2)U^\dagger$ (4) $U(I_1Z_2)U^\dagger$.

   (b) Consider the following quantum circuit:



   The input state is $|00\rangle$, which is stabilized by $S_0 = \langle IZ, ZI\rangle$. Give the generators of the stabilizers describing the state after the Hadamard S_1 and after the controlled-NOT gate S_2. Work this out by using the fact that U acting on a state stabilized by S produces a state stabilized by $USU^\dagger$.

6. Please write down the difference between quantum error correction and classical error correction.

7. Find a parity check matrix H for the $[6, 2]$ repetition code defined by the generator

matrix G. Then verify that $HG = 0$.

$$G = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

8. Please give a parity check matrix H for the $[7, 4]$ Hamming code, and write down its distance.

9. For 9-qubit Shor code, its logical bit code is

$$|0\rangle_L = (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)/2\sqrt{2},$$

$$|1\rangle_L = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)/2\sqrt{2}.$$

(1) Please give all the generators of the stabilizers;

(2) Please draw the encoding quantum circuit;

(3) Show that the operations

$$\bar{Z} = X_1X_2X_3X_4X_5X_6X_7X_8X_9 \text{ and } \bar{X} = Z_1Z_2Z_3Z_4Z_5Z_6Z_7Z_8Z_9$$

act as logical Z and X operations on a Shor-code encoded qubit.

10. Single qubit quantum operations $\mathcal{E}(\rho)$ model quantum noise which is corrected by quantum error correction codes.

(1) Construct operation elements for $\mathcal{E}$ such that upon input of any state ρ replaces it with the completely randomized state $I/2$.

(2) The action of the bit flip channel can be described by the quantum operation $\mathcal{E}(\rho) = (1-p)\rho + pX\rho X$. Show that this may be given an alternate operator-sum

representation as $\mathcal{E}(\rho) = (1-2p)\rho + 2pP_+\rho P_+ + 2pP_-\rho P_-$, where P_+ and P_- are projectors onto the $+1$ and -1 eigenstates of X, $(|0\rangle+|1\rangle)/\sqrt{2}$ and $(|0\rangle-|1\rangle)/\sqrt{2}$, respectively.

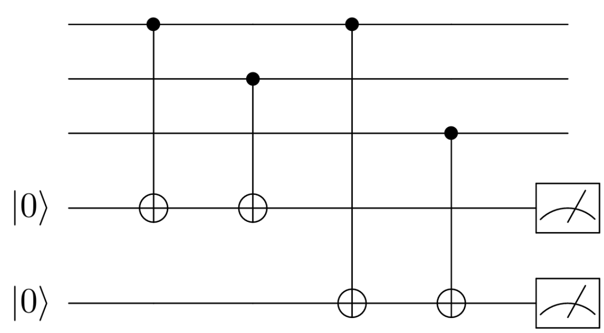
11. Suppose $|\psi\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$ is a general single qubit state encoded in the bit flip code. Then, due to errors it is mapped to the following mixed state:

$$\rho = (1-p)\rho_0 + \frac{p}{3}(X_1\rho_0X_1 + X_2\rho_0X_2 + X_3\rho_0X_3)$$

where $\rho_0 = |\psi\rangle\,\langle\psi|$, and $X_1 = \sigma_x \otimes I \otimes I$ and so on.

(1) Write out the state ρ in bra-ket form.

(2) Compute the state that is produced when the bit-flip code error detection circuit



is executed with the state of the first three qubits being ρ. What are the probabilities of getting the four possible measurement results (00, 01, 10 and 11) when the ancilla are measured?

(3) The correction gates for each of the ancilla measurement results are:

$$00 \rightarrow \text{no correction}$$

$$01 \rightarrow X_3$$

$$10 \rightarrow X_2$$

$$11 \rightarrow X_1$$

Confirm that when the correct correction gate is applied, you can recover the original state $|\psi\rangle$.

12. Consider the code $|0\rangle_L = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, $|1\rangle_L = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Show that an arbitrary superposition of the logical codewords, i.e., $\alpha|0\rangle_L + \beta|1\rangle_L$ is invariant under errors of the form $e^{-i\theta\sigma_z/2} \otimes e^{-i\theta\sigma_z/2}$.