量子信息导论
PHYS5251P

中国科学技术大学
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第二章 量子纠缠

1. **量子纠缠的概念与内涵**
2. 量子纠缠判据
3. 量子纠缠检验Entanglement witness
4. 纠缠量化
5. 多体纠缠
6. Shannon entropy，Von Neumann entropy
7. 纠缠提纯

量子纠缠

纠缠态的定义

$$\mathcal{H}_{AB} \ni |\Psi\rangle \neq |\psi\rangle \otimes |\varphi\rangle, \quad |\psi\rangle \in \mathcal{H}_A, \ |\varphi\rangle \in \mathcal{H}_B$$

Quantum Entanglement: from Magic to a Physical Resource

Einstein-Podolski-Rosen: An entangled wavefunction does not describe the physical reality in a complete way

Schrödinger: For an entangled state the best possible knowledge of the whole does not include the best possible knowledge of its parts

Mermin: a correlation that contradicts the theory of elements of reality

Peres: a trick that quantum magicians use to produce phenomena that cannot be imitated by classical magicians

Bell : a correlation that is stronger than any classical correlation

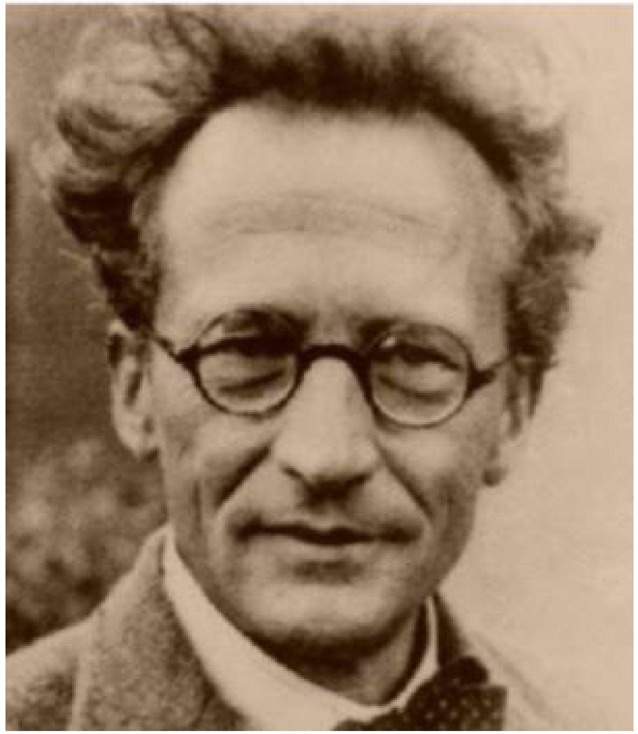
Bennett : a resource that enables quantum teleportation

Shor : a global structure of the wavefunction that allows for faster algorithms

Ekert : a tool for secure communication

量子纠缠

"Entanglement is *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought".



- Quantum computation
- Quantum teleportation
- Dense coding
- Quantum cryptography
- Quantum error correction

E. Schrödinger, Proc. Cambridge Philos. Soc. **31**, 555 (1935)

量子纠缠

$$\rho^{AB} \neq \rho^{A} \otimes \rho^{B}$$

可分离态

$$\rho^{AB} = \sum_{r=1}^{m} p_r \rho_r^A \otimes \rho_r^B \ , \ p_r > 0 \ , \ \sum_r p_r = 1$$

LOCC操作
“local operations and classical communication”

局域操作: unitary dynamic actions, measurements, and all other local manipulations

经典通信**:** exchange information via classical communication

量子纠缠

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

A pure or mixed quantum state which is not separable is called entangled. An entangled quantum state thus contains non-classical correlations, which are also called quantum correlations or EPR correlations.

量子纠缠

Pure state: Tensor Product

$$\begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} \otimes \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \alpha_0\beta_0 \\ \alpha_0\beta 1 \\ \alpha_1\beta_0 \\ \alpha_1\beta 1 \end{pmatrix} \begin{matrix} \longrightarrow \\ \not\longleftarrow \end{matrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Separable　　　　　　　　Entangled

Schrödinger in 1935 (or earlier)

"When two systems, …… enter into temporary physical interaction due to known forces between them, and …… separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives [the quantum states] have become entangled."

Schrödinger (Cambridge Philosophical Society)

量子纠缠性质

$$\rho = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\frac{\langle 00| + \langle 11|}{\sqrt{2}}\right)$$
$$= \frac{|00\rangle\langle 00| + |11\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 11|}{2}$$

$$\rho^1 = \mathrm{tr}_2(\rho)$$
$$= \frac{\mathrm{tr}_2(|00\rangle\langle 00|) + \mathrm{tr}_2(|11\rangle\langle 00|) + \mathrm{tr}_2(|00\rangle\langle 11|) + \mathrm{tr}_2(|11\rangle\langle 11|)}{2}$$
$$= \frac{|0\rangle\langle 0|\langle 0|0\rangle + |1\rangle\langle 0|\langle 0|1\rangle + |0\rangle\langle 1|\langle 1|0\rangle + |1\rangle\langle 1|\langle 1|1\rangle}{2}$$
$$= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}$$
$$= \frac{I}{2}.$$

This strange property, that the joint state of a system can be completely known, yet a subsystem be in mixed states, is another hallmark of quantum entanglement.

量子纠缠

高维最大纠缠态

$$|\psi\rangle = U_A \otimes U_B |\Phi_d^+\rangle_{AB}$$

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle |i\rangle$$

相对论定域性与量子非定域性

L

粒子A ⟷ 粒子B

纠缠粒子对

相对论定域性

测量时间：Δt

类空间隔：$L > c\Delta t$

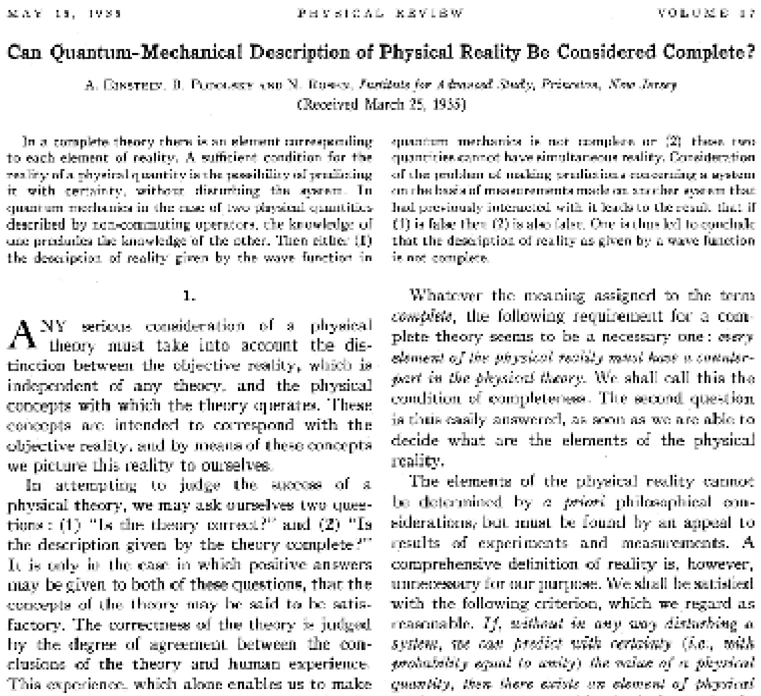
⇒ 对一个粒子的测量不会对另一个粒子产生影响

量子非定域性

量子纠缠

⇒ 对一个粒子的测量会瞬间改变另一个粒子的状态

EPR & Bohm



MAY 15, 1935 PHYSICAL REVIEW VOLUME 47

Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?

A. Einstein, B. Podolsky and N. Rosen, *Institute for Advanced Study, Princeton, New Jersey*

(Received March 25, 1935)

In a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system. In quantum mechanics in the case of two physical quantities described by non-commuting operators, the knowledge of one precludes the knowledge of the other. Then either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality. Consideration of the problem of making predictions concerning a system on the basis of measurements made on another system that had previously interacted with it leads to the result that if (1) is false then (2) is also false. One is thus led to conclude that the description of reality as given by a wave function is not complete.

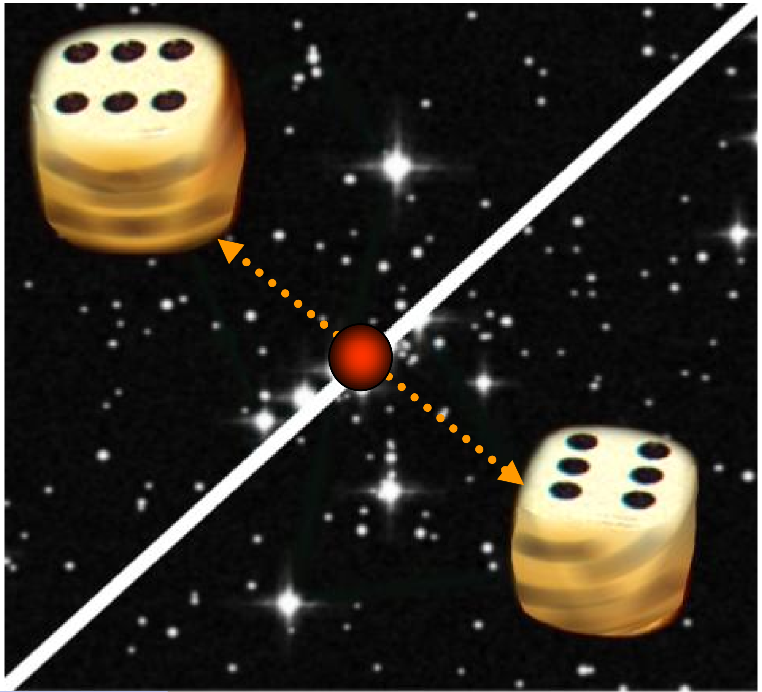
1.

ANY serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) "Is the theory correct?" and (2) "Is the description given by the theory complete?" It is only in the case in which positive answers may be given to both of these questions, that the concepts of the theory may be said to be satisfactory. The correctness of the theory is judged by the degree of agreement between the conclusions of the theory and human experience. This experience, which alone enables us to make

Whatever the meaning assigned to the term *complete*, the following requirement for a complete theory seems to be a necessary one: *every element of the physical reality must have a counterpart in the physical theory*. We shall call this the condition of completeness. The second question is thus easily answered, as soon as we are able to decide what are the elements of the physical reality.

The elements of the physical reality cannot be determined by *a priori* philosophical considerations, but must be found by an appeal to results of experiments and measurements. A comprehensive definition of reality is, however, unnecessary for our purpose. We shall be satisfied with the following criterion, which we regard as reasonable. *If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical*

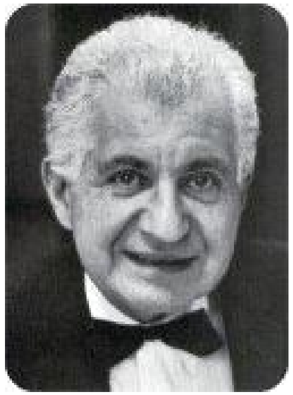


“遥远地点之间的诡异互动” ——爱因斯坦

Plausible Propositions of EPR

- Perfect Correlation (Quantum Prediction)
- Locality
- Reality
- Completeness

David Bohm



Boris Podolsky

Nathan Rosen

Einstein, Podolsky, and Rosen, Phys. Rev. 47, 777 (1935)

Quantum states

- Superposition Principle in Quantum Mechanics

A

the system can be in: $|0\rangle$

or: $|1\rangle$

or: $a|0\rangle + b|1\rangle$

Mathematically: such a state is a vector in $\mathcal{C}^2$

$$|1> = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |0> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so that:

$$\alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$ where $$|\alpha|^2 + |\beta|^2 = 1$$

Two or more systems:

a state $|\Psi\rangle$ of the system can be in: $|00\rangle, |01\rangle, |10\rangle, |11\rangle,$

or: $a|00\rangle + b|01\rangle + g|10\rangle + d|11\rangle$

where $$|00\rangle = |0\rangle \otimes |0\rangle$$

The system is entangled if

$$|\Psi\rangle \neq |\Psi\rangle_A \otimes |\Psi\rangle_B$$

Example: Bohm state $$|\Psi^-\rangle = 1/\sqrt{2}\,(|01\rangle - |10\rangle)$$

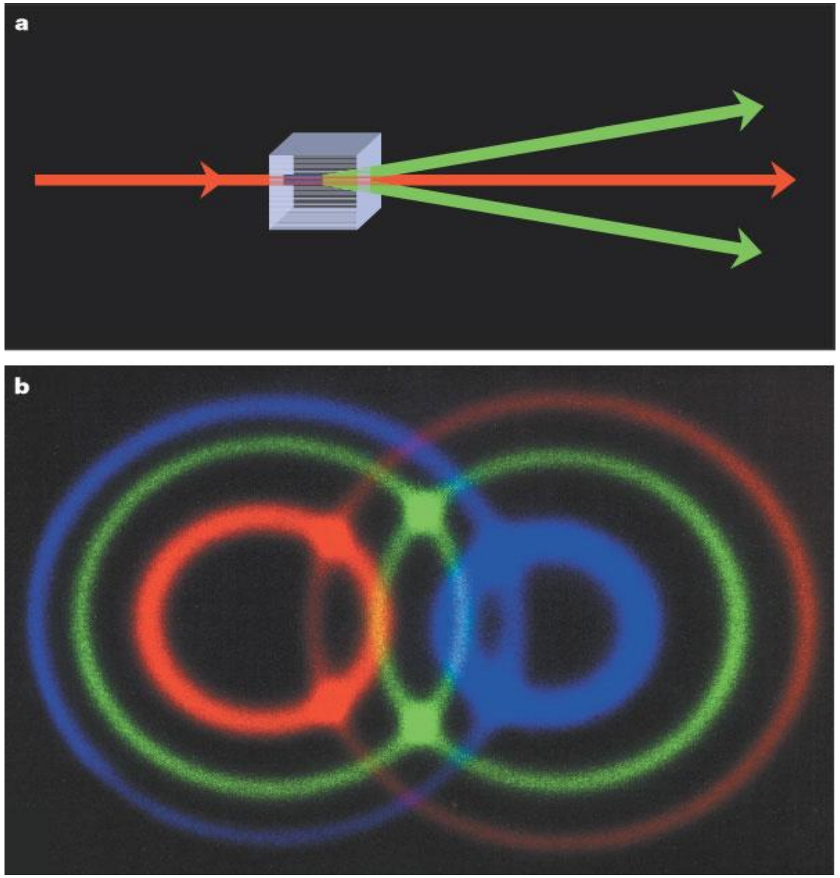
i.e. EPR (Einstein, Podolsky and Rosen) pair

Entangled states

- Non-local correlations among the separated parts

- Failing to interpret with the LHV theory

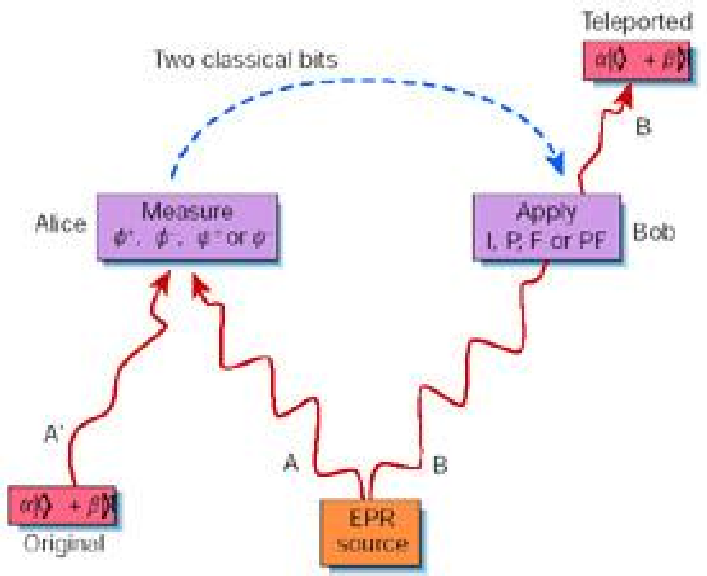
- Bell´s theorem (test non-locality)



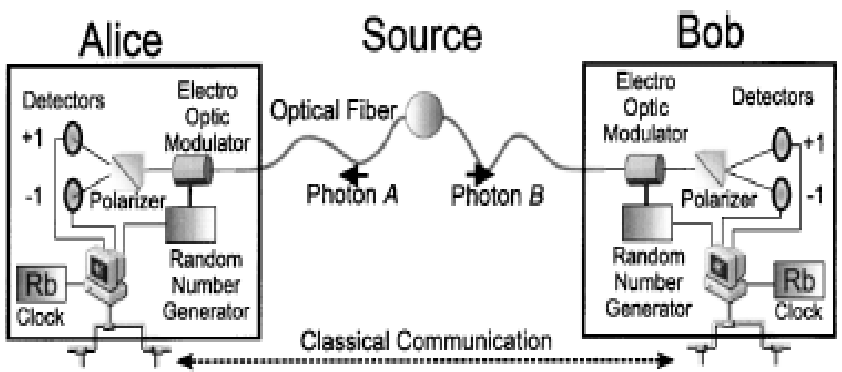
EPR pair

Applications （basic resources）

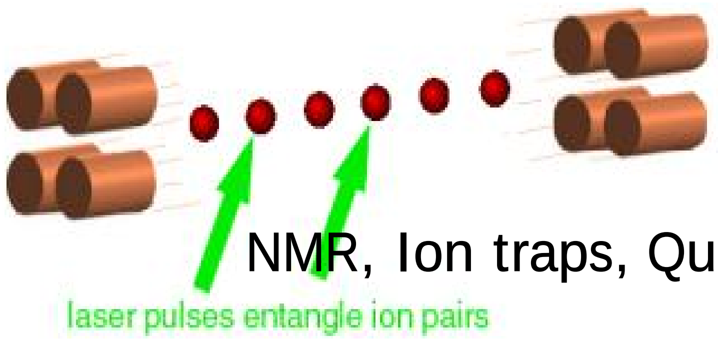
- Quantum teleportation



- Quantum communication (Quantum Key Distribution)



- Quantum computation



NMR, Ion traps, Quantum dots, Josephson junctions etc.

-Shor`s algorithm for factorization

-Grover`s algorithm for database search

-Quantum simulations (Feymann,Lloyd)

Becoming key resources for present and future quantum informartion processing!

量子纠缠应用：
superdense coding

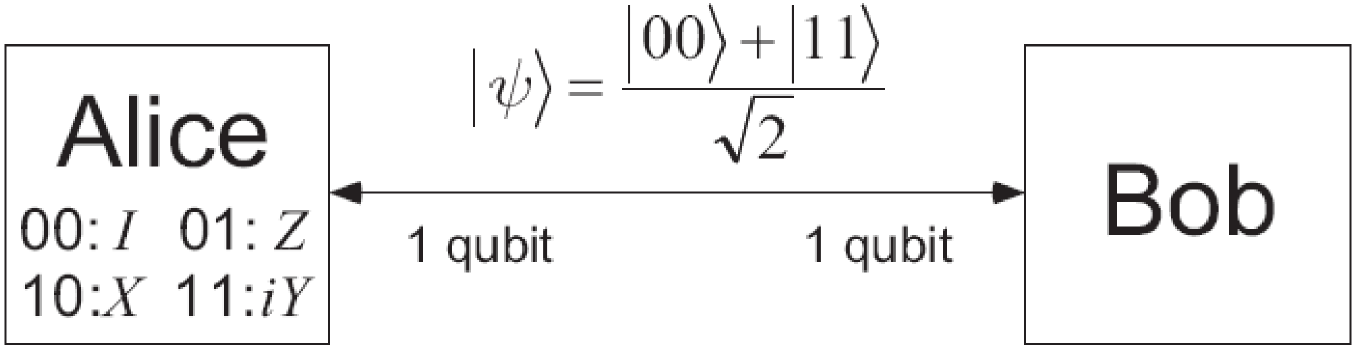


Figure 2.3. The initial setup for superdense coding, with Alice and Bob each in possession of one half of an entangled pair of qubits. Alice can use superdense coding to transmit two classical bits of information to Bob, using only a single qubit of communication and this preshared entanglement.

编码和解码

$$00 : |\psi\rangle \rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$01 : |\psi\rangle \rightarrow \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

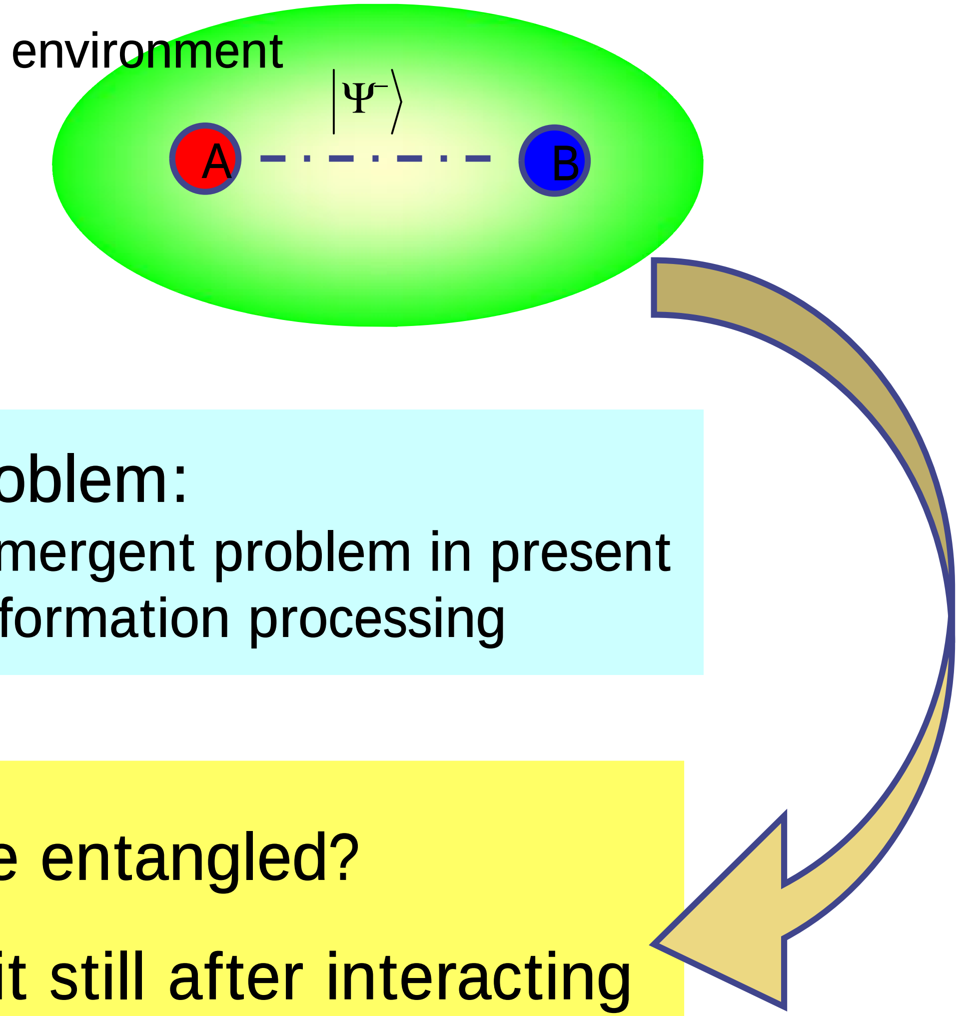
$$10 : |\psi\rangle \rightarrow \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

$$11 : |\psi\rangle \rightarrow \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

量子纠缠应用

Secret communication
Factorization (Shor)
Database search (Grover)
Quantum simulations

Decoherence



The separability problem:
one of the basic and emergent problem in present and future quantum information processing

Is a quantum state entangled?

How entangled is it still after interacting with a noisy environment?

Density matrix of quantum states

A number of states $|\psi_i\rangle$ with respective probabiliti p_i

define: $$\rho \equiv \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

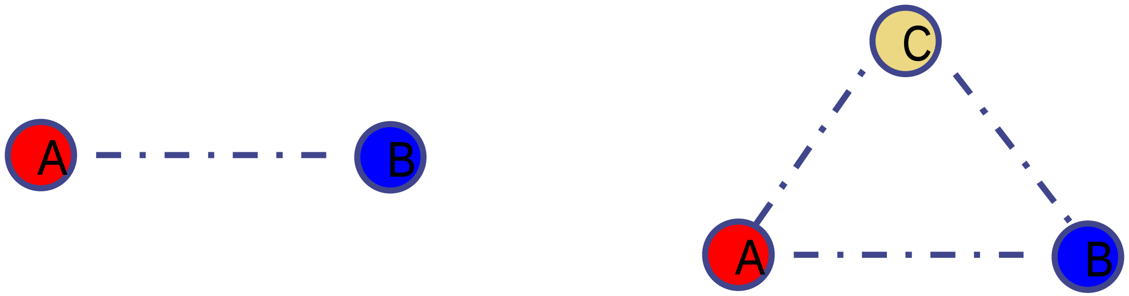
we call $\{p_i, |\psi_i\rangle\}$ an ensemble of pure states,
where

$$\rho \geq 0, tr\rho = 1, \rho = \rho^\dagger$$

Pure states: $$\rho^2 = \rho = |\psi_i\rangle\langle\psi_i|$$

Mixed states: $$\rho^2 \neq \rho$$

Separability



entangled?

Pure states

- Product states(separable):

$$|\Psi_{AB}\rangle = |\psi_A\rangle|\psi_B\rangle$$

density matrix

$$\rho_{AB} = \rho_A \otimes \rho_B,\ \rho_A = |\psi\rangle_A\langle\psi|,\ \rho_B = |\psi\rangle_B\langle\psi|$$

- Examples：

Product state: $|\Psi\rangle = |00\rangle$

Entangled state: $|\Psi\rangle = c_0|00\rangle + c_1|11\rangle \quad c_0, c_1 \neq 0$

Mixed states

Physical definition:

a separable state is a quantum state which can be prepared in a *local* or *classical* way (Local operations and classical communications: LOCC),

this is equivalent to:

$$\rho_{AB\cdots Z} = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \cdots \otimes \rho_i^Z$$

(Werner 89)

Otherwise, it is entangled

Problem：there are infinite possible decomposition,

$$r_{AB} = \sum_i q_i r^i_{AB}$$

does there exist decomposition like formula ?

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Separability criterion for multipartite pure state

A pure state is separable if and only if

$$\rho_{AB\cdots Z} = \rho_A \otimes \rho_B \otimes \cdots \otimes \rho_Z$$

where

$$\begin{aligned}\rho_A &= Tr_{B,C,\cdots,Z}(\rho_{AB\cdots Z}),\\ \rho_B &= Tr_{A,C,\cdots,Z}(\rho_{AB\cdots Z}),\\ &\vdots\\ \rho_Z &= Tr_{A,B,\cdots,Y}(\rho_{AB\cdots Z}),\end{aligned}$$

are the reduced density matrices for the subsystems A,B,...,Z respectively.

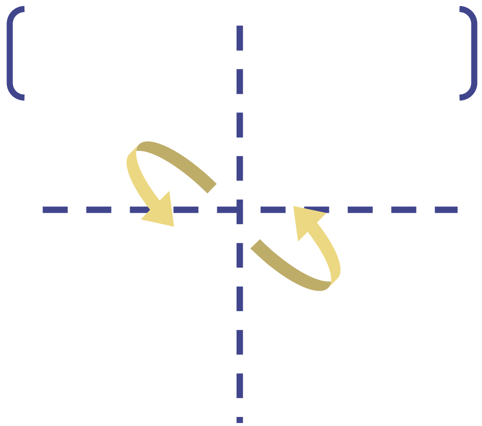
A strong separability criterion for mixed state

Positive partial transpositions(PPT) Peres PRL **77**, 1413 (1996)

$$\rho = \sum_i p_i \rho_A^i \otimes \rho_B^i \geq 0 \quad \Rightarrow \quad \rho^{T_A} = \sum_i p_i (\rho_A^i)^T \otimes \rho_B^i \geq 0$$

An example of 2x2 state:

$$r = \left(\begin{array}{cc|cc} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ \hline r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{array}\right)$$



$$r^{T_A} = \left(\begin{array}{cc|cc} r_{11} & r_{12} & r_{31} & r_{32} \\ r_{21} & r_{22} & r_{41} & r_{42} \\ \hline r_{13} & r_{14} & r_{33} & r_{34} \\ r_{23} & r_{24} & r_{43} & r_{44} \end{array}\right)$$

例子

量子态 $\Psi = |00\rangle + |11\rangle$ 其密度矩阵为

$$\rho = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

部分转置给出

$$\rho^{T_A} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$$

为非正半定的。本征值为{-1/2, 1/2, 1/2, 1/2}

重要结果

Horodecki *et al*. (PLA,1996)

$2 \otimes 2,\ 2 \otimes 3$ cases: PPT $\Leftrightarrow$ Separable

Horodeckis, Phys. Lett. A **223**,1 (1996)

部分转置的量子态可表为

$$\langle m|\langle\mu|\varrho_{AB}^{T_B}|n\rangle|\nu\rangle \equiv \langle m|\langle\nu|\varrho_{AB}|n\rangle|\mu\rangle$$

对于可分离态，也应为一个密度矩阵，
应有非负的本征谱

更一般的结果

Necessary and Sufficient Condition for Separability

For any positive (P) but not completely positive (CP) map,

$$\Lambda : \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_{A'})$$

one should have $$[I_A \otimes \Lambda_B](\varrho_{AB}) \geqslant 0$$

for any separable states.

$$[I_A \otimes \Lambda_B](\varrho_{AB}) = \begin{pmatrix} \Lambda(\varrho_{00}) & \cdots & \Lambda(\varrho_{0d_A-1}) \\ \Lambda(\varrho_{10}) & \cdots & \Lambda(\varrho_{1\ d_A-1)}) \\ \cdots & \cdots & \cdots \\ \Lambda(\varrho_{d_A-10}) & \cdots & \Lambda(\varrho_{d_A-1d_A-1}) \end{pmatrix}$$

Here $$\varrho_{ij} \equiv \langle i| \otimes \mathrm{I} \,|\varrho_{AB}|\, j\rangle \otimes I$$

低维情形

当取$2 \otimes 2, 2 \otimes 3$ 情形时，

所有的正映射是可分解的

$$\Lambda^{\rm dec} = \Lambda_{CP}^{(1)} + \Lambda_{CP}^{(2)} \circ T$$

对于$2 \otimes 2$情形，我们还有

$$\det(\varrho_{AB}^{\Gamma}) \geqslant 0$$

为可分性的充要条件

Majorization判据

If a state is separable then the inequalities

$$\lambda(\rho) \prec \lambda(\rho_A), \quad \lambda(\rho) \prec \lambda(\rho_B)$$

Holds.

Here $\lambda(\rho)$ is a vector of eigenvalues of ρ ;
$\lambda(\rho_A)$ and $\lambda(\rho_B)$ are defined similarly.

$x \prec y$ means $\Sigma_{i=1}^{k} x_i^{\downarrow} \leqslant \Sigma_{i=1}^{k} y_i^{\downarrow},\ 1 \leqslant k \leqslant d$

Nielsen, M. A., and J. Kempe, 2001, Phys. Rev. Lett. 86, 5184

Reduction criterion

定义映射

$$\Lambda^{red}(\varrho) = I\,\mathrm{Tr}(\varrho) - \varrho$$

对于可分态应有

$$[I_A \otimes \Lambda_B^{\mathrm{red}}](\varrho_{AB}) \geqslant 0$$

化简后，即得

$$\varrho_A \otimes I - \varrho_{AB} \geqslant 0$$

- 此判据弱于PPT准则，但是强于Majorization判据
- 违背此准则，一定是可提纯的

Cerf, N. J., C. Adami, and R. M. Gingrich, 1999, Phys. Rev. A 60, 898.

Horodecki, M., and P. Horodecki, 1999, Phys. Rev. A 59, 4206.

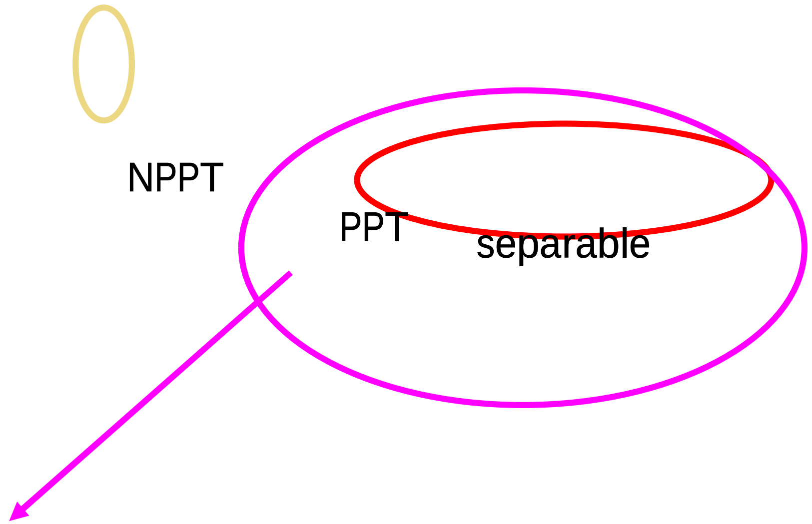
Hiroshima, T., 2003, Phys. Rev. Lett. 91, 057902.

其它判据

- LOO(Local Orthogonal Observables)判据
- Covariance matrix criterion
- Local uncertainty relations判据
- Range criterion

Status for the separability problem before 2002

Generic state



-Low rank

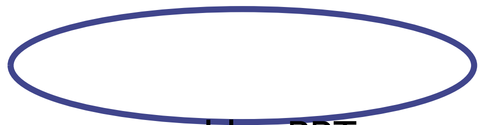
-Operational necessary or sufficient conditions (Lewenstein, Horodecki, Albeverio, Fei *et al.*,2000, 2001)

Bound entangled states (BES) which can not be distilled to be EPR pair: un-distillable

The main progress：

- Bell inequalities (Bell, 1964)
- Entanglement of formation for two qubits (Wootters,1998)
- The reduction criterion (Horodecki, Cerf *et al.* 1999)
- Low rank cases (Lewenstein,Cirac,Horodecki, Albeverio, Fei *et al.* 2000, 2001)
- The necessary and sufficient criterion(Y.D. Zhang and C.Z. Li 2000,2001)
- The majorization criterion (Nielsen and Kempe,2001)
- Entanglement witnesses (Horodecki, Terhal, Lewenstein *et al.*,1996,2000)
- PPT extension (Doherty *et al.*,2002)

2x2 and 2x3



Horodeckis, Phys. Lett. A **223**,1 (1996)

Disadvantages:

- Only a few are operational and computational, even they are, most of them are weaker than PPT.
- unable to distinguish bound entangled states (BES)
- some of them are complicated

A Matrix Realignment Method for Recognizing Entanglement

Define realignment operation:

If Z is an mxm block matrix with block size nxn,

$$Z = \begin{pmatrix} Z_{11} & \cdots & Z_{1m} \\ \vdots & \ddots & \vdots \\ Z_{m1} & \cdots & Z_{mm} \end{pmatrix} \longrightarrow \tilde{Z} = \begin{pmatrix} vec(Z_{11})^T \\ \vdots \\ vec(Z_{m1})^T \\ \vdots \\ vec(Z_{1m})^T \\ \vdots \\ vec(Z_{mm})^T \end{pmatrix}$$

$$A = [a_{ij}] \longrightarrow vec(A) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \\ \vdots \\ a_{1m} \\ \vdots \\ a_{mm} \end{pmatrix}$$

A 2x2 example:

$$\rho = \left(\begin{array}{cc|cc} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \hline \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{array}\right) \longrightarrow \tilde{\rho} = \left(\begin{array}{cccc} \rho_{11} & \rho_{21} & \rho_{12} & \rho_{22} \\ \hline \rho_{31} & \rho_{41} & \rho_{32} & \rho_{42} \\ \hline \rho_{13} & \rho_{23} & \rho_{14} & \rho_{24} \\ \hline \rho_{33} & \rho_{43} & \rho_{34} & \rho_{44} \end{array}\right)$$

The realignment criterion

For any bipartite separable state, we have

$$\|\tilde{\rho}\| \leq 1$$

necessary criterion for separability

Here $\|\tilde{r}\|$ is the sum of all the singular values of $\tilde{r}$, or sum of the square roots of eigenvalue for $\tilde{\rho}\tilde{\rho}^{\dagger}$.

Kai Chen, Ling-An Wu, *Quantum Information and Computation* 3, 193-202 (2003)

Recognizing entangled states

[redacted] → ρ is entangled

sufficient criterion for entanglement

This criterion is strong enough to distinguish most of BES in the literature!

Examples

Distinguish completely

- d＝2 Werner state
- the Bell diagonal states
- isotropic states in arbitrary dimensions
- most of BES (PPT fails）

1. d＝2 Werner state

$$\rho = x|\Psi^-\rangle\langle\Psi^-| + (1-x)\frac{Id}{4}, 0 \leq x \leq 1$$

$$\rho = \begin{pmatrix} \frac{1-x}{4} & 0 & 0 & -\frac{x}{2} \\ 0 & \frac{1+x}{4} & 0 & 0 \\ 0 & 0 & \frac{1+x}{4} & 0 \\ -\frac{x}{2} & 0 & 0 & \frac{1-x}{4} \end{pmatrix},$$

ρ is entangled iff

$$\frac{1}{3} < x \leq 1 \Longleftrightarrow \|\tilde{\rho}\| > 1$$

2. BES of 3x3 (weak inseparable PPT state)

a. BES constructed from unextendible product bases (UPB)

Bennett *et al.*, PRL82 (1999) 5385

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle), \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle,$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle), \quad |\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle,$$

$$|\psi_4\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle),$$

$$\rho = \frac{1}{4}(\mathbb{1} - \sum_{i=0}^{4} |\psi_i\rangle\langle\psi_i|)$$ [redacted]

b. 3x3 BES constructed by Horodecki

Horodecki, PLA232,333(1997)

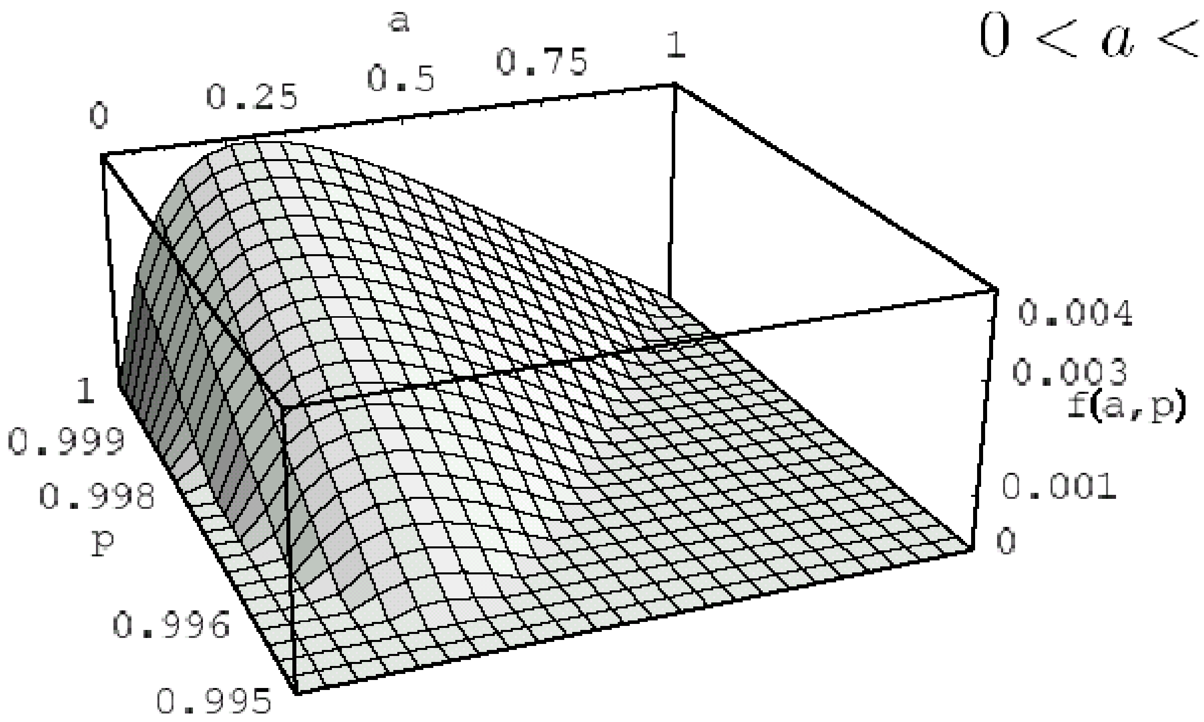
$$\rho = \frac{1}{8a+1}\begin{bmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{bmatrix},$$

$$0 < a < 1$$

Let

$$\rho_p = p\rho + (1-p)Id/9$$

$$f(a,p) = \max(0, \log\|\widetilde{\rho_p}\|)$$



When a=0.236，f(a,p) maintains its maximum

c. seven parameter family of PPT entangled states

Bruß.Peres, PRA61,030301(R)(2000)

$$\rho = N \sum_{j=1}^{4} |V_j\rangle\langle V_j|,$$

where

$$N = 1/\sum_j \langle V_j, V_j\rangle$$

$$|V_1\rangle = |m, 0, s;\ 0, n, 0;\ 0, 0, 0\rangle,$$

$$|V_2\rangle = |0, a, 0;\ b, 0, c;\ 0, 0, 0\rangle,$$

$$|V_3\rangle = |n^*, 0, 0;\ 0, -m^*, 0;\ t, 0, 0\rangle,$$

$$|V_4\rangle = |0, b^*, 0;\ -a^*, 0, 0;\ 0, d, 0\rangle,$$

The criterion could detect entanglement in about 22% of these BES satisfied $r = r^{T_A}$ by numerical calculation.

d. Entangled state in three-party system

Bi-separable with respect to A|BC,B|CA and C|AB

Bennett *et al.*, PRL82,5385(1999)

$$\rho_{ABC} = \frac{1}{8}(I - \sum_{i=1}^{4} |\psi_i\rangle\langle\psi_i|)$$

where ψ_i is

$$|0,1,+\rangle, |1,+,0\rangle, |+,1,0\rangle, |-,-,-\rangle$$

and $|\pm\rangle = 1/\sqrt{2}(|0\rangle \pm |1\rangle)$

define

$$\mathcal{R} : Z_{AB} \longrightarrow \tilde{Z}_{AB}$$

then $||(I_A \otimes \mathcal{R}_{BC})\rho_{ABC}|| = 1.086$

Horodecki et al., Open Syst. Inf. Dyn. 13, 103 (2006)

The generalized partial transposition operations (GPT operations)

Define the operations:

$$\mathcal{T}_r : A \longrightarrow \text{row transposition of } A$$

$$\Longleftrightarrow A \longrightarrow (vec(A))^t$$

$$\mathcal{T}_c : A \longrightarrow \text{column transposition of } A$$

$$\Longleftrightarrow A \longrightarrow vec(A)$$

$$\mathcal{T}_c\mathcal{T}_r \text{ or } \mathcal{T}_r\mathcal{T}_c : A \longrightarrow A^t$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\mathcal{T}_r(A) = \left(\begin{array}{cc|cc} a_{11} & a_{21} & a_{12} & a_{22} \end{array} \right)$$

$$\mathcal{T}_c(A) = \begin{pmatrix} a_{11} \\ a_{21} \\ \hline a_{12} \\ a_{22} \end{pmatrix}$$

The GPT Criterion

For any *n*-partite separable state, we have

$$||\rho^{\mathcal{T}_{\mathcal{Y}}}|| \leq 1, \qquad \forall \mathcal{Y} \subset \{\underbrace{r_A, c_A, r_B, c_B, \cdots, r_Z, c_Z}_{2n}\}$$

Where $\mathcal{T}_{r_k}$ or $\mathcal{T}_{c_k}$ $(k = A, B, \cdots, Z)$ means transpositions with respect to the row or column for the *k*th subsystem.

If $\exists \mathcal{Y},$

Kai Chen, Ling-An Wu,
Physics Letters A 306, 14-20 (2002)

$$||\rho^{\mathcal{T}_{\mathcal{Y}}}|| > 1 \longrightarrow \rho$$ is entangled

The Generalized reduction criterion

Define an operation:

$$\rho_{AB} \longrightarrow \widetilde{\rho_{AB}} = ab\mathbb{I}_{mn} - a\mathbb{I}_m \otimes \rho_B - b\rho_A \otimes \mathbb{I}_n + \rho_{AB},$$

For any *mxn* bipartite separable state, one has

$$||\widetilde{\rho_{AB}}^{\mathcal{T}_{\mathcal{Y}}}|| \leq h_a h_b, \qquad \forall \mathcal{Y} \subset \{r_A, c_A, r_B, c_B\},$$

Where h_a and h_b are simple functions of *a,b,m* and *n*.

If $\exists a, b$

S. Albeverio, K. Chen, S.M. Fei, Phys. Rev. A 68, 062313 (2003)

$$||\widetilde{\rho_{AB}}^{\mathcal{T}_{\mathcal{Y}}}|| > h_a h_b \longrightarrow \rho \text{ is entangled}$$

Two special cases:

1. In the case of *a=1 and b=0*, or *a=0 and b=1*, this criterion reduces to the reduction criterion (Horodecki, Cerf *et al.* 1999)
2. In the case of *a=0 and b=0*, this criterion reduces to the GPT criterion

An 3x3 BES constructed by Horodecki

(Horodecki, PLA232,333(1997))

Let

$$\rho=\frac{1}{8c+1}\times\begin{bmatrix} c & 0 & 0 & 0 & c & 0 & 0 & 0 & c \\ 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & c & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+c}{2} & 0 & \frac{\sqrt{1-c^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 \\ c & 0 & 0 & 0 & c & 0 & \frac{\sqrt{1-c^2}}{2} & 0 & \frac{1+c}{2} \end{bmatrix}$$

1. When *a=0*，this criterion detect all the BES for *0<c<1* while *b=0* or *b=2/3*.
2. When *a=1*, it also detect all BES *0<c<1* while *b=-1/3* or *b=1*.

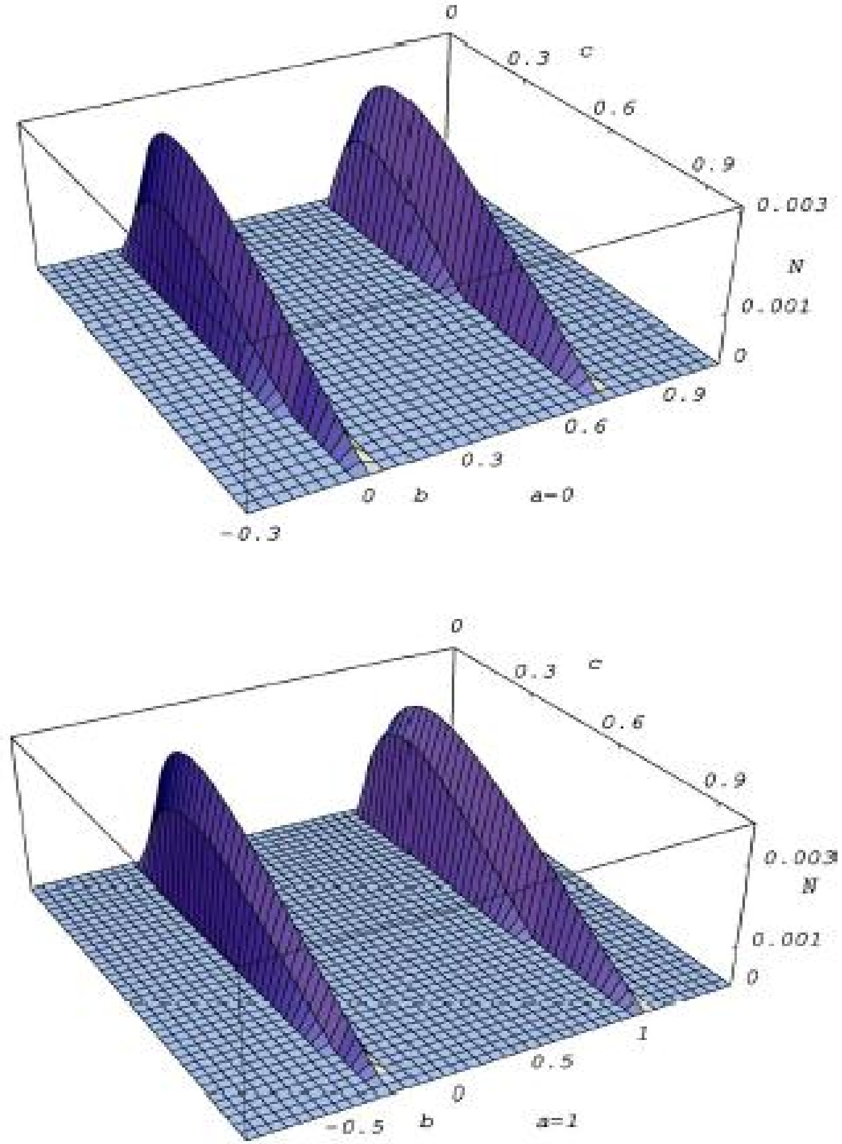


FIG. 2. Depiction of $N=\max\{\|\widetilde{\rho_{AB}}^{T(c_A,r_B)}\|-h_a h_b,0\}$ for a Horodecki 3×3 bound entangled state as a function of b and c when $a=0$ (the top figure) and $a=1$ (the bottom figure), respectively.

Positive maps connected to entanglement witnesses (EW)

Jamiołkowski isomorphism

$$W_\Lambda = [I \otimes \Lambda](P_d^+) \qquad P_d^+ = |\Phi_d^+\rangle\langle\Phi_d^+|$$

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle, \quad d = \dim \mathcal{H}_A$$

其中

$$\Lambda(|i\rangle\langle j|) = \langle i|W|j\rangle$$

不满足

$$(Id_A \otimes \Lambda)\rho \geq 0 \longrightarrow \rho \text{ 纠缠的}$$

Universal construction of the witness operator

1. Universal construction of the witness operator from the realignment criterion

$$W = Id - (\Re^{-1}(U^* V^T))^T$$

where U,V are unitary matrices that yield the singular value decomposition (SVD) of $\mathcal{R}(\rho)$ i.e.,$\mathcal{R}(\rho) = U \Sigma V^\dagger$

2. Universal construction of the witness operator from the *PPT* criterion

$$W = Id - (VU^+))^{T_A}$$

where U,V are unitary matrices that yield the singular value decomposition (SVD) †of ρ^{T_A} i.e.$\rho^{T_A} = U \Sigma V^\dagger$

Kai Chen, Ling-An Wu, *Phys. Rev. A 69, 022312 (2004)*

An BES (weak inseparable PPT state) constructed from unextendible product bases (UPB) (Bennett *et al.*, PRL82, 5385 (1999))

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle), \quad |\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle,$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle), \quad |\psi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle,$$
$$|\psi_4\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle),$$

$$\rho = \frac{1}{4}(\mathbb{1} - \sum_{i=0}^{4} |\psi_i\rangle\langle\psi_i|)$$

Consider

$$\rho_p = p\rho + (1-p)Id/9$$ *(through a depolarizing channel)*

1. Realignment criterion recognize entanglement for $p>88.97\%$
2. An optimal witness can only recognize entanglement for $p>94.88\%$ (B.M. Terhal, Phys. Lett. A 271 (2000) 319, O. Gühne *et al.*, Phys. Rev. A 66,062305 (2002).)
3. An EW constructed from the realignment criterion gives $p>88.41\%$
4. An PM obtained from EW constructed from p=0.3 gives $p>87.44\%$

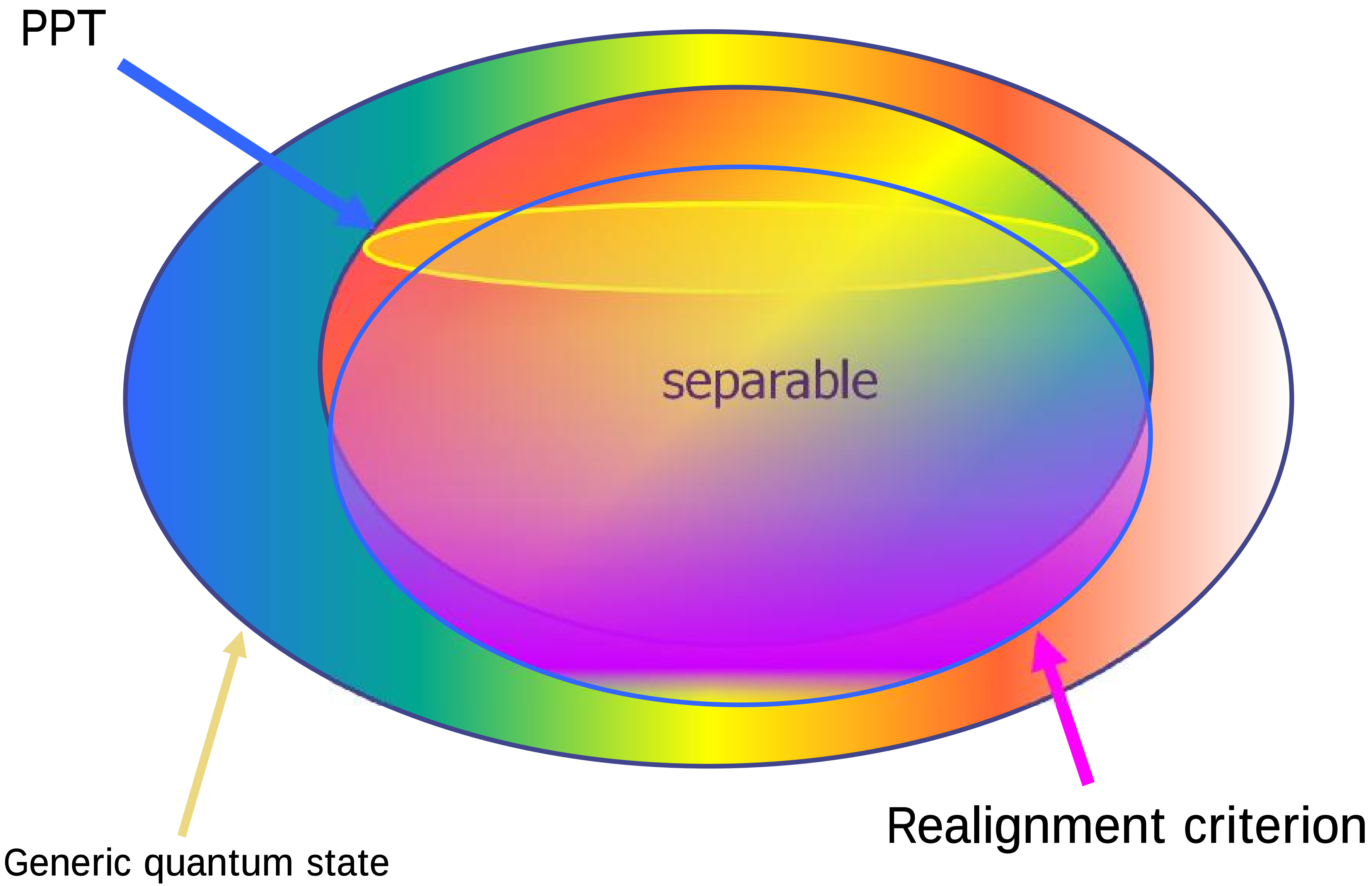
Results

1. Entanglement witness operators generated from the realignment criterion and PPT criterion are more powerful than the two criteria to identify entanglement
2. Positive map (not completely positive) constructed from these entanglement witnesses (EW) are further powerful than the EWs

Significance

1. Offer a more power operational method to recognize entanglement, in particular, the bounded entanglement
2. Provide a powerful new method to detect entanglement, since the entanglement witnesses are physical observables and may be measured locally
3. Gives a new systematic way to obtain positive but non-CP maps

Comparison of separability criteria



量子纠缠可分性问题展望

- The separability of a quantum state and quantitative character for entanglement become two of the most basic problems in quantum Information theory

- Multipartite systems and higher dimensions make a richer structure but with more complexity

- The PPT criterion, realignment criterion, its generalizations and the corresponding witness operators and positive maps significantly expand our ability to recognize directly the entanglement

- The final solution needs better ideas and is still full of challenge

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6. Shannon entropy，Von Neumann entropy
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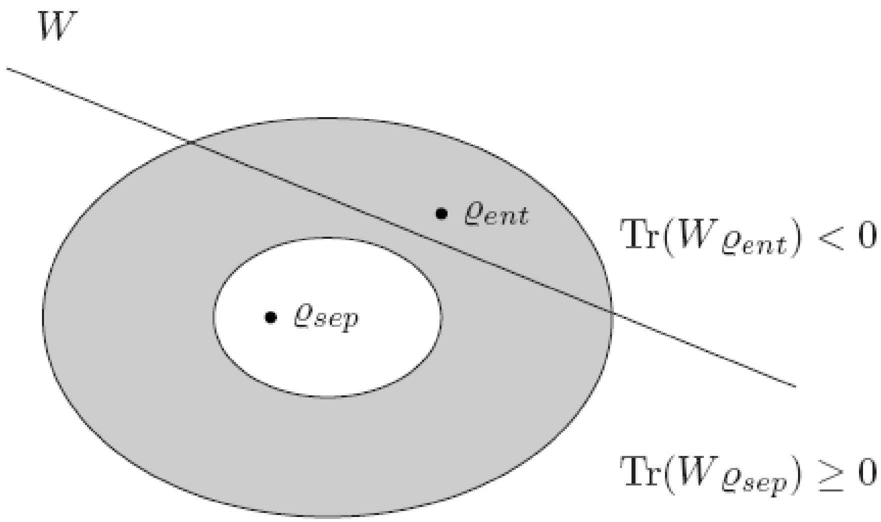
Entangelement witness (EW)

定义映射

$$\mathrm{Tr}(W\varrho_{AB}) \geqslant 0$$

W为可观测量，满足

- 至少有一个负本征值
- 对于所有直积态，应有 $\langle\psi_A|\langle\phi_B|W|\psi_A\rangle|\phi_B\rangle \geqslant 0$

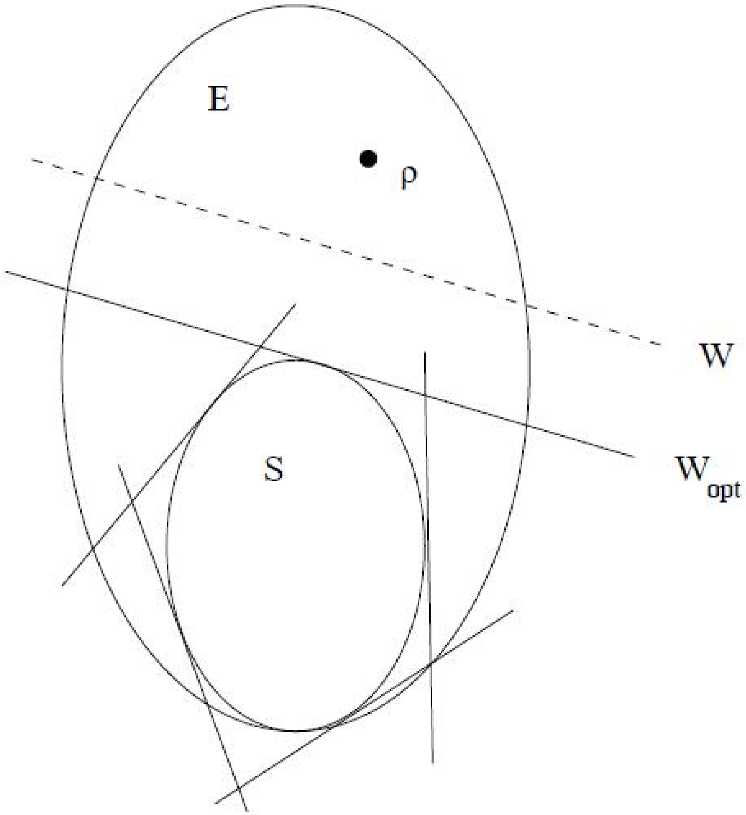


Terhal, B. M., 2000, Phys. Lett. A 271, 319

Entanglement witness (EW)

Hahn-Banach theorem

Let S be a convex, compact set, and let $\rho \notin S$,
then there exists a hyper-plane that separates ρ from S



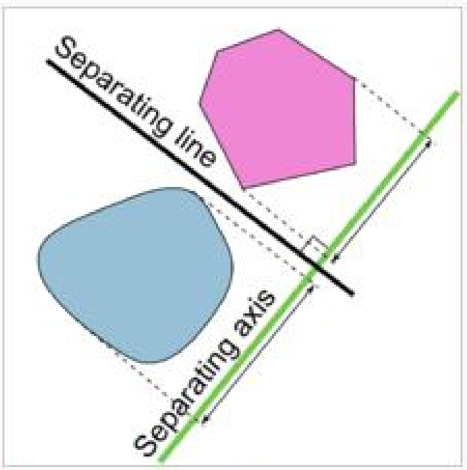


Illustration of the separating axis theorem.

Hermann Minkowski

Entanglement witness例子

交换算符

$$V=\sum_{i,j=0}^{d-1}|i\rangle\langle j|\otimes|j\rangle\langle i|$$

$$\langle\psi_A|\langle\phi_B|V|\psi_A\rangle|\phi_B\rangle=|\langle\psi_A|\phi_B\rangle|^2\geqslant 0$$

$$V=P^{(+)}-P^{(-)}$$

对称子空间和反对称子空间

$$P^{(+)}=\frac{1}{2}(I+V)\quad\text{and}\quad P^{(-)}=\frac{1}{2}(I-V)$$

具有本征值-1

Entanglement witness例子

四体cluster态

$$|C_4\rangle = \frac{(|0000\rangle_{1234} + |0011\rangle_{1234} + |1100\rangle_{1234} - |1111\rangle_{1234})}{2}$$

构造

$$\mathcal{W} = \frac{[4I^{\otimes 4} - (XXIZ + XXZI + IIZZ + IZXX + ZIXX + ZZII)]}{2}$$

只需要两个实验settings即可

$$XXZZ \text{ and } ZZXX$$

<W>的负值意味着真正的4体纠缠

Choi-Jamiołkowski 同构

定义EW

$$W_\Lambda = [I \otimes \Lambda](P_d^+)$$

其中

$$P_d^+ = |\Phi_d^+\rangle\langle\Phi_d^+|$$

$$|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle, \quad d = \dim \mathcal{H}_A$$

Jamiołkowski, 1972; Choi, 1982

References for the realignment criterion

1. Kai Chen, Ling-An Wu,
   Quantum Information and Computation 3, 193-202 (2003);
   Physics Letters A 306, 14-20 (2002);
   Phys. Rev. A 69, 022312 (2004);

2. S. Albeverio, K. Chen, S.M. Fei, *Phys. Rev. A* 68, 062313 (2003);

3. O. Rudolph, *Quantum Information Processing* 4, 219-239 (2005);

4. Horodecki, M., P. Horodecki, and R. Horodecki, *Open Syst. Inf. Dyn.* 13, 103 (2006).

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纠缠量化

Good entanglement measures

- 对于可分离态为0

- **No increase under LOCC**

$$E(\Lambda_{LOCC}(\varrho)) \leq E(\varrho)$$

- **Continuity**

$$E(\varrho) - E(\sigma) \to 0 \quad \text{for} \quad ||\varrho - \sigma|| \to 0$$

纠缠量化

Good entanglement measures

- **Convexity**

$$E(\lambda\varrho + (1-\lambda)\sigma) \leq \lambda E(\varrho) + (1-\lambda)E(\sigma)$$

- **Normalization**

$$E(P_+^d) = \log d$$

纠缠量化

Bad!

- **Additivity**

$\sigma_1 \in \mathcal{H}_{A1} \otimes \mathcal{H}_{B1}$ *and* $\sigma_2 \in \mathcal{H}_{A2} \otimes \mathcal{H}_{B2}$. *Then*

$$E_F(\sigma_1 \otimes \sigma_2) = E_F(\sigma_1) + E_F(\sigma_2)$$

- **The strong superadditivity**

density matrix σ *over a quadripartite system* $\mathcal{H}_{A1} \otimes \mathcal{H}_{A2} \otimes \mathcal{H}_{B1} \otimes \mathcal{H}_{B2}$

$$E_F(\sigma) \geq E_F(\mathrm{Tr}_2\sigma) + E_F(\mathrm{Tr}_1\sigma)$$

M. B. Hastings, Nature Physics 5, 255 - 257 (2009); Los Alamos National Laboratory

P. W. Shor, Comm. Math. Phys. 246, 453–472 (2004); AT&T

纠缠度量引出的4个等价问题

- *additivity of the minimum entropy output of a quantum channel;*
- *additivity of the Holevo capacity of a quantum channel;*
- *additivity of the entanglement of formation;*
- *strong superadditivity of the entanglement of formation.*

P. W. Shor, Comm. Math. Phys. 246, 453–472 (2004); AT&T

纠缠常用的度量

Distillable Entanglement

$$E_D(\rho) := \sup\left\{r : \lim_{n\to\infty}\left[\inf_{\Psi} \mathrm{tr}\left|\Psi(\rho^{\otimes n}) - \Phi(2^{rn})\right|\right] = 0\right\}$$

$\Phi(K)$ is the density operator corresponding to the maximally entangled state vector in K dimensions,

$$\Phi(K) = |\psi_K^+\rangle\langle\psi_K^+|$$

Ψ is a general trace preserving LOCC operation

物理含义： At what rate may we obtain maximally entangled states (of two qubits) from an input supply of states of the form ρ.

Plenio, M. B., and S. Virmani, 2006

纠缠常用的度量

Entanglement Cost

$$E_C(\rho) = \inf\left\{r : \lim_{n\to\infty}\left[\inf_{\Psi} D(\rho^{\otimes n}, \Psi(\Phi(2^{rn})))\right] = 0\right\}$$

$\mathrm{D}(\sigma, \eta)$ is a suitable measure of distance

$$\text{i.e. } D(\sigma, \eta) = tr|\sigma - \eta|$$

物理含义：For a given state ρ this measure quantifies the maximal possible rate r at which one can convert blocks of 2-qubit maximally entangled states into output states that approximate many copies of ρ, such that the approximations become vanishingly small in the limit of large block sizes.

Plenio, M. B., and S. Virmani, 2006

纯态纠缠度量

定义纯态的纠缠度量
Entropy of Entanglement

$$E(|\psi\rangle\langle\psi|) := S(\mathrm{tr}_A|\psi\rangle\langle\psi|) = S(\mathrm{tr}_B|\psi\rangle\langle\psi|)$$

其中

$$S(\rho) = -\mathrm{tr}[\rho \log_2 \rho]$$

为von-Neumann entropy

对于纯态 $E_D(\rho)$ and $E_C(\rho)$ are identical

混合态纠缠

定义混合态的量子纠缠度量

$$E(\varrho) = \inf \sum_i p_i E(\psi_i), \quad \sum_i p_i = 1, \quad p_i \geqslant 0$$

$\{p_i, \psi_i\}$ 满足 $\varrho = \Sigma_i p_i |\psi_i\rangle\langle\psi_i|$

Uhlmann, 1998

纠缠最常用的度量Entanglement of Formation

$$E_F(\rho) := \inf\{\sum_i p_i E(|\psi_i\rangle\langle\psi_i|) \ : \ \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|\}$$

其中 $E(|\psi\rangle\langle\psi|) = S(\mathrm{tr}_B\{|\psi\rangle\langle\psi|\})$

Bennett, DiVincenzo, *et al.* 1996

Two qubits纠缠度量

定义纯态的concurrence

$$C = \sqrt{2(1 - \mathrm{Tr}\rho^2)}$$

$$C(\psi) = 2a_1 a_2$$ 其中a_1,a_2为Schmidt系数

等价地
$$C = \langle \psi | \theta | \psi \rangle \qquad \theta\psi = \sigma_y \otimes \sigma_y \psi^*$$

S. Hill and W.K. Wootters, Phys. Rev. Lett. 78, 5022–5025 (1997)

Two qubits纠缠度量

定义

$$\tilde{\rho} = \theta\rho\theta \qquad \omega = \sqrt{\rho}\sqrt{\tilde{\rho}}$$

则混合态的concurrence

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$$

其中$\lambda_1, \lambda_2, \lambda_3, \lambda_4$为$\omega$的以递减顺序排列的奇异值

则Entanglement of Formation (EoF)为

$$E_F(\rho) = H\left(\frac{1 + \sqrt{1 - C^2(\rho)}}{2}\right)$$

W.K. Wootters, Phys. Rev. Lett. 80, 2245–2248 (1998)

其中 $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$

其它纠缠度量

Negativity

$$N(\rho) := \frac{||\rho^{T_B}|| - 1}{2}$$

其中 $||X|| := \mathrm{tr}\sqrt{X^\dagger X}$

或者Logarithmic Negativity

$$E_N(\rho) := \log_2 ||\rho^{T_B}||$$

均是Entanglement Monotones,但是后者非凸。

G. Vidal and R. F. Werner, Phys. Rev. A 65, 032314 (2002)

纠缠度量的一般构造
－Convex roof measures

混合态的纠缠度量

$$E(\varrho)=\inf\sum_i p_i E(\psi_i),\quad \sum_i p_i=1,\quad p_i\geqslant 0$$

Monotonicity under LOCC: Entanglement cannot increase under local operations and classical communication.

For any LOCC operation, we have

$$E(\Lambda(\rho))\leqslant E(\rho) \qquad \Lambda(\rho)=\sum_i A_i\otimes B_i(\rho)A_i^\dagger\otimes B_i^\dagger$$

距离形式的纠缠度量

定义纠缠度量

$$E_{\mathcal{D},\mathcal{S}}(\varrho) = \inf_{\sigma \in \mathcal{S}} \mathcal{D}(\varrho, \sigma)$$

其中距离*D*满足 $\mathcal{D}(\rho,\sigma) \geqslant \mathcal{D}(\Lambda(\rho),\Lambda(\sigma))$

例如relative entropy of entanglement

$$S(\varrho|\sigma) = \mathrm{Tr}\,\varrho(\log_2 \varrho - \log_2 \sigma)$$

$$E_R = \inf_{\sigma \in \mathrm{SEP}} \mathrm{Tr}\varrho(\log_2 \varrho - \log_2 \sigma)$$

V. Vedral, "The role of relative entropy in quantum information theory", Rev. Mod. Phys. 74, 197 (2002)

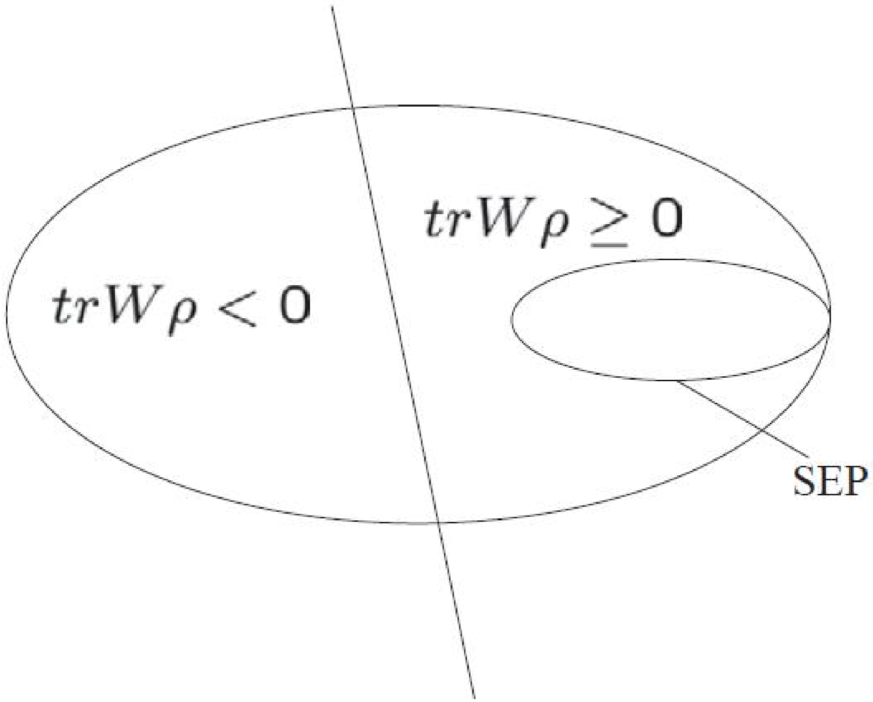
Entanglement Witness Monotones

Entanglement Witness

$$\forall\, \rho \in SEP \quad \mathrm{tr}\{W\rho\} \geq 0$$

$$and$$

$$\exists \rho \text{ s.t.} \quad \mathrm{tr}\{W\rho\} < 0.$$



定义度量

$$E_{wit}(W) = \max\{0, -\mathrm{tr}\{W\rho\}\}$$

纠缠度量大小如何计算？

推广的Concurrence

$$C(|\psi\rangle) = \sqrt{2(1 - \mathrm{Tr}\rho_A^2)}$$

$$C(\rho) \equiv \min_{\{p_i|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle)$$

纯态

$$|\psi\rangle = \sum_i \sqrt{\mu_i}\,|a_i b_i\rangle$$

$$C^2(|\psi\rangle) = 2\left(1 - \sum_i \mu_i^2\right) = 4\sum_{i<j}\mu_i\mu_j$$

where $\sqrt{\mu_i}$ $(i = 1, \ldots, m)$ are the Schmidt coefficients

结论

Theorem.—For any $m \otimes n (m \leq n)$ mixed quantum state ρ, the concurrence $C(\rho)$ satisfies

$$C(\rho) \geq \sqrt{\frac{2}{m(m-1)}}(\max(\|\rho^{T_A}\|, \|\mathcal{R}(\rho)\|) - 1).$$

 K. Chen, S. Albeverio, S.M. Fei, Phys. Rev. Lett. 95 (2005) 040504

纠缠度量大小如何计算？

Entanglement of Formation

纯态 $E(|\psi\rangle) = S(\rho_A)$ $\rho_A \equiv \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$

$$S(\rho_A) \equiv -\sum_{i=1}^{m} \mu_i \log_2 \mu_i = H(\vec{\mu})$$

结论 $$E(\rho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle)$$

$$E(\rho) \geq \begin{cases} 0, & \Lambda = 1, \\ H_2[\gamma(\Lambda)] + [1 - \gamma(\Lambda)]\log_2(m-1), & \Lambda \in [1, \frac{4(m-1)}{m}], \\ \frac{\log_2(m-1)}{m-2}(\Lambda - m) + \log_2 m, & \Lambda \in [\frac{4(m-1)}{m}, m], \end{cases}$$

$$R(\Lambda) = H_2[\gamma(\Lambda)] + [1 - \gamma(\Lambda)]\log_2(m-1),$$

$$\gamma(\Lambda) = \frac{1}{m^2}[\sqrt{\Lambda} + \sqrt{(m-1)(m-\Lambda)}]^2, \quad \Lambda = \max(\|\rho^{T_A}\|, \|\mathcal{R}(\rho)\|)$$

K. Chen, S. Albeverio, S.M. Fei, Phys. Rev. Lett. 95 (2005) 210501
S.M. Fei, X. Li-Jost, Phys. Rev. A 73 (2006) 024302

Ordering by Entanglement

$$E(\rho) \geqslant E(\sigma) \overset{?}{\Longrightarrow} E'(\rho) \geqslant E'(\sigma)$$

Not generally!

There are many different types of entanglement, and in one state we have more entanglement of one type, while in the other state there is more entanglement of some other type

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多粒子纠缠度量

Three-tangle or residual tangle

$$\tau(A{:}B{:}C) = \tau(A{:}BC) - \tau(AB) - \tau(AC)$$

Coffman *et al.* 2000

where two-tangles on the right-hand side are squares of concurrence

For a $2 \times n$ dimensional systems

$$\tau(\rho) = \left\{ \inf \sum_i p_i C^2 (|\psi_i\rangle\langle\psi_i|) \right\}$$

满足 $\tau(A:B) + \tau(A:C) + \tau(A:D) + ... \leq \tau(A:BCD...)$

Monogamy of Entanglement

For any tripartite state of systems A, B, C,
if one has

$$E(A:B) + E(A:C) \leqslant E(A:BC)$$

then

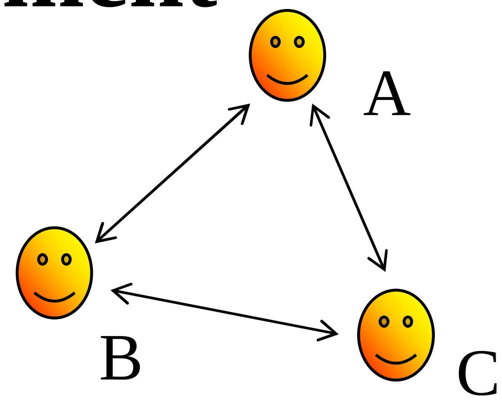
$$E(A:B_1) + E(A:B_2) + \cdots + E(A:B_N) \leqslant E(A:B_1 \cdots B_N).$$

已知结果 $E_{sq}(A:B) + E_{sq}(A:C) \leqslant E_{sq}(A:BC)$

Koashi and Winter 2004

E_f and E_c are not monogamous

Monogamy of Entanglement



Pure three qubit state $|\phi\rangle_{ABC}$

Concurrence $C_{AB}^2 + C_{AC}^2 \le C_{A(BC)}^2$

$$\rho_{AB} = Tr_C(|\phi\rangle_{ABC}\langle\phi|) \qquad \rho_{AC} = Tr_B(|\phi\rangle_{ABC}\langle\phi|)$$

Negativity $\mathcal{N} = \|\rho^{T_A}\| - 1$

$$\mathcal{N}_{AB}^2 + \mathcal{N}_{AC}^2 \le \mathcal{N}_{A(BC)}^2$$

High dimensional case

Y.C. Ou, H. Fan and S.M. Fei, Phys. Rev. A, 78 (2008) 012311.

第二章 量子纠缠

1. 量子纠缠的概念与内涵
2. 量子纠缠判据
3. 量子纠缠检验Entanglement witness
4. 纠缠量化
5. 多体纠缠
6. **Shannon entropy，Von Neumann entropy**
7. 纠缠提纯

Shannon entropy

Operationally as the minimum number of bits needed to communicate a message produced by a classical statistical source associated to a random variable X.

- *The Shannon entropy of X quantifies how much information we gain, on average, when we learn the value of X.*
- *The entropy of X measures the amount of uncertainty about X before we learn its value.*

Shannon entropy

A measure of our uncertainty before we learn the value of X

A measure of how much information we have gained after we learn the value of X.

$$H(X) \equiv H(p_1, \ldots, p_n) \equiv -\sum_x p_x \log p_x$$

Shannon's noiseless coding theorem:

It can be used to quantify the resources needed to store information

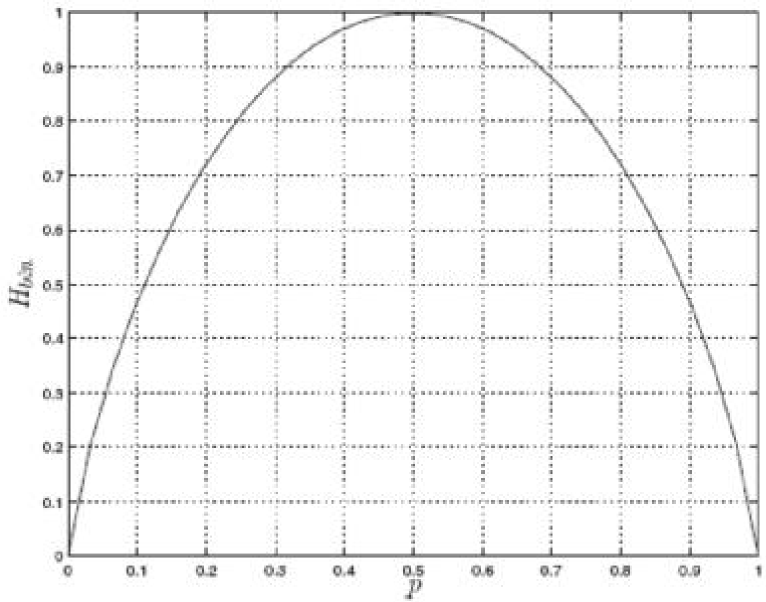
熵的基本性质

定义binary entropy

$$H_{\text{bin}}(p) \equiv -p\log p - (1-p)\log(1-p)$$

concavity

$$H(qp_{\text{U}} + (1-q)p_{\text{A}}) \geq qH(p_{\text{U}}) + (1-q)H(p_{\text{A}})$$



熵的变体

The relative entropy

$$H(p(x)\|q(x)) \equiv \sum_x p(x)\log\frac{p(x)}{q(x)} \equiv -H(X) - \sum_x p(x)\log q(x)$$

A good measure of distance between two distributions

$H(p(x)\|q(x)) \geq 0$, with equality if and only if $p(x) = q(x)$ for all x

熵的变体

Joint entropy

$$H(X,Y) \equiv -\sum_{x,y} p(x,y) \log p(x,y)$$

The joint entropy measures our total uncertainty about the pair (X, Y).

Conditional entropy

$$H(X|Y) \equiv H(X,Y) - H(Y)$$

A measure of how uncertain we are, on average, about the value of X, given that we know the value of Y.

熵的变体

Mutual information

$$H(X:Y) \equiv H(X) + H(Y) - H(X,Y)$$

Measuring how much information X and Y have in common.

Useful equality

$$H(X:Y) = H(X) - H(X|Y)$$

Shannon熵的基本性质

(1) $H(X,Y) = H(Y,X)$, $H(X:Y) = H(Y:X)$.

(2) $H(Y|X) \geq 0$ and thus $H(X:Y) \leq H(Y)$, with equality if and only if Y is a function of X, $Y = f(X)$.

(3) $H(X) \leq H(X,Y)$, with equality if and only if Y is a function of X.

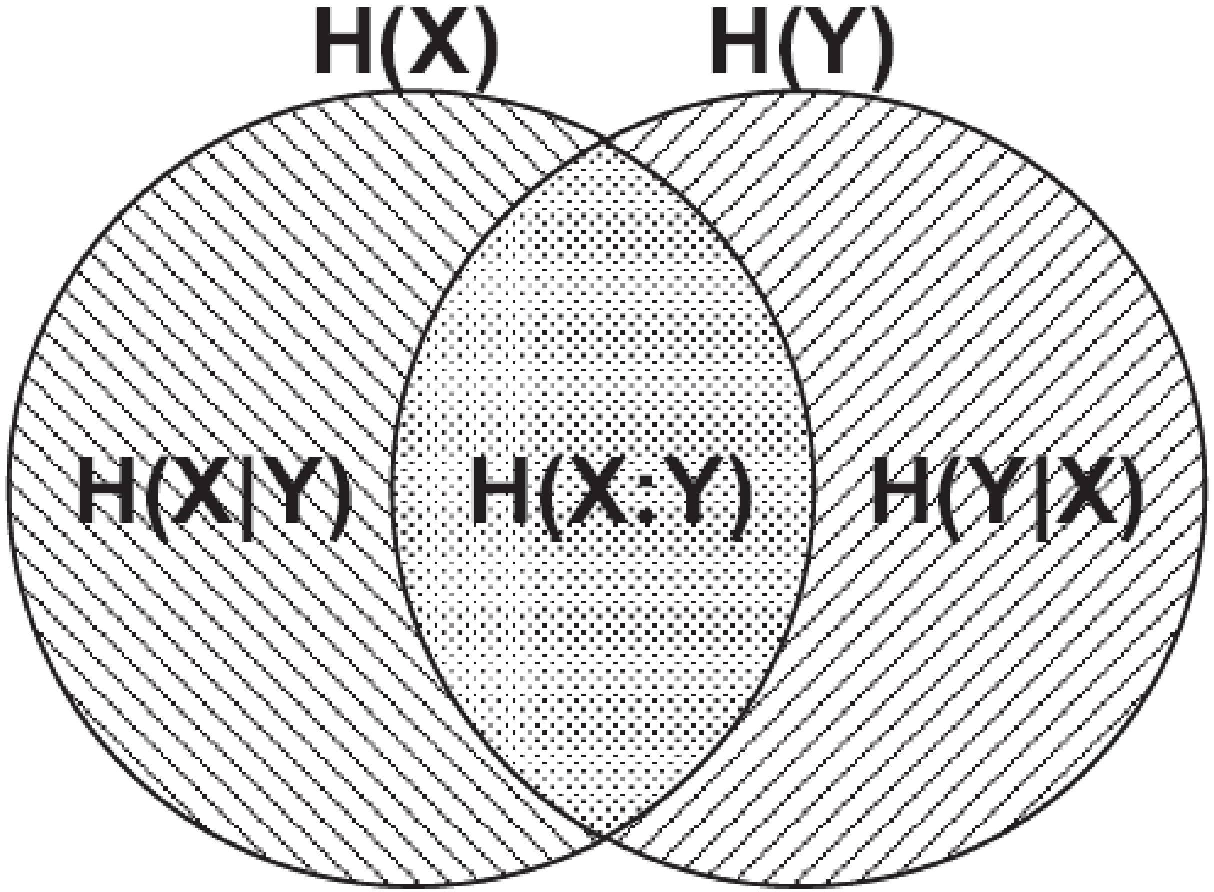
(4) **Subadditivity:** $H(X,Y) \leq H(X) + H(Y)$ with equality if and only if X and Y are independent random variables.

(5) $H(Y|X) \leq H(Y)$ and thus $H(X:Y) \geq 0$, with equality in each if and only if X and Y are independent random variables.

(6) **Strong subadditivity:** $H(X,Y,Z) + H(Y) \leq H(X,Y) + H(Y,Z)$, with equality if and only if $Z \to Y \to X$ forms a Markov chain.

(7) **Conditioning reduces entropy:** $H(X|Y,Z) \leq H(X|Y)$.

Shannon系列熵关系图



Von Neumann entropy

$$S(\rho) = -\mathrm{tr}[\rho \log_2 \rho]$$

$$S(\rho) = -\sum_x \lambda_x \log \lambda_x$$

λ_x *are the eigenvalues of* ρ

Relative entropy

$$S(\rho||\sigma) \equiv \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma)$$

V. Vedral, "The role of relative entropy in quantum information theory", Rev. Mod. Phys. 74, 197 (2002)

Von Neumann entropy基本性质

(1) The entropy is non-negative. The entropy is zero if and only if the state is pure.

(2) In a d-dimensional Hilbert space the entropy is at most $\log d$. The entropy is equal to $\log d$ if and only if the system is in the completely mixed state I/d.

(3) Suppose a composite system AB is in a pure state. Then $S(A) = S(B)$.

(4) Suppose p_i are probabilities, and the states ρ_i have support on orthogonal subspaces. Then

$$S\left(\sum_i p_i \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i).$$

(5) **Joint entropy theorem:** Suppose p_i are probabilities, $|i\rangle$ are orthogonal states for a system A, and ρ_i is any set of density operators for another system, B. Then

$$S\left(\sum_i p_i |i\rangle\langle i| \otimes \rho_i\right) = H(p_i) + \sum_i p_i S(\rho_i).$$

Von Neumann entropy和测量

A projective measurement described by projectors P_i

则有

$$\rho' = \sum_i P_i \rho P_i$$

The system after the measurement is at least as great as the original entropy

$$S(\rho') \geq S(\rho)$$

with equality if and only if $\rho = \rho$.

Subadditivity and concavity

Suppose distinct quantum systems A and B have a joint state ρ_{AB},

则有

$$S(A,B) \leq S(A) + S(B)$$

$$S(A,B) \geq |S(A) - S(B)|$$

concavity

$$S\left(\sum_i p_i \rho_i\right) \geq \sum_i p_i S(\rho_i)$$

Note that equality holds if and only if all the states ρ_i for which $p_i > 0$ are identical; that is, the entropy is a strictly concave function of its inputs.

Von Neumann entropy重要性质

混合量子态的熵性质

$$\sum_i p_i S(\rho_i) \leq S\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i S(\rho_i) + H(p_i)$$

For any trio of quantum systems, A,B,C, the inequalities hold

$$S(A) + S(B) \leq S(A, C) + S(B, C)$$

$$S(A, B, C) + S(B) \leq S(A, B) + S(B, C)$$

Von Neumann entropy重要性质

定义

$$
\begin{aligned}
&(\textit{entropy}) && S(A) = -\mathrm{tr}(\rho^A \log \rho^A)\\
&(\textit{relative entropy}) && S(\rho\|\sigma) = -S(\rho) - \mathrm{tr}(\rho \log \sigma)\\
&(\textit{conditional entropy}) && S(A|B) = S(A,B) - S(B)\\
&(\textit{mutual information}) && S(A:B) = S(A) + S(B) - S(A,B)
\end{aligned}
$$

(1) **Conditioning reduces entropy**: Suppose ABC is a composite quantum system. Then $S(A|B,C) \leq S(A|B)$.

(2) **Discarding quantum systems never increases mutual information**: Suppose ABC is a composite quantum system. Then $S(A:B) \leq S(A:B,C)$.

(3) **Quantum operations never increase mutual information**: Suppose AB is a composite quantum system and $\mathcal{E}$ is a trace-preserving quantum operation on system B. Let $S(A:B)$ denote the mutual information between systems A and B before $\mathcal{E}$ is applied to system B, and $S(A':B')$ the mutual information after $\mathcal{E}$ is applied to system B. Then $S(A':B') \leq S(A:B)$.

Von Neumann entropy重要性质

Subadditivity of the conditional entropy

$$S(A,B|C,D) \leq S(A|C) + S(B|D)$$

$$S(A,B|C) \leq S(A|C) + S(B|C)$$

$$S(A|B,C) \leq S(A|B) + S(A|C)$$

Monotonicity of the relative entropy

$$S(\rho^A \| \sigma^A) \leq S(\rho^{AB} \| \sigma^{AB})$$

where ρ^{AB} and σ^{AB} be any two density matrices of a composite system AB.

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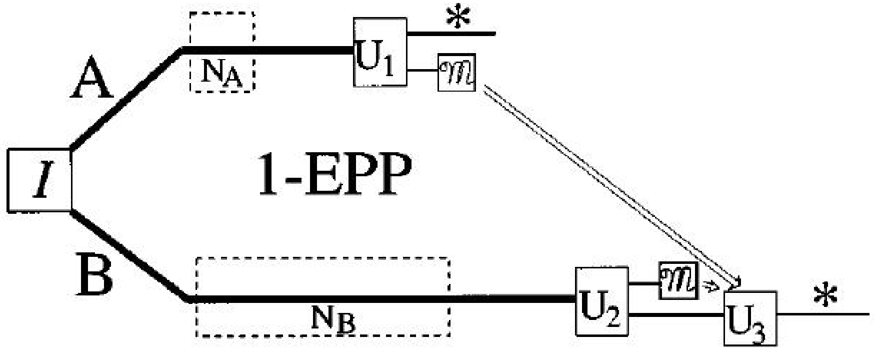
Distillable entanglement

Distillable entanglement: The asymptotic yield of arbitrarily pure singlets that can be prepared locally from mixed state by entanglement purification protocols (EPPs) involving one-way or two-way communication between Alice and Bob.



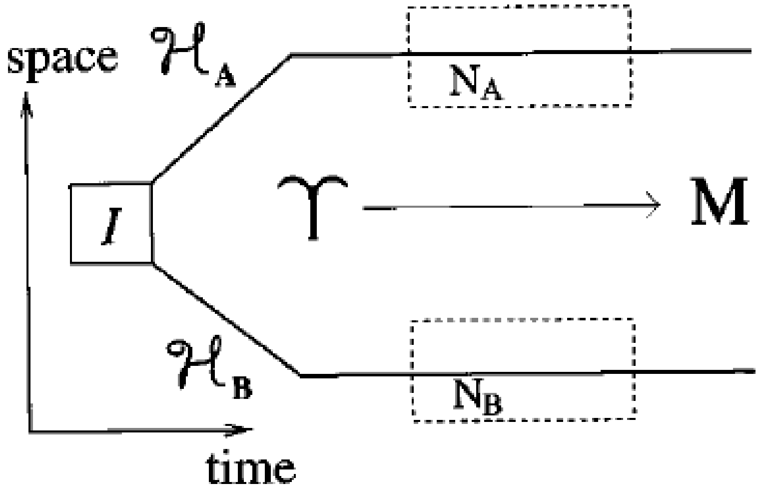


FIG. 3. One-way entanglement purification protocol (1-EPP). In 1-EPP there is only one stage; after unitary transformation U_1 and measurement $\mathcal{M}$, Alice sends her classical result to Bob, who uses it in combination with his measurement result to control a final transformation U_3. The unidirectionality of communication allows the final, maximally entangled state (*) to be separated both in space and in time.

Bennett, C. H., D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, 1996, Phys. Rev. A 54, 3824.

Distillable entanglement

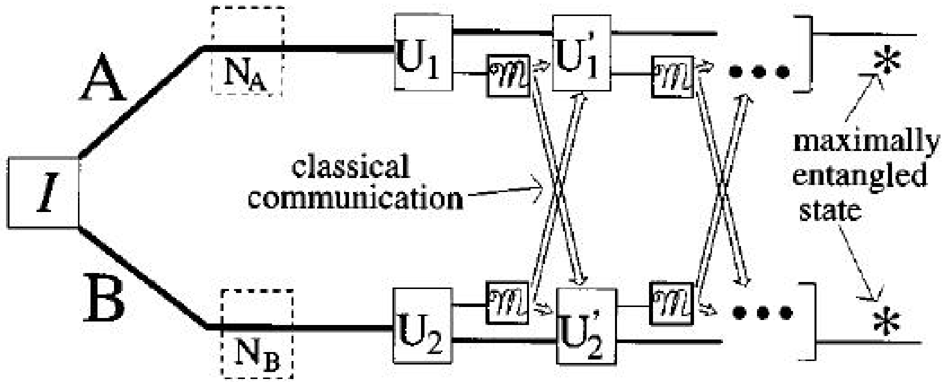
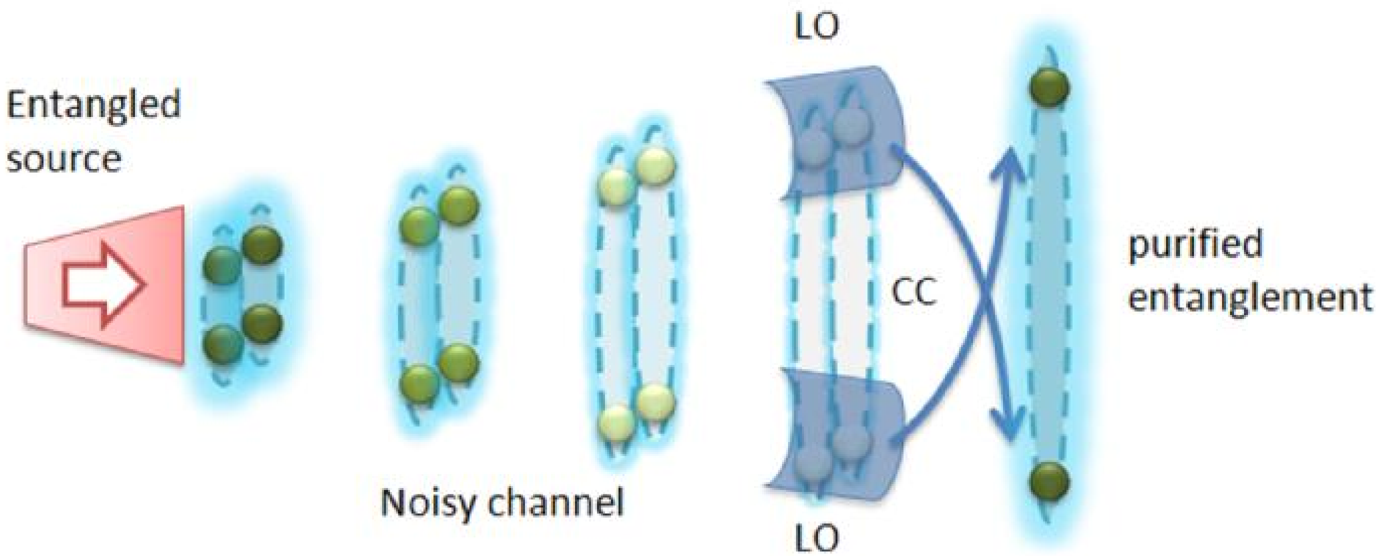


FIG. 2. Entanglement purification protocol involving two-way classical communication (2-EPP). In the basic step of 2-EPP, Alice and Bob subject the bipartite mixed state to two local unitary transformations U_1 and U_2. They then measure some of their particles $\mathcal{M}$, and interchange the results of these measurements (classical data transmission indicated by double lines). After a number of stages, such a protocol can produce a pure, near-maximally-entangled state (indicated by *'s).

Bennett, C. H., D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, 1996, Phys. Rev. A 54, 3824.

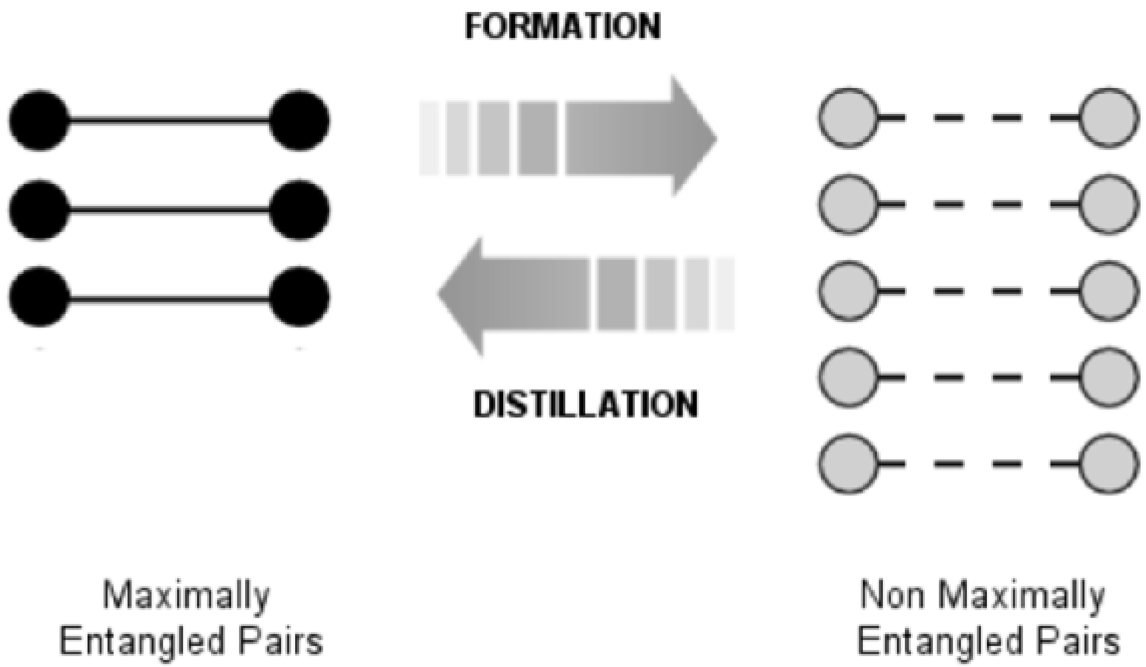
Entanglement distillation

- 应用LOCC操作 (局域操作和经典通信)
  - 利用一对或者多对纠缠资源
  - 牺牲一部分纠缠资源
- 在噪声信道分发和长距离的量子通信后



Entanglement distillation

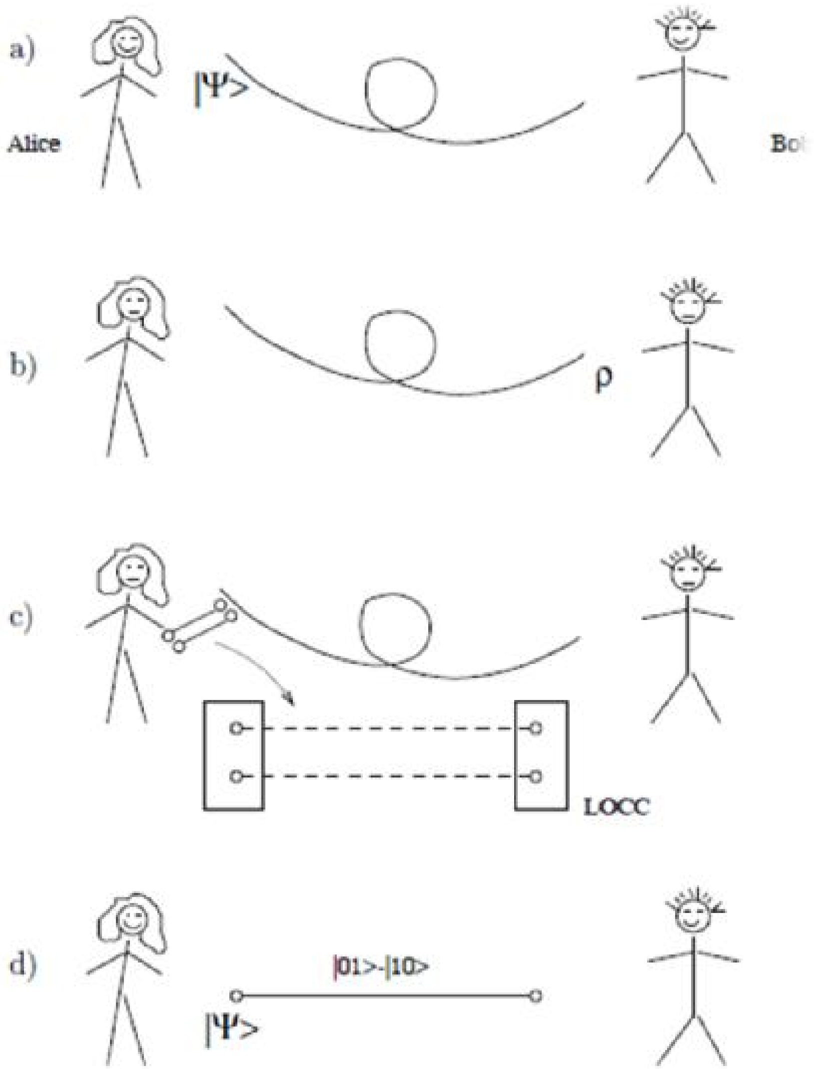
A certain number of maximally entangled EPR pairs is manipulated by local operations and classical communication and converted into pairs in some state. The asymptotic conversion rate is known as the entanglement of formation.



Vlatko Vedral, Introduction to Quantum Information Science, Oxford University Press, 2006

The converse of formation is the distillation of entanglement.
The asymptotic rate of conversion of pairs in the state into maximally entangled states is known as the entanglement of distillation.

Distillation scheme



Dagmar Bruss 2001

FIG. 2. Providing a noiseless channel via distillation: a) Alice wants to send the message $|\psi\rangle$ to Bob. b) Bob receives ϱ instead, as the channel is noisy. c) Alice sends one subsystem of a maximally entangled state through the noisy channel to Bob, and repeats this with a second pair. They employ a distillation protocol. d) Alice and Bob have created a maximally entangled singlet which they can use as a noiseless teleportation channel.

纠缠提纯方法

One-way hashing distillation protocol

Bell diagonal states B_{diag} are naturally parametrized by the probability distribution of mixing *p*.

$$E_D(\rho_{B\text{diag}}) \geqslant 1 - H(\{p\})$$

The n copies of the two-qubit Bell diagonal state B_{diag} can be viewed as a classical mixture of strings of n Bell states. Typically, there are only about $2^{nH(\{p\})}$ such strings that are likely to occur (Cover and Thomas, 1991).

纠缠提纯方法

Two-way recurrence distillation protocol

Two-step procedure:

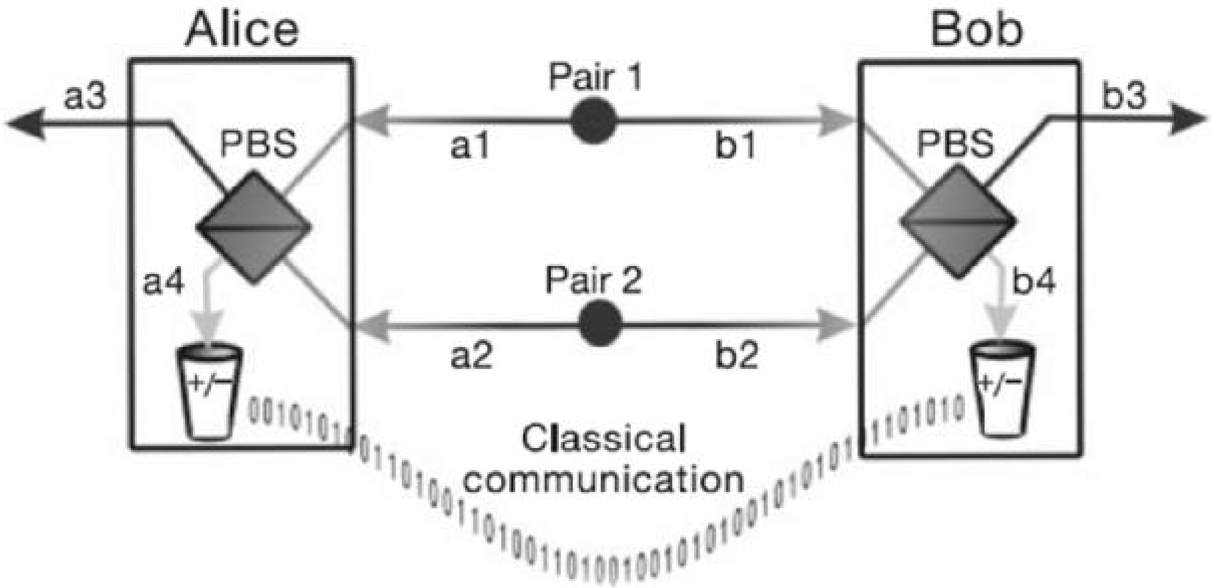
- In the first step Alice and Bob take two pairs, and apply locally a controlled NOT gate. Then they measure the target pair in a bit basis. If the outcomes are different they discard the source pair failure, otherwise they keep it.
- In the latter case, a second step can be applied: they twirl the source pair to the Werner state.

$$F'(F) = \frac{F^2 + \frac{1}{9}(1-F)^2}{F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2}$$

$$F = \mathrm{Tr}\,\rho|\phi^+\rangle\langle\phi^+|$$

If only $F>1/2$, the above recursive map converges to 1 for a sufficiently large initial number of copies.

纯化纠缠实验



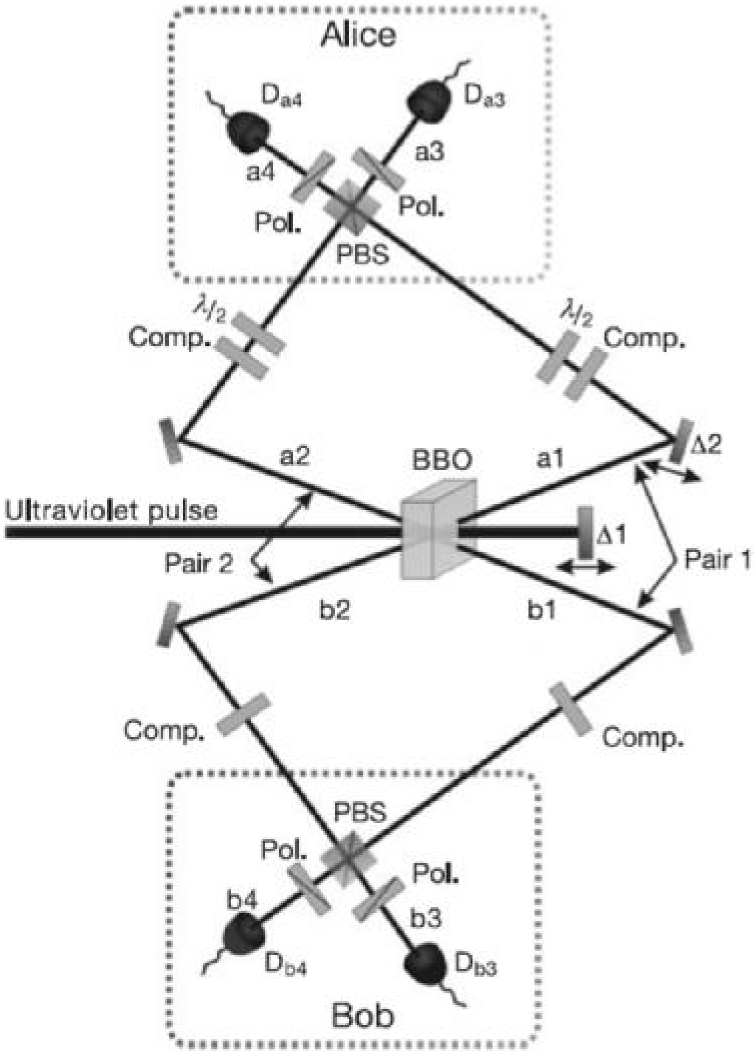
$$\rho_{ab} = F|\Phi^+\rangle_{ab}\langle\Phi^+| + (1-F)|\Psi^-\rangle_{ab}\langle\Psi^-|$$

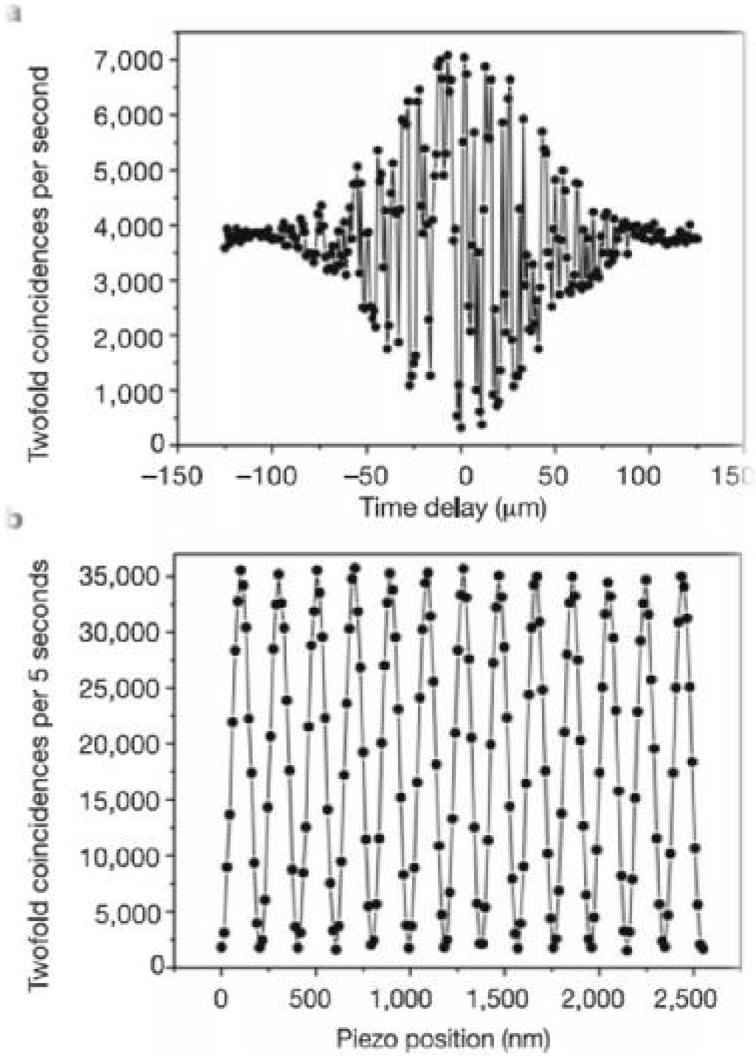
$$|\Phi^\pm\rangle_{ab} = \frac{1}{\sqrt{2}}(|H\rangle_a|H\rangle_b \pm |V\rangle_a|V\rangle_b)$$

$$|\Psi^\pm\rangle_{ab} = \frac{1}{\sqrt{2}}(|H\rangle_a|V\rangle_b \pm |V\rangle_a|H\rangle_b)$$

Jian-Wei Pan *et al.* Nature 410, 1067 (2001)

纯化纠缠实验





Jian-Wei Pan *et al.* Nature 423, 417 (2003)

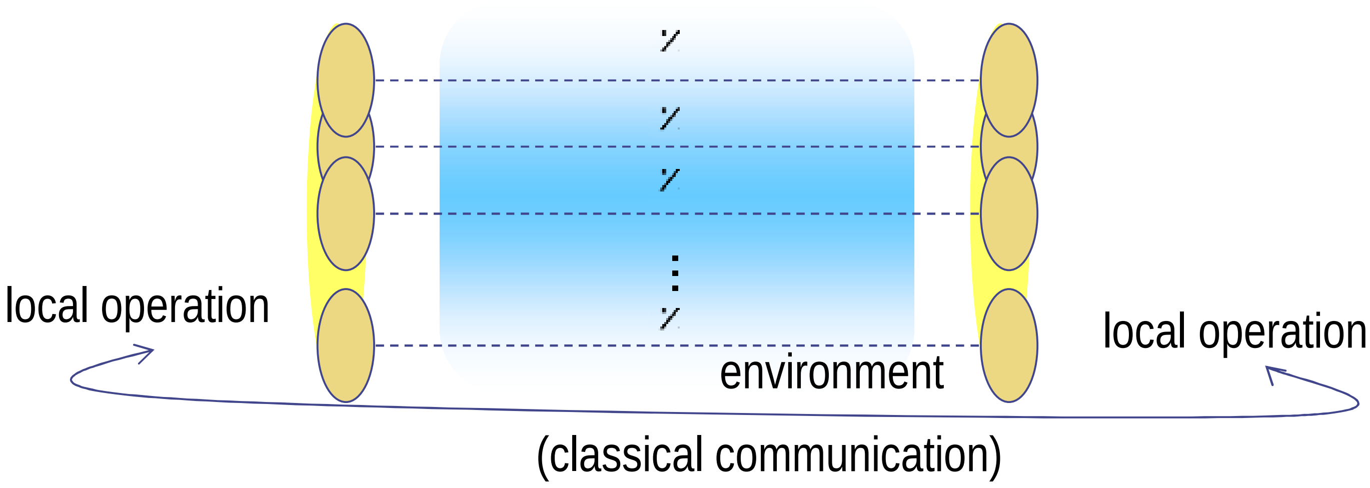
$$\rho'_{ab} = F' |\Phi^+\rangle_{ab}\langle\Phi^+| + (1 - F')|\Psi^-\rangle_{ab}\langle\Psi^-|$$

$$F' = F^2/[F^2 + (1 - F)^2] > F \text{ (for } F > 1/2)$$

One photon pair of fidelity 92% could be obtained from two pairs, each of fidelity 75%.

Entanglement distillation

思想

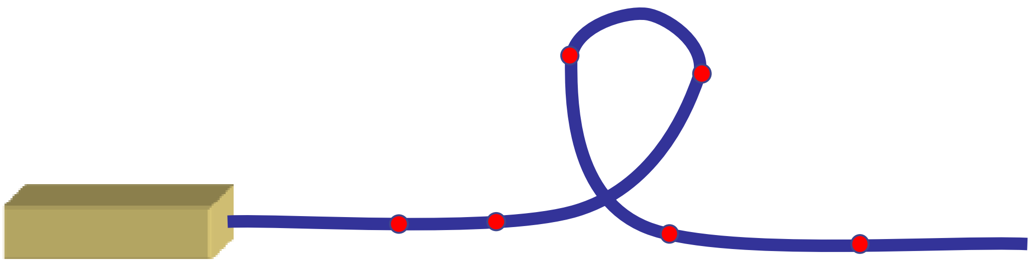


or:

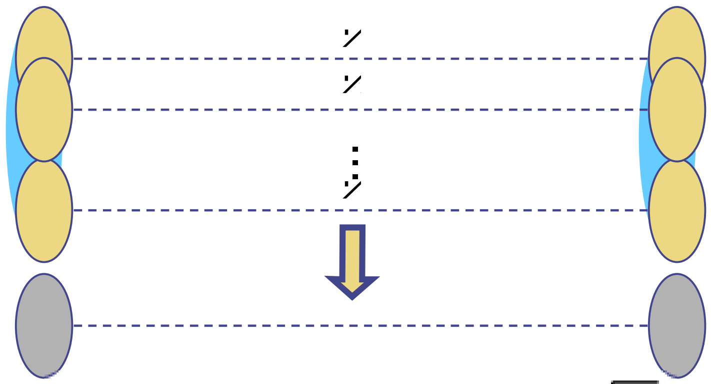
提纯出

$$(|00\rangle + |11\rangle)/\sqrt{2}$$

From Cirac



Distillability



$$(|00\rangle + |11\rangle)/\sqrt{2}$$

Can we distill MES using LOCC?

PPT states cannot be distilled. Thus, there are bound entangled states.

(Horodecki 97)

There seems to be NPT states that cannot be distilled.

(DiVincezo *et al.*, Dur *et al.*, 2000)

From Cirac

Distillability

- Qubits: $H = C^2 - C^2$

All entangled two-qubit states are distillable

$\rho^{T_A} \geq 0$ | $\rho^{T_A} < 0$

distillable

- Higher dimensions: $H = C^3 - C^3$

$\rho^{T_A} \geq 0$ | $\rho^{T_A} < 0$

?

NPT

distillable

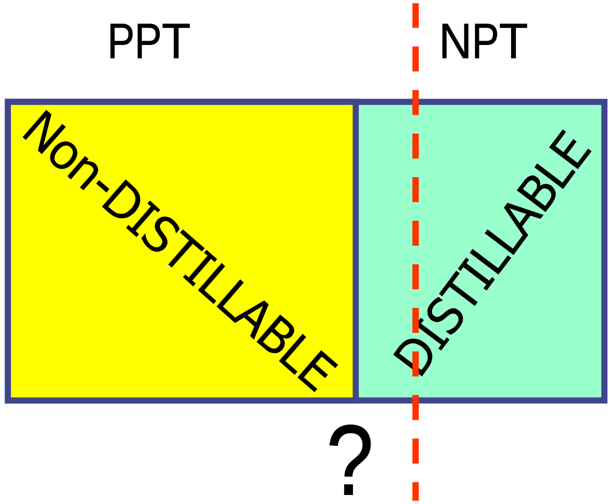
Idea: find A, B such that they project onto $C^2 - C^2$ with $\rho^{T_A} < 0$

there is a strong evidence that they are not distillable: for any finite N, all projections onto $C^2 - C^2$ have $\rho^{T_A} \geq 0$

From Cirac

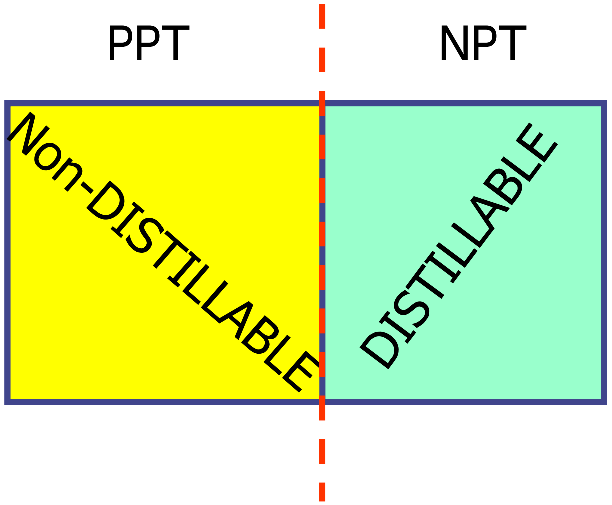
What is known?

In general



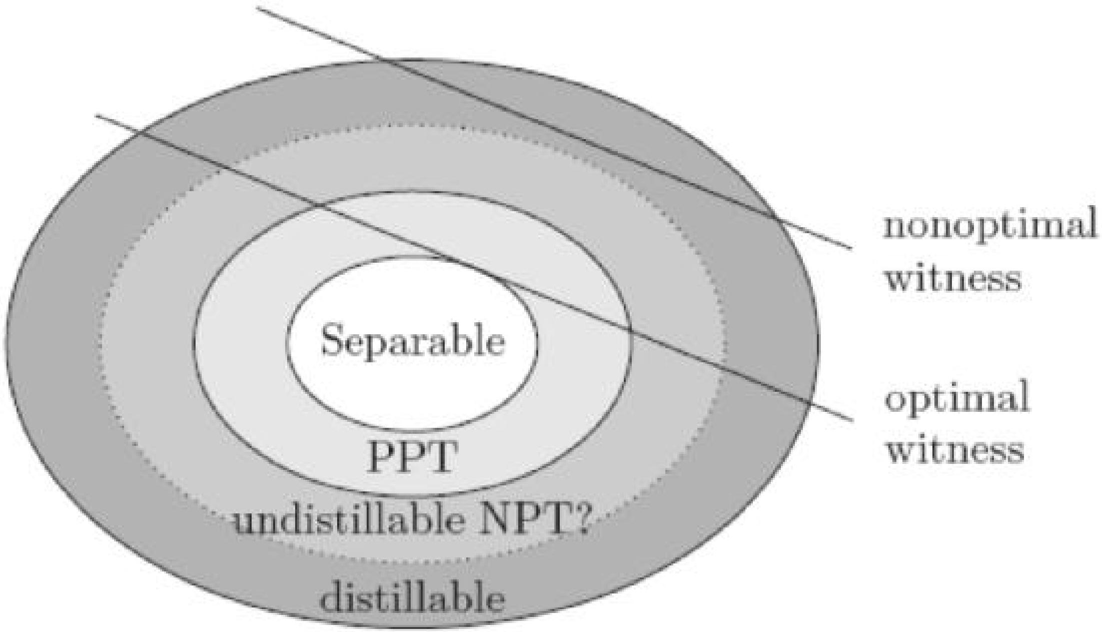
2xN

(Horodecki 97, Dur *et al*. 2000)



From Cirac

纠缠提纯总结



- 所有两比特纠缠态可提纯
- 需要发展更好的可提纯协议
- One-way and two-way
- 多个copies

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谢谢

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