PHYS5251P: Exercise 1, Fall 2025, USTC
'Introduction to Quantum Information'

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1. Express each of the Pauli operators in the outer product notation with respect to the $\{|0\rangle, |1\rangle\}$ basis. Write down the commutation relations and anti-commutation relations for the Pauli operators.

2. Show that if $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{i\phi}|1\rangle$ then $|\psi\rangle\langle\psi| = \frac{1}{2}(I + s_x\sigma_x + s_y\sigma_y + s_z\sigma_z)$. Show that $s = (s_x, s_y, s_z)$ (the Bloch vector) has unit length, and so $|\psi\rangle\langle\psi|$ can be represented by a point on the unit sphere (Bloch sphere).

3. Let $\vec{v}$ be any real, three-dimensional unit vector and θ be a real number. Prove that

$$\exp(\mathrm{i}\theta\vec{v}\cdot\vec{\sigma}) = \cos(\theta)I + \mathrm{i}\sin(\theta)\vec{v}\cdot\vec{\sigma},$$

where $\vec{v}\cdot\vec{\sigma} = \sum_{i=1}^{3} v_i\sigma_i$ and σ_i are Pauli matrices.

4. Prove that for any 2-dimension linear operator A,

$$A = \frac{1}{2}Tr(A)I + \frac{1}{2}\sum_{k=1}^{3} Tr(A\sigma_k)\sigma_k,$$

in which σ_k are Pauli matrices.

5. Prove that an operator ρ is the density operator associated to some ensemble $\{p_i, |\psi_i\rangle\}$ if and only if it satisfies the conditions:

(1). (Trace condition) ρ has trace equal to one,

(2). (Positive condition) ρ is a positive operator.

6. Consider an experiment in which we prepare the state $|0\rangle$ with the probability $|C_0|^2$, and the state $|1\rangle$ with the probability $|C_1|^2$. How to describe this type of quantum state? Compare the differences and similarities between it with the state $C_0|0\rangle + C_1 e^{i\theta}|1\rangle$.

7. In the context of optical experiments using polarized quantum states, consider preparing the state $C_0|0\rangle + C_1 e^{i\theta}|1\rangle$ from the initial $|0\rangle$ state using a combination of half-wave plates and quarter-wave plates. To achieve arbitrary single-qubit unitary transformations, how can we determine the minimum number of such wave plates required and what is the corresponding operational procedure?

8. Suppose I make a beam of vertically polarized light, and pass it through an ideal polarizer with a vertical axis. The light beam will be completely transmitted. Now suppose I put a second polarizer after the first one, at an angle θ, the transmitted fraction will drop to $\cos^2\theta$. Now suppose I use two ideal polarizers after the first one, at angles of $45°$ and $90°$: what will be the transmitted fraction in this case? Now suppose I use a sequence of n polarizers, equally spaced up to $90°$ (so that for the case $n = 3$ the first polarizer is at $0°$ and the next three are at $30°$, $60°$ and $90°$ respectively). What is the transmission for general values of n? What is the value in the limit $n \to \infty$?

9. Let ρ be a density operator.

(1). Show that ρ can be written as

$$\rho = \frac{I + \boldsymbol{r} \cdot \boldsymbol{\sigma}}{2}$$

where $\boldsymbol{r}$ is a real three-dimensional vector and $||\boldsymbol{r}|| \leq 1$.

(2). Show that $Tr(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

(3). Show that a state ρ is a pure state if and only if $||\boldsymbol{r}|| = 1$.

10. Suppose a 2-qubit pure state is of the form $|\Phi\rangle = \sum_{ij} a_{ij} |i\rangle |j\rangle$. By defining $A_{ij} = a_{ij}$ where A_{ij} are elements of a matrix A, calculate the reduced density matrices ρ_A and ρ_B.

11. Suppose a 2-qubit pure state is of the form $|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}} |0\rangle (\frac{1}{2} |0\rangle + \frac{\sqrt{3}}{2} |1\rangle) + \frac{1}{\sqrt{2}} |1\rangle (\frac{\sqrt{3}}{2} |0\rangle + \frac{1}{2} |1\rangle)$.

    (1). Calculate the reduced density matrices ρ_A and ρ_B.

    (2). Perform Schmidt decomposition of $|\Phi\rangle_{AB}$.

12. Suppose $|\psi\rangle$ and $|\phi\rangle$ are two pure states of a composite quantum system AB, with identical Schmidt coefficients. Show that there are unitary transformations U on system A and V on system B such that $|\psi\rangle = (U \otimes V) |\phi\rangle$.

13. Suppose ABC is a three component quantum system. Show by example that there are pure quantum states ψ of such systems which can not be written in the form

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle$$

    where λ_i are real numbers, and $|i_A\rangle, |i_B\rangle, |i_C\rangle$ are orthonormal bases of the respective systems.

14. Consider a single qubit and unit vectors $\vec{n_k}, k \in 1, 2, \ldots, N$ such that $\sum_k \lambda_k \vec{n_k} = 0$ for $\lambda_k \in (0, 1)$ and $\sum_k \lambda_k = 1$. Show that a measurement on a qubit defined by

$$F_k = 2\lambda_k |\uparrow_{\vec{n_k}}\rangle \langle\uparrow_{\vec{n_k}}|$$

    is a POVM.

15. A symmetrical informationally-complete POVM (also known as a SIC POVM) on $\mathbb{C}^d$ is a set of d^2 rank-1 projections

$$\{|\psi_1\rangle\langle\psi_1|, \dots, |\psi_{d^2}\rangle\langle\psi_{d^2}|\} \subset Proj(\mathbb{C}^d)$$

such that

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{d\delta_{ij} + 1}{d + 1}$$

for any $i, j \in \{1, \dots, d^2\}$. Construct a SIC POVM on $\mathbb{C}^2$. (Consider the vertices of a regular tetrahedron in the Bloch sphere.)

16. Suppose $\{|\psi_i\rangle\}, \{|\tilde{\psi}_i\rangle\}$ are two sets of normalized states in space H and they satisfy the conditions that $\langle\psi_i|\psi_j\rangle = \langle\tilde{\psi}_i|\tilde{\psi}_j\rangle$ for $\forall i, j$, prove that there exist a transformation U, such that $U|\psi_i\rangle = |\tilde{\psi}_i\rangle$, and construct the transformation U.