PHYS5251P: Exercise 2, Fall 2025, USTC
‘Introduction to Quantum Information’

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1. The four Bell states have the following mathematical expressions on the basis $\{|0\rangle, |1\rangle\}$ (the eigenstates of σ_z),

$$
\begin{aligned}
|\Phi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \\
|\Psi^{\pm}\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).
\end{aligned}
$$

   (1) Prove that the four Bell states can be transformed to each other using single qubit rotations $\{I, \sigma_x, \sigma_y, \sigma_z\}$.

   (2) Show that each of the four Bell states is an eigenstate of the observables $\{\sigma_{1x}\sigma_{2x}, \sigma_{1y}\sigma_{2y}, \sigma_{1z}\sigma_{2z}\}$ and write down the corresponding eigenvalues.

2. For the singlet state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, prove that Alice and Bob's outcomes are always anti-correlated when they measure two particles respectively along the same direction.

3. Let $\sigma_\theta \equiv \cos\theta\sigma_z + \sin\theta\sigma_x$. Define $|+_\theta\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$ and $|-_\theta\rangle = -\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle$.

   (1) Verify that $|+_\theta\rangle$ is the eigenket of σ_θ with eigenvalue $+1$, and $|-_\theta\rangle$ is the eigenket of σ_θ with eigenvalue -1.

(2) $R_y(\theta) = \exp\left(\frac{-i\theta\sigma_y}{2}\right)$ represents the counterclockwise rotation of angle θ around the y-axis. Verify that $\sigma_\theta = R_y(\theta)\sigma_z R_y(-\theta)$.

(3) Using the definitions of $|+_\theta\rangle$ and $|-_\theta\rangle$, show that for any θ,

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|+_\theta +_\theta\rangle + |-_\theta -_\theta\rangle).$$

4. Suppose $|\psi\rangle$ is a pure state of a composite system AB. Prove that there exist orthonormal states $|i_A\rangle$ for system A, and orthonormal states $|i_B\rangle$ for system B such that

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle,$$

where λ_i are non-negative real numbers satisfying $\sum_i \lambda_i^2 = 1$ known as ***Schmidt coefficients*** .

5. Prove that a state $|\psi\rangle$ of a composite system AB is a product state if and only if it has Schmidt number 1. Prove that $|\psi\rangle$ is a product state if and only if ρ^A (and thus ρ^B) are pure states.

6. Prove that for any state ρ_{AB} we have

$$Tr[\rho_{AB}(O_A \otimes I_B)] = Tr[Tr_B[\rho_{AB}]O_A]$$

for all observables O_A.That is, the partial trace is the reduced state on subsystem A.

7. (1) Can every projective measurement (also called projector valued measurement, PVM) be phrased as a POVM? Either prove that this is always the case or show a counterexample.

(2) Can every POVM be phrased as a PVM on the same Hilbert space? Argue the answer, and give an illustrative example.

8. (1) What conditions should a good entanglement measures meet?

(2) Describe the definition of distillable entanglement and entanglement cost and their relationship.

(3) Write down the monogamy of entanglement and describe its physical meanings.

9. PPT (Positive Partial Transposition) criterion is a strong separability criterion for quantum states, which is very convenient and practical for entanglement detection.

   (1) Describe the PPT criterion and the realignment criterion.

   (2) For the 2-qubit state $\rho = p\,|\psi^-\rangle\,\langle\psi^-| + (1-p)\frac{\mathbb{I}}{4}$, where, $0 \le p \le 1$, $|\psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$, calculate the lower bound of p for ρ to be an entangled state using PPT criterion and realignment criterion respectively.

10. (1) For the 3-qubit W state $|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, if one particle is lost, what's the reduced density matrix of the remaining two particles?

    (2) For the n-qubit W state $|W_n\rangle = \frac{1}{\sqrt{n}}(|10\cdots 0\rangle + |01\cdots 0\rangle + \cdots + |00\cdots 1\rangle)$, if $(n-2)$ particles are lost, what's the reduced density matrix of the remaining two particles? Use the PPT criterion to find out whether the remaining two particles are entangled or not.

11. An entanglement witness(EW) is a functional which distinguishes a specific entangled state from separable ones.

    (1) Describe the definition of the Entanglement Witness (EW).

    (2) For the mixed state $\rho = p\frac{\mathbf{I}}{8} + (1-p)|GHZ\rangle\langle GHZ|$ $(0 \le p \le 1)$, calculate p's upper bound when ρ is an entangled state using the entanglement witness $\mathcal{W} = \frac{1}{2}\mathbf{I} - |GHZ\rangle\langle GHZ|$.

12. Consider the density matrix $\rho_w = r|\phi^+\rangle\langle\phi^+| + \frac{1-r}{4}I_4$, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is Bell state and $0 \le r \le 1$. Calculate the concurrence of ρ_w.

13. For the 2-qubit state $\rho = p|\Psi^-\rangle\langle\Psi^-| + (1-p)\frac{\mathbb{I}}{4}$, where $0 \le p \le 1$, $|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$, calculate the EOF (Entanglement of Formation) of ρ.

14. (1) Prove that $0 \le S(\rho) \le \log D$, where D is the number of non-zero eigenvalues of ρ.

(2) Prove that for any two positive definite matrices A and B we have $\log(A \otimes B) = \log(A) \otimes I + I \otimes \log(B)$.

15. (1) Prove the subadditivity of the von Neumann entropy

$$|S(A) - S(B)| \leq S(A, B) \leq S(A) + S(B).$$

(2) Prove the concavity of the von Neumann entropy

$$S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i).$$

(3) Suppose ABC is a composite quantum system. Prove that

$$S(A|B, C) \leq S(A|B).$$

16. (1) Calculate the von Neumann entropy of the following density matrix,

(a) $\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, (b) $\rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, (c) $\rho_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, (d) $\rho_4 = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

(2) Consider the states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B),$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} |0\rangle (\frac{1}{2} |0\rangle + \frac{\sqrt{3}}{2} |1\rangle) + \frac{1}{\sqrt{2}} |1\rangle (\frac{\sqrt{3}}{2} |0\rangle + \frac{1}{2} |1\rangle).$$

Calculate the Von Neumann entropy of ρ_{1_A} and ρ_{2_A}.

17. Suppose $\{P_i\}$ is a complete set of orthogonal projectors and ρ is a density operator. Prove that the entropy of the state $\rho' \equiv \sum_i P_i \rho P_i$ of the system after the measurement is at least as great as the orignal entropy, $S(\rho') \geq S(\rho)$, with equality if and only if $\rho = \rho'$.

18. Consider a composed system $A \otimes B \otimes C$ with a shared state ρ_{ABC}, we can define a conditional version of the mutual information between A and B as

$$I(A : B|C) = S(A|C) + S(B|C) - S(AB|C) = S(A|C) - S(A|BC).$$

Consider the so-called cat state shared by four qubits, that is defined as $|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$. Calculate the mutual information between qubits A and B changes with the knowledge of the remaining qubits, namely: $I(A:B), I(A:B|C), I(A:B|CD)$.