

PHYS5251P: Exercise 3, Fall 2025, USTC

‘Introduction to Quantum Information’

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1. Please describe the EPR paradox introduced by Einstein, Podolsky, Rosen at 1935, and explain the contradiction between quantum theory and local realism theory.
2. Derive the Bell’s theorem without inequalities from the GHZ state

$$|\psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle).$$

3. (1) Calculate the amount of entanglement of the state $\rho = \lambda |\phi^+\rangle \langle \phi^+| + (1 - \lambda) |\psi^+\rangle \langle \psi^+|$, ($0 \leq \lambda \leq 1$) with negativity measure, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.
(2) Derive the value scope for λ when the state ρ is entangled using negativity measure.
4. Consider a 2-qubit quantum state $\rho_{AB} = \frac{1}{8}I + \frac{1}{2} |\psi^-\rangle \langle \psi^-|$, where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$.
(1) Give the spectral decomposition of ρ_{AB} .
(2) Suppose one measures $\vec{n} \cdot \vec{\sigma}_A$ and measures $\vec{m} \cdot \vec{\sigma}_B$ with $\vec{n} \cdot \vec{m} = \cos \theta$, calculate the probability that both outcomes are +1.
(3) Use the realignment criterion to find out whether ρ_{AB} is entangled or not.

5. Suppose Alice and Bob share the two-qubit state $|\psi_{AB}\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$. Recall that a quantum measurement is specified by a set of operators $\{M_0, M_1\}$ such that $\sum_k M_k^\dagger M_k = I$.

(1) Give a and b such that the quantum measurement outcome from operator $M_0 = a|0\rangle\langle 0| + b|1\rangle\langle 1|$ acting on $|\psi_{AB}\rangle$ produces the post-measurement result $(|00\rangle + |11\rangle)/\sqrt{2}$ with probability $1/4$.

(2) Give an operator M_1 such that $M_0^\dagger M_0 + M_1^\dagger M_1 = I$. With what probability does the corresponding outcome occur, acting on $|\psi_{AB}\rangle$, and what is the post-measurement result?

6. In the basis that diagonalizes $\sigma_z^A \otimes \sigma_z^B$, write out the 4×4 unitary matrix U that maps an arbitrary tensor product state $|\psi\rangle_A \otimes |\phi\rangle_B$ to $|\phi\rangle_A \otimes |\psi\rangle_B$ and verify that $U^2 = I$.

7. Let $\vec{n}$ be a normalized real vector in three dimensions and let θ be real. Prove that the equality

$$f(\theta \vec{n} \cdot \vec{\sigma}) = \frac{f(\theta) + f(-\theta)}{2} I + \frac{f(\theta) - f(-\theta)}{2} \vec{n} \cdot \vec{\sigma}$$

holds for any function $f(\cdot)$.

8. Consider a particle with initial state $|0\rangle$. We perform N sequential measurements $\sigma_k \equiv \vec{n}_k \cdot \vec{\sigma}$ with $\vec{n}_k = \left(\sin\left(\frac{k\pi}{2N}\right), 0, \cos\left(\frac{k\pi}{2N}\right)\right)$ and $k = 1, 2, \dots, N$. What's the probability that all outcomes are $+1$? What if $N \rightarrow \infty$?

9. Let ρ be a single qubit density matrix, and recall the Bloch sphere representation

$$\rho = \frac{I + \mathbf{r} \cdot \boldsymbol{\sigma}}{2},$$

where $\mathbf{r}$ is a real three-dimensional vector and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Express the von Neumann entropy of ρ in terms of $\mathbf{r}$ and the binary entropy function $H(p) = -p \log p - (1-p) \log(1-p)$.

10. (1) Prove the CHSH inequality

$$|E(A_1B_1) + E(A_1B_2) + E(A_2B_1) - E(A_2B_2)| \leq 2,$$

in which $E(A_iB_j)$ is the expectation value of the correlation experiment A_i, B_j .

(2) For the singlet state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$

prove that the correlation function $E(A_iB_j)_{\text{quantum}} = \langle \psi^- | A_i \otimes B_j | \psi^- \rangle \equiv \langle \psi^- | (\vec{a}_i \cdot \vec{\sigma}) \otimes (\vec{b}_j \cdot \vec{\sigma}) | \psi^- \rangle$ is

$$E(A_iB_j)_{\text{quantum}} = -\vec{a}_i \cdot \vec{b}_j.$$

(3) What's the maximal violation of the CHSH inequality allowed by quantum mechanics? Give the corresponding quantum state and specify the measurement operators.

11. (Tsirelson's inequality) Suppose $Q = \vec{q} \cdot \vec{\sigma}$, $R = \vec{r} \cdot \vec{\sigma}$, $S = \vec{s} \cdot \vec{\sigma}$, $T = \vec{t} \cdot \vec{\sigma}$, where $\vec{q}$, $\vec{r}$, $\vec{s}$ and $\vec{t}$ are real unit vectors in three dimensions and $\vec{\sigma} = (\sigma_x \sigma_y \sigma_z)$. Show that

$$(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 = 4I + [Q, R] \otimes [S, T].$$

Use this result to prove that

$$\langle Q \otimes S \rangle + \langle R \otimes S \rangle + \langle R \otimes T \rangle - \langle Q \otimes T \rangle \leq 2\sqrt{2}.$$

12. Consider the CHSH game with the following choices for Alice's and Bob's observables:

$$P : \sigma_z^A$$

$$Q : \cos(\frac{\pi}{4})\sigma_z^A + \sin(\frac{\pi}{4})\sigma_x^A$$

$$R : \sigma_z^B$$

$$S : \cos(\frac{\pi}{4})\sigma_z^B - \sin(\frac{\pi}{4})\sigma_x^B,$$

where $\sigma_z^A = \sigma_z^A \otimes I^B$ is an observable on Alice's qubit only, and so on. Let the two-qubit state shared by Alice and Bob be an imperfect entangled state:

$$\rho = p \frac{I}{4} + (1-p) |\psi^-\rangle \langle \psi^-|,$$

where $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Calculate the CHSH quantity:

$$|CHSH| = |E(P, R) + E(Q, R) + E(P, S) - E(Q, S)|$$

for this state, as a function of p . For what values of p is the CHSH inequality violated?

13. Given the state of the form $|\Psi\rangle = \alpha|00\rangle + \beta|11\rangle$ (with real α and β), the Bell parameter can be expressed by

$$\mathcal{B} = E_{a_1 b_1} + E_{a_2 b_1} + E_{a_1 b_2} - E_{a_2 b_2}$$

where $E_{a_i b_j} = \langle \Psi | (\vec{a}_i \cdot \vec{\sigma}) \otimes (\vec{b}_j \cdot \vec{\sigma}) | \Psi \rangle$ with unit vectors $\vec{a}_1, \vec{a}_2, \vec{b}_1, \vec{b}_2$.

- (1) Denote the length of $\vec{a}_1 + \vec{a}_2$ as $2 \cos \theta$ with $0 \leq \theta \leq \pi/2$. Define two new unit vectors $\vec{c}_1, \vec{c}_2$ by $\vec{a}_1 + \vec{a}_2 = 2\vec{c}_1 \cos \theta, \vec{a}_1 - \vec{a}_2 = 2\vec{c}_2 \sin \theta$. Derive that

$$\max_{\theta} \mathcal{B} = 2[\langle \Psi | (\vec{c}_1 \cdot \vec{\sigma}) \otimes (\vec{b}_1 \cdot \vec{\sigma}) | \Psi \rangle^2 + \langle \Psi | (\vec{c}_2 \cdot \vec{\sigma}) \otimes (\vec{b}_2 \cdot \vec{\sigma}) | \Psi \rangle^2]^{1/2}.$$

- (2) We take coplanar vectors, say all in the x - z plane: $\vec{b}_1 = (\sin \phi, 0, \cos \phi), \vec{b}_2 = (\sin \phi', 0, \cos \phi'), \vec{c}_1 = (\sin \gamma, 0, \cos \gamma), \vec{c}_2 = (-\cos \gamma, 0, \sin \gamma)$. Maximize over ϕ, ϕ', γ to derive the value of $\mathcal{B}_{max}$.

14. Consider the CHSH game in which the referee chooses questions $r, s \in \{0, 1\}$ uniformly, and Alice and Bob must each answer a single bit: a for Alice, b for Bob, in which $a, b \in \{0, 1\}$. They win if $a \oplus b = r \wedge s$ and lose otherwise.

- (1) Give the maximum probability of winning with the classical strategy.
- (2) Suppose Alice and Bob share a maximum quantum entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, please derive the maximum probability of winning and give the corresponding quantum strategy.

15. Consider the GHZ game in which the referee chooses questions $rst \in \{000, 011, 101, 110\}$ uniformly, and Alice, Bob and Charles must each answer a single bit: a for Alice, b for Bob, c for Charles, in which $a, b, c \in \{0, 1\}$. They win if $a \oplus b \oplus c = r \vee s \vee t$ and lose otherwise. Suppose Alice, Bob and Charles share a GHZ state $|\psi\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle)$, give a quantum strategy that maximize probability of winning.
16. Two players, Alice and Bob, are required to independently fill a 3×3 magic square. As shown in Fig. 1, the referee randomly sends two queries $x, y \in \{0, 1, 2\}$ to Alice and Bob, respectively. Here, x labels rows and y labels columns. Alice and Bob are required to reply with three numbers with specific conditions. Denote Alice's answers in a row as $[a_0^x, a_1^x, a_2^x]$ and Bob's answers in a column as $[b_0^y, b_1^y, b_2^y]$, where $a_i^x, b_j^y \in \{-1, +1\}$ for any $i, j \in \{0, 1, 2\}$. Alice's answers must satisfy $\prod_i a_i^x = +1$, while Bob's should satisfy $\prod_j b_j^y = -1$ for any x and y . During the game, Alice and Bob are forbidden to communicate with each other. They win the game if the overlapped entry of Alice's row and Bob's column is always the same, i.e., $a_y^x = b_x^y$ for each x and y .
- (1) Give the maximum probability of winning with the classical strategy.
 - (2) Suppose Alice and Bob share a maximum quantum entangled state $|\phi\rangle_{A_1 A_2 B_1 B_2} = |\psi\rangle_{A_1 B_1} \otimes |\psi\rangle_{A_2 B_2}$ with $|\psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, and Alice has systems $A_1 A_2$ and Bob has $B_1 B_2$. Please derive the maximum probability of winning and give the corresponding quantum strategy.

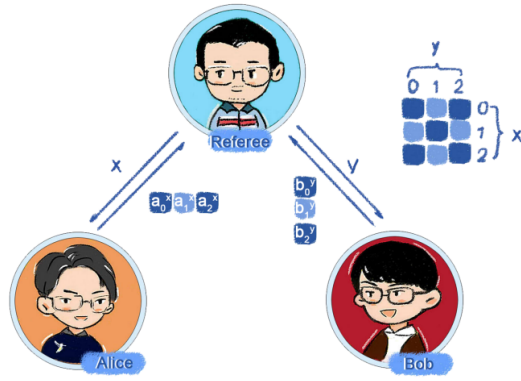


FIG. 1. The Mermin-Peres magic square game.