PHYS5251P: Exercise 6, Fall 2025, USTC
'Introduction to Quantum Information'

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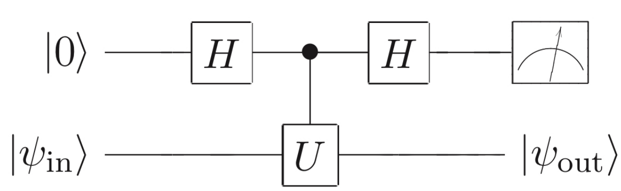
1. Let ρ be an arbitrary 2-qubit density operator. Suppose we perform a projective measurement of the second qubit in the computational basis. Let ρ' be the density matrix which would be assigned to the system after the measurement by an observer who did not learn the measurement result. Prove that the reduced density matrix for the first qubit is not affected by the measurement, i.e.,

$$\mathrm{tr}_2(\rho') = \mathrm{tr}_2(\rho).$$

2. Suppose we have a single qubit operator U with eigenvalues ± 1, so that U is both Hermitian and unitary, so it can be regarded both as an observable and a quantum gate. Suppose we wish to measure the observable U. That is, we desire to obtain a measurement result indicating one of the two eigenvalues, and leaving a post-measurement state which is the corresponding eigenvector. Show that the following circuit implements a measurement of U.



3. Consider a general single-qubit unitary $U = R_z(\beta)R_y(\gamma)R_z(\delta)$, built up from a com-

bination of single-qubit y and z-rotations of the following form,

$$R_z(\theta) = \exp(-i\theta\sigma_z/2), R_y(\theta) = \exp(-i\theta\sigma_y/2).$$

Our goal is to show that a two-qubit controlled-U gate can be built from a combination of $CNOT$ gates and single-qubit rotations. Here, A, B and C are single-qubit gates, chosen as the following combinations of single-qubit rotations,

$$A = R_z(\beta)R_y(\gamma/2), B = R_y(-\gamma/2)R_z((-\beta-\delta)/2), C = R_z((\delta-\beta)/2).$$

(1) Show that $ABC = I$ and $AXBXC = U$ hold, where X is one of the Pauli matrices. (Hint:You might want to use the identity $XYX = -Y$ to first show that $XR_y(\theta)X = R_y(-\theta)$.)

(2) Based on this, give the circuit that uses only the A, B, C and $CNOT$ gates, which can realize the controlled-U gate.

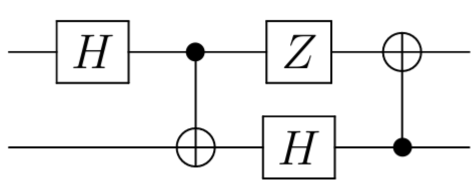
4. The $\pi/8$-gate can be denoted as $T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$.

(1) Calculate $TXT^\dagger$ and $TYT^\dagger$, express the results in terms of Pauli matrices X, Y and Z.

(2) Write down a set of universal quantum logic gates that contains the $\pi/8$-gate.

5. A single-qubit rotation with rotation axis $\vec{n}$ can be written as $R_{\vec{n}}(\theta) = exp(-i\theta\vec{n} \cdot \vec{\sigma}/2) = \cos(\theta/2)I - i\sin(\theta/2)(n_xX + n_yY + n_zZ)$. Calculate $THTH$ using $T = e^{-i\pi/8Z}$, and find suitable θ and $\vec{n} = (n_x, n_y, n_z)$ for it.

6. Consider the following quantum circuit C:



(1) Write down the matrix of the unitary operation U corresponding to C, with respect to the computational basis.

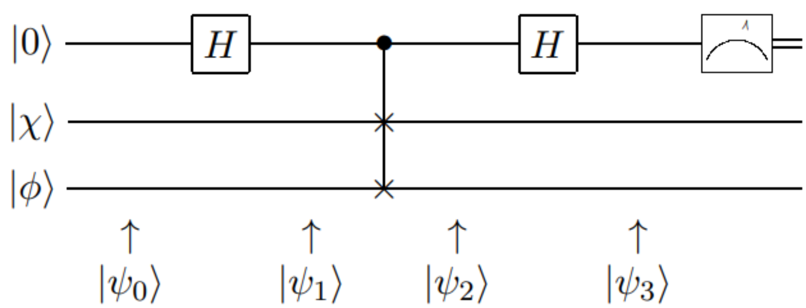
(2) Write down a quantum circuit corresponding to the inverse operation U^{-1}.

(3) If C is applied to the initial state $|0\rangle|0\rangle$ and is followed by a measurement of each qubit in the computational basis, what is the distribution on measurement outcomes?

7. Please construct the quantum SWAP gate to swap two qubits using the C-NOT gate.

8. Please design a quantum circuit which converts the state $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ into four Bell states.

9. Consider the following three-qubit quantum circuit, in which $|\chi\rangle$ and $|\phi\rangle$ are arbitrary qubit states:

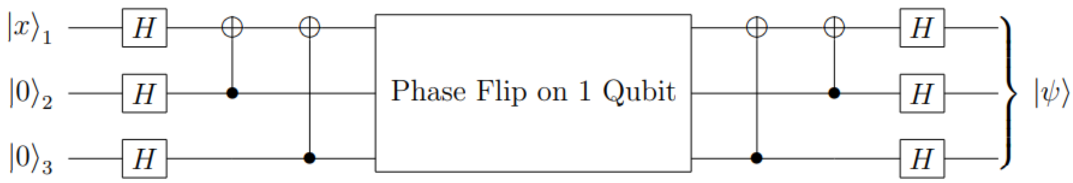


(1) Give the intermediate states of the circuit, $|\psi_0\rangle$, $|\psi_1\rangle$, $|\psi_2\rangle$, $|\psi_3\rangle$.

(2) If the measurement result is zero, what is the state of the bottom two qubits?

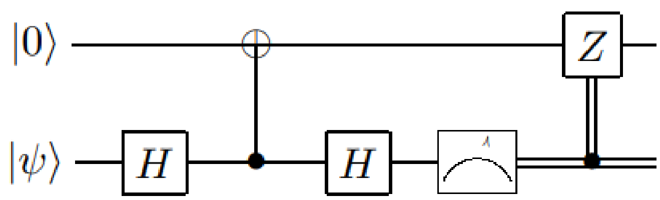
(3) If $\langle\chi|\phi\rangle = \alpha$, with what probability is the measurement result zero?

10. Verify that the following circuit is the appropriate encoder/decoder circuit for the 3 qubit phase flip code. In other words, exhibit a measurement on the two ancillae in the circuit's output $|\psi\rangle$ that will detect whether a phase flip error occurred on one of the three qubits.



11. Please write down the DiVincenzo criterion that quantum computer implementation must satisfy.

12. Let $|\psi\rangle = a|0\rangle + b|1\rangle$ and consider the following circuit. What is the output state of the top qubit?



13. Evaluate the output of the following quantum circuit.