量子信息导论
PHYS5251P

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第一章 量子体系

1. 量子态表示
2. 密度矩阵（纯态、混合态）
3. 量子不可克隆定理
4. **Schmidt分解**
5. 密度矩阵（表示、操作）
6. 量子测量

Schmidt decomposition

任意两体纯态

$$|\Phi\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij}|i\rangle\otimes|j\rangle$$

假设 n ≤m

存在一组正交态$|\mu_1\rangle, |\mu_2\rangle, \ldots, |\mu_n\rangle$ 以及$|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_n\rangle$ 使得

$$|\Phi\rangle = \sum_{c=1}^{n}\sqrt{p_c}\,|\mu_c\rangle\otimes|\varphi_c\rangle$$

$|\varphi_1\rangle, |\varphi_2\rangle, \ldots, |\varphi_n\rangle$为 $Tr_1|\Phi\rangle\langle\Phi|$ 的本征态

参考QCQI §2.5, M.A. Nielsen and I.L. Chuang

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密度矩阵矩阵元的表示

$$I=\sigma_0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad \sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

也即

$$I=|0\rangle\langle 0|+|1\rangle\langle 1| \qquad \sigma_1=|0\rangle\langle 1|+|1\rangle\langle 0|$$

$$\sigma_3=|0\rangle\langle 0|-|1\rangle\langle 1| \qquad \sigma_2=-i|0\rangle\langle 1|+i|1\rangle\langle 0|$$

$$|0\rangle\langle 0|=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}=\frac{1}{2}(I+\sigma_3) \qquad |0\rangle\langle 1|=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=\frac{1}{2}(\sigma_1+i\sigma_2)$$

$$|1\rangle\langle 1|=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=\frac{1}{2}(I-\sigma_3) \qquad |1\rangle\langle 0|=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}=\frac{1}{2}(\sigma_1-i\sigma_2)$$

密度矩阵性质

$$
\begin{aligned}
\varrho^2 &= \varrho * \varrho \\
&= \sum_i p_i |\psi_i\rangle\langle\psi_i| \sum_j p_j |\psi_j\rangle\langle\psi_j| \\
&= \sum_i p_i p_j |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j| \\
&= \sum_i p_i^2 |\psi_i\rangle\langle\psi_i| \, .
\end{aligned}
$$

因此 $Tr\rho^2 = \sum_i p_i^2$ 混合程度，最小值？

$$\rho = \frac{1}{d}\mathbb{1}$$

纯态 $\sum_i p_i^2 = 1$

混合态 $\mathrm{tr}(\varrho^2) < 1$

密度矩阵性质

所有N维Hilbert空间H中的密度矩阵构成一个凸集

$$\rho(\lambda)=\lambda\rho_1+(1-\lambda)\rho_2$$

(i) ρ is positive: $\langle\varphi|\rho|\varphi\rangle \geq 0$, $\forall\ |\varphi\rangle \in \mathcal{H}_d$ (and thus Hermitian, $\rho^\dagger=\rho$)

(ii) $tr[\rho]=1$

一般地

$$\rho=\sum_{l=1}^{k} r_l\rho_l$$

仍然为一个密度矩阵

密度矩阵演化

作用一个幺正变换**U**在初态 $|\psi\rangle$ ，我们有

$$U|\psi\rangle$$

密度矩阵变为

$$U|\psi\rangle\left(U|\psi\rangle\right)^{\dagger}=U|\psi\rangle\langle\psi|U^{\dagger}$$

密度矩阵演化

作用一个幺正变换**U**在一个系综上 $\{q_k, |\psi_k\rangle\}$，我们有 $\{q_k, U|\psi_k\rangle\}$

密度矩阵变为

$$\sum_k q_k U|\psi_k\rangle\langle\psi_k|U^{t}$$

$$= U\left(\sum_k q_k |\psi_k\rangle\langle\psi_k|\right)U^{\dagger}$$

$$= U\rho U^{\dagger}$$

系综分解

任意系综生成和表示

$$\rho = \sum_a p_a|\psi_a\rangle\langle\psi_a| = \sum_a |\tilde{\psi}_a\rangle\langle\tilde{\psi}_a|, \quad |\tilde{\psi}_a\rangle := \sqrt{p_a}|\psi_a\rangle,$$

正交归一基矢下的系综分解

$$\rho = \sum_{n=1}^{d} \lambda_n |n\rangle\langle n| = \sum_{n=1}^{d} |\tilde{n}\rangle\langle\tilde{n}|$$

可以证明，同一密度矩阵的两组系综之间存在一个幺正变换关系

$$|\tilde{\psi}_i\rangle = \sum_j u_{ij}|\tilde{\varphi}_j\rangle$$

Trace操作

$$\mathrm{tr}(\varrho) = \sum_k \langle k|\varrho|k\rangle$$

对于混合态 $\varrho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

$$\begin{aligned}
\mathrm{tr}(\varrho) &= \sum_k \langle k| \sum_i p_i |\psi_i\rangle\langle\psi_i||k\rangle \\
&= \sum_i p_i \ \mathrm{tr}(|\psi_i\rangle\langle\psi_i|) \\
&= \sum_i p_i \\
&= 1\,.
\end{aligned}$$

意味着对于各种量子态的测量输出总的概率为1

Trace操作性质

矩阵对角元求和

$$Tr\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = a_{00} + a_{11} + a_{22}$$

特性

$$Tr[xA + yB] = xTr[A] + yTr[B]$$

$$Tr[AB] = Tr[BA]$$

$$Tr[ABC] = Tr[CAB]$$

$$Tr[UAU^{\dagger}] = Tr[A]$$

$$Tr[A] = \sum_i \langle \varphi_i | A | \varphi_i \rangle$$

正交基展开 $|\varphi_i\rangle$

Trace操作

$$Tr(|\psi\rangle\langle\varphi|) = \langle\varphi|\psi\rangle^*$$

可分离态情形

$$\rho_A \otimes \rho_B = \sum_{ij} (p_i |i_A\rangle\langle i_A|) \otimes (q_i |j_B\rangle\langle j_B|)$$

$$Tr_B(\rho_A \otimes \rho_B) = \rho_A \otimes Tr(\rho_B)$$

Trace操作

纠缠态情形

$$Tr(|\psi\rangle\langle\varphi|) = \langle\varphi|\psi\rangle^*$$

$$|\psi_{AB}\rangle = \sum_i \alpha_i |i_A\rangle|i_B\rangle$$

$$\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}| = \sum_i \alpha_i |i_A\rangle|i_B\rangle \sum_j \alpha_j^* \langle j_A|\langle j_B|$$

我们得到

$$\begin{aligned} \mathrm{tr}_B \varrho_{AB} &= \sum_{i,j} \alpha_i \alpha_j^* |i_A\rangle\langle j_A| \langle j_B||i_B\rangle \\ &= \sum_{i,j} \alpha_i \alpha_j^* |i_A\rangle\langle j_A| \delta_{ij} \\ &= \sum_i |\alpha_i|^2 |i_A\rangle\langle i_A| \,. \end{aligned}$$

Trace操作

一个例子

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0_A0_B\rangle + |1_A1_B\rangle)$$

我们得到

$$Tr_B(|\Phi^+\rangle\langle\Phi^+|) = Tr_B\left(\frac{1}{2}(|0_A0_B\rangle+|1_A1_B\rangle)(\langle 0_A0_B|+\langle 1_A1_B|)\right)$$

$$=\frac{1}{2}(|0_A\rangle\langle 0_A|\otimes Tr(|0_B\rangle\langle 0_B|)+|1_A\rangle\langle 0_A|\otimes Tr(|1_B\rangle\langle 0_B|)+|0_A\rangle\langle 1_A|\otimes Tr(|0_B\rangle\langle 1_B|)+|1_A\rangle\langle 1_A|\otimes Tr(|1_B\rangle\langle 1_B|))$$

$$=\frac{1}{2}(|0_A\rangle\langle 0_A|+|1_A\rangle\langle 1_A|)$$

量子态纯化**Purification**

$$\varrho = \sum_i p_i |i\rangle\langle i|$$

$$|\psi\rangle = \sum_i \sqrt{p_i}|i_A\rangle \otimes |i_B\rangle$$

Trace操作 作用在矩阵上

$$
\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{Tr_2} \begin{bmatrix} Tr\begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} & Tr\begin{bmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{bmatrix} \\ Tr\begin{bmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{bmatrix} & Tr\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \end{bmatrix}
$$

$$
= \begin{bmatrix} a_{00} + a_{11} & a_{02} + a_{13} \\ a_{20} + a_{31} & a_{22} + a_{33} \end{bmatrix}
$$

约化密度矩阵 —练习

$$|\Phi\rangle_{AB} = \sum_{ij} a_{ij} |ij\rangle_{AB}$$

$$A_{ij} = a_{ij}$$

$$\rho_A = ?$$

$$\rho_B = ?$$

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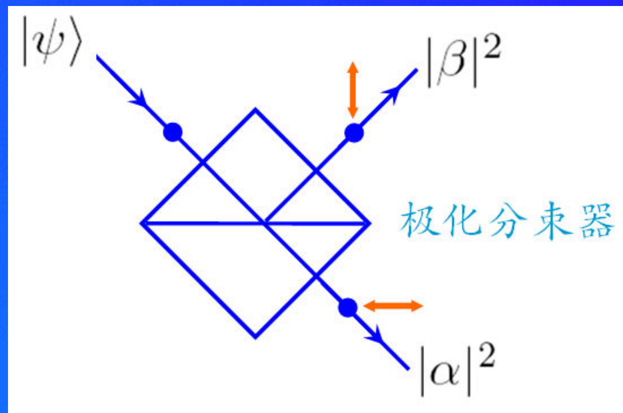
量子测量

$$|\phi\rangle = \sum_{i=1}^{n} a_i |\beta_i\rangle, \sum_{i=1}^{n} |a_i|^2 = 1$$

沿基矢 $\{\beta_i\}_{i=1}^{n}$ 进行测量

几率幅 $a_i = \langle \beta_i | \phi \rangle$

几率 $|a_i|^2$



量子测量 如对于任意叠加态

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$$

Von Neumann测量

Von Neumann测量是投影测量的一种类型。给定一组正交基 $\{|\psi_k\rangle\}$，如果我们对于量子态 $|\Phi\rangle = \sum \alpha_k |\psi_k\rangle$ $\{|\psi_k\rangle\}$ 实施沿基矢 $|\psi_k\rangle$ 的 Von Neumann测量，则有

$$|\alpha_k|^2 = |\langle\psi_k|\Phi\rangle|^2 = \langle\psi_k|\Phi\rangle\langle\Phi|\psi_k\rangle$$

$$= Tr(\langle\psi_k|\Phi\rangle\langle\Phi|\psi_k\rangle) = Tr(|\psi_k\rangle\langle\psi_k||\Phi\rangle\langle\Phi|)$$

Von Neumann测量

例如考虑对于量子态 $|\Phi\rangle = (\alpha|0\rangle + \beta|1\rangle)$

实施相对于基矢 $\left\{\frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}\right\}$ 的Von Neumann测量

注意到 $|\Phi\rangle = \frac{\alpha+\beta}{\sqrt{2}}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) + \frac{\alpha-\beta}{\sqrt{2}}\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right)$

因此我们有测得 $\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$ 的几率为 $\frac{|\alpha+\beta|^2}{2}$

Von Neumann测量

实际上

$$\left(\frac{\langle 0|+\langle 1|}{\sqrt{2}}\right)|\Phi\rangle=\frac{\alpha+\beta}{\sqrt{2}}$$

$$\langle\Phi|\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)=\frac{\alpha^*+\beta^*}{\sqrt{2}}$$

$$\left(\frac{\langle 0|+\langle 1|}{\sqrt{2}}\right)|\Phi\rangle\langle\Phi|\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)$$

$$=Tr\left(\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\left(\frac{\langle 0|+\langle 1|}{\sqrt{2}}\right)|\Phi\rangle\langle\Phi|\right)=\frac{|\alpha+\beta|^2}{2}$$

投影测量(Projective measurements)

可观测量M存在一个谱分解

$$M=\sum_m mP_m$$

满足

$$P_mP_n=P_m\delta_{m,n}$$

$$\sum_m P_m=I$$

获得结果m的几率为

$$p(m)=\mathrm{tr}(P_m\varrho)$$

系统的量子态变为

$$\frac{P_m\varrho P_m}{p(m)}$$

投影测量(Projective measurements)

可观测量M存在一个谱分解

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$\sigma_2 = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$$

$$\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$$

更一般地定义可观测量

$$\vec{v} \cdot \vec{\sigma} \equiv v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$$

称为自旋沿着 $\vec{v}$ 轴方向的测量

广义测量

Quantum measurements are described by a collection $\{M_m\}$ *of measurement operators. These are operators acting on the state space of the system being measured. The index* m *refers to the measurement outcomes that may occur in the experiment.*

对于量子态，结果m发生的几率为

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

测量后的系统处于

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

广义测量

其中测量算符满足完备性条件

$$\sum_m M_m^\dagger M_m = I$$

可见

$$1 = \sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle$$

例子

我们有投影算子 $M_0 = |0\rangle\langle 0|$ 和 $M_1 = |1\rangle\langle 1|$

满足

$$M_0^2 = M_0, M_1^2 = M_1$$

$$I = M_0^\dagger M_0 + M_1^\dagger M_1 = M_0 + M_1$$

从而

$$p(0) = \langle\psi|M_0^\dagger M_0|\psi\rangle = \langle\psi|M_0|\psi\rangle = |a|^2$$

同理 $p(1) = |b|^2$

测量之后量子态处于

$$\frac{M_0|\psi\rangle}{|a|} = \frac{a}{|a|}|0\rangle$$

$$\frac{M_1|\psi\rangle}{|b|} = \frac{b}{|b|}|1\rangle.$$

广义测量是可实现的

任意广义的量子测量均可以通过纠缠该系统与一个辅助系统，应用一个幺正变换并实施投影测量来实现

量子系统A，测量算符M_m, 辅助系统B

定义
$$U|\psi\rangle|0\rangle = \sum_m M_m|\psi\rangle|m\rangle$$

可见
$$\langle\varphi|\langle 0|U^\dagger U|\psi\rangle|0\rangle = \sum_m \langle\varphi|M_m^\dagger M_m|\psi\rangle = \langle\varphi|\psi\rangle$$

实施投影测量
$$P_m \equiv I\otimes|m\rangle\langle m|$$

从而有输出m的几率为
$$\begin{aligned} p(m) &= \langle\psi|\langle 0|U^\dagger P_m U|\psi\rangle|0\rangle \\ &= \sum_{m',m''}\langle\psi|M_{m'}^\dagger\langle m'|(I_Q\otimes|m\rangle\langle m|)M_{m''}|\psi\rangle|m''\rangle \\ &= \langle\psi|M_m^\dagger M_m|\psi\rangle, \end{aligned}$$

广义测量是可实现的

任意广义的量子测量均可以通过纠缠该系统与一个辅助系统，应用一个幺正变换并实施投影测量来实现

量子系统A，测量算符M_m, 辅助系统B

测量之后的联合量子态处于

$$\frac{P_m U|\psi\rangle|0\rangle}{\sqrt{\langle\psi|U^\dagger P_m U|\psi\rangle}} = \frac{M_m|\psi\rangle|m\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

要测量的系统处于

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

POVM测量

——Positive Operator-Valued Measure

量子系统A，测量算符M_m,

对于量子态，结果m发生的几率为

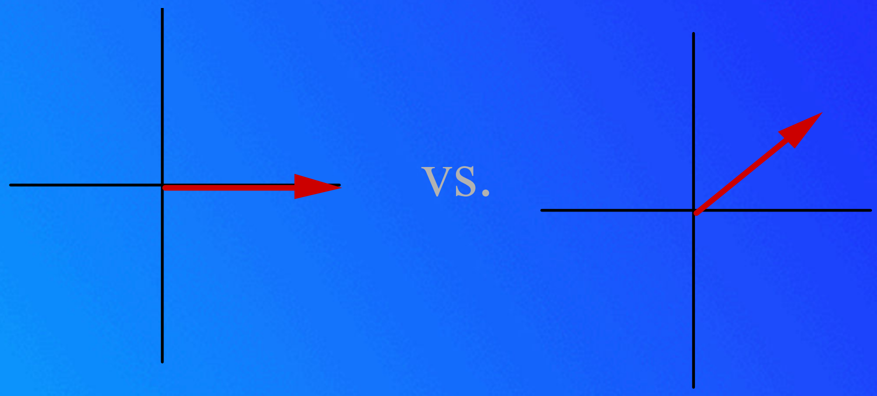
$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

定义 $$E_m \equiv M_m^\dagger M_m$$

我们有 $$\sum_m E_m = I \text{ and } p(m) = \langle\psi|E_m|\psi\rangle$$

可见集合E_m本身就可以决定所有可能的不同测量输出的几率

How to distinguish non-orthogonal states optimally



H-polarized photon **45^o-polarized photon**

From Steinberg

Use generalized (POVM) quantum measurements.

[see, e.g., Y. Sun, J. Bergou, and M. Hillery, Phys. Rev. A 66, 032315 (2002).]

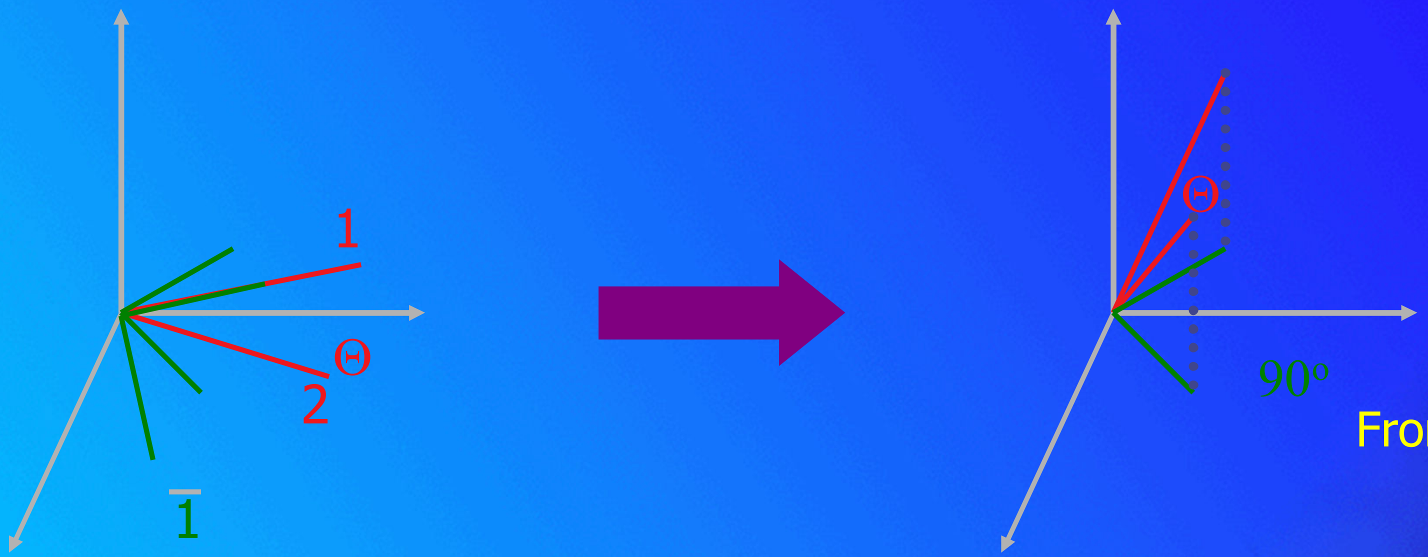
The view from the laboratory:

A measurement of a two-state system can only yield two possible results.

If the measurement isn't guaranteed to succeed, there are three possible results: (1), (2), and ("I don't know").

Therefore, to discriminate between two non-orth. states, we need to use an expanded (3D or more) system. To distinguish 3 states, we need 4D or more.

The geometric picture



From Steinberg

Two non-orthogonal vectors

The same vectors rotated so their *projections* onto x-y are orthogonal

(The z-axis is “inconclusive”)

A test case

Consider these three non-orthogonal states:

$$|\psi_1\rangle_{in} = \begin{pmatrix} \sqrt{2/3} \\ 0 \\ 1/\sqrt{3} \end{pmatrix} \; ; \; |\psi_2\rangle_{in} = \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix} \; ; \; |\psi_3\rangle_{in} = \begin{pmatrix} 0 \\ -1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}$$

Projective measurements can distinguish these states with *certainty* no more than 1/3 of the time.

(No more than one member of an orthonormal basis is orthogonal to *two* of the above states, so only one pair may be ruled out.)

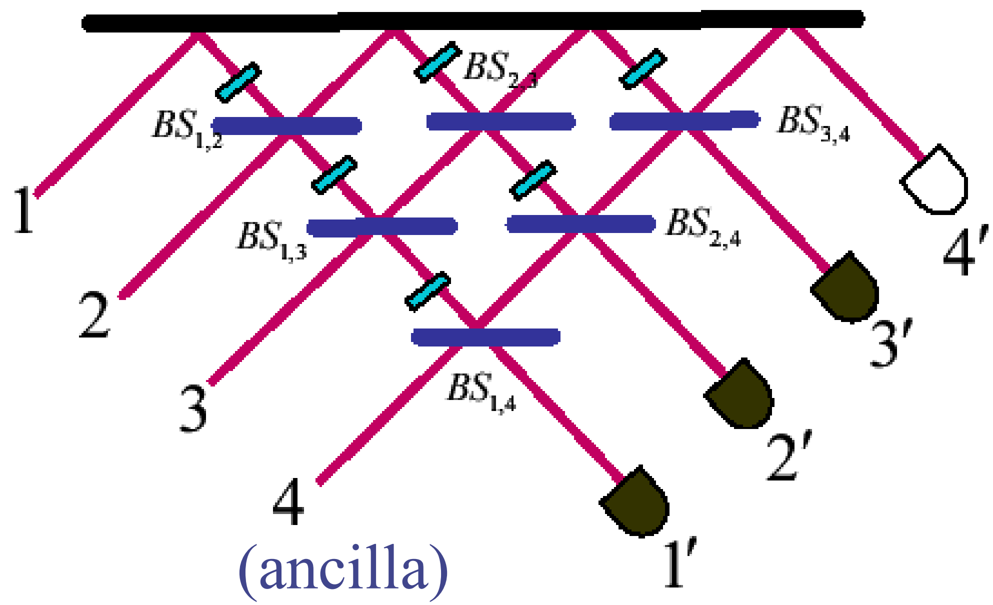
But a unitary transformation in a 4D space produces:

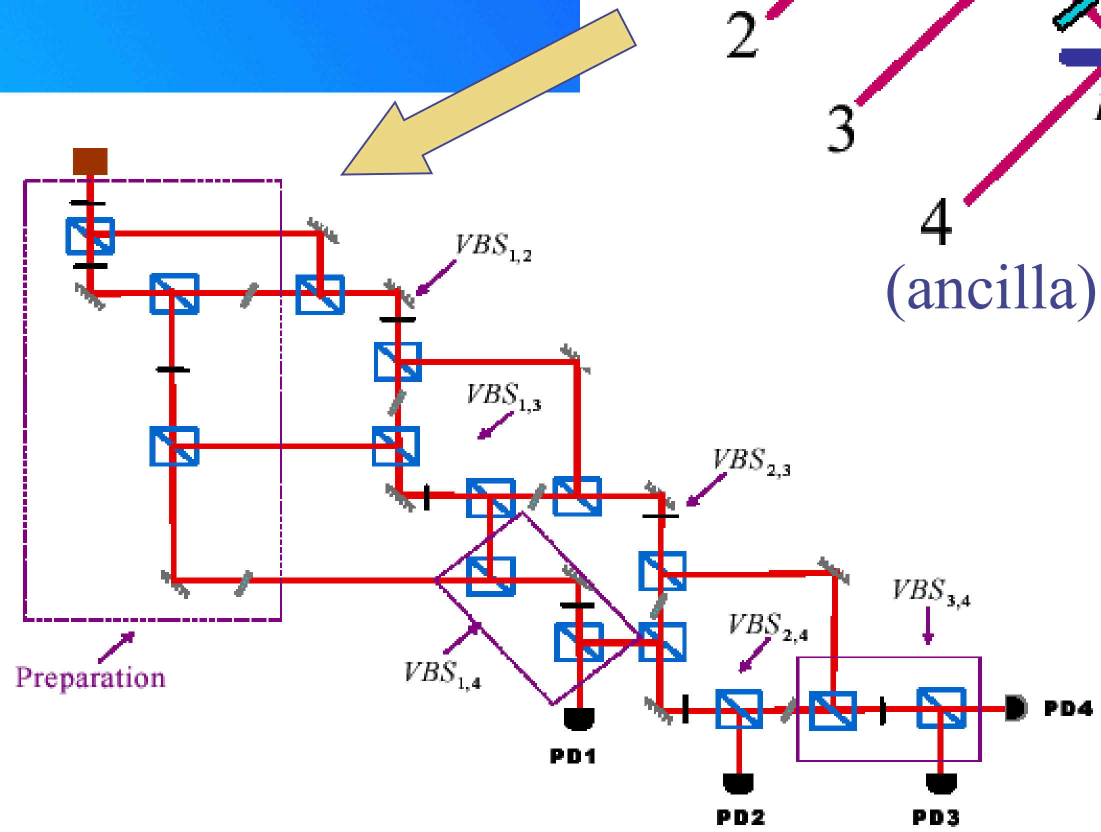
From Steinberg

$$|\psi_1\rangle_{out} = \begin{pmatrix} 1/\sqrt{3} \\ 0 \\ 0 \\ \sqrt{2/3} \end{pmatrix} \quad |\psi_2\rangle_{out} = \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \quad |\psi_3\rangle_{out} = \begin{pmatrix} 0 \\ -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

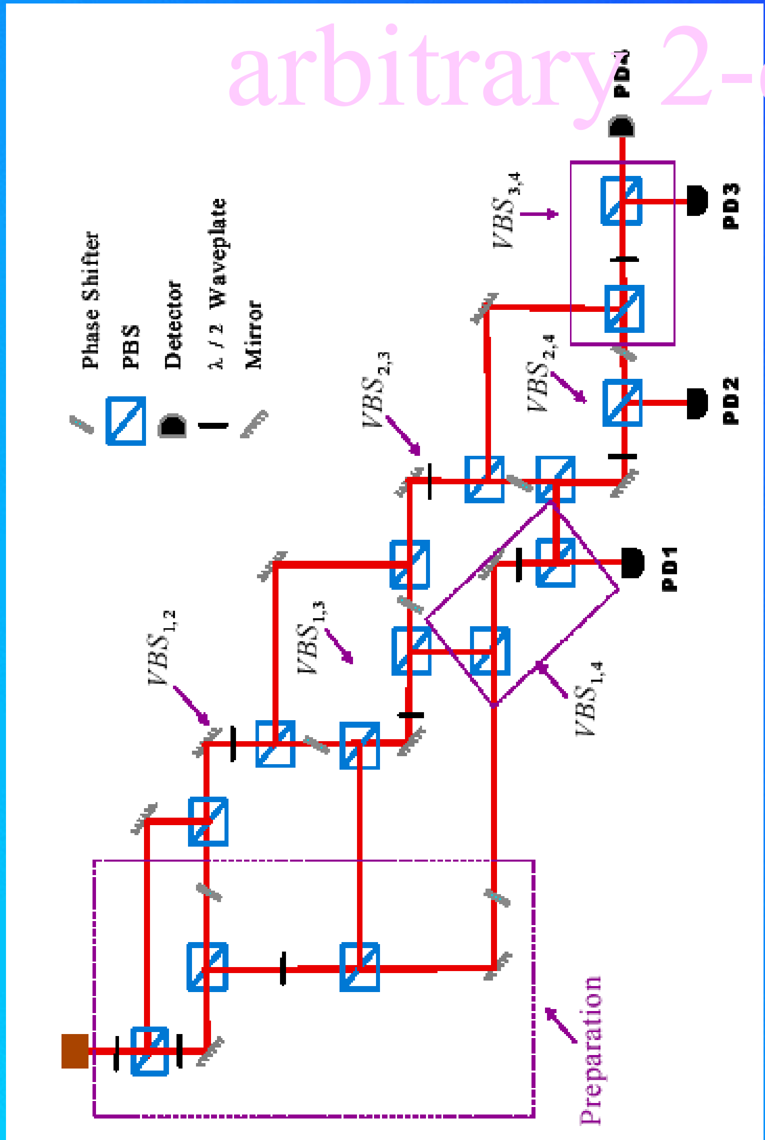
…and these states can be distinguished with certainty up to 55% of the time

Experimental schematic



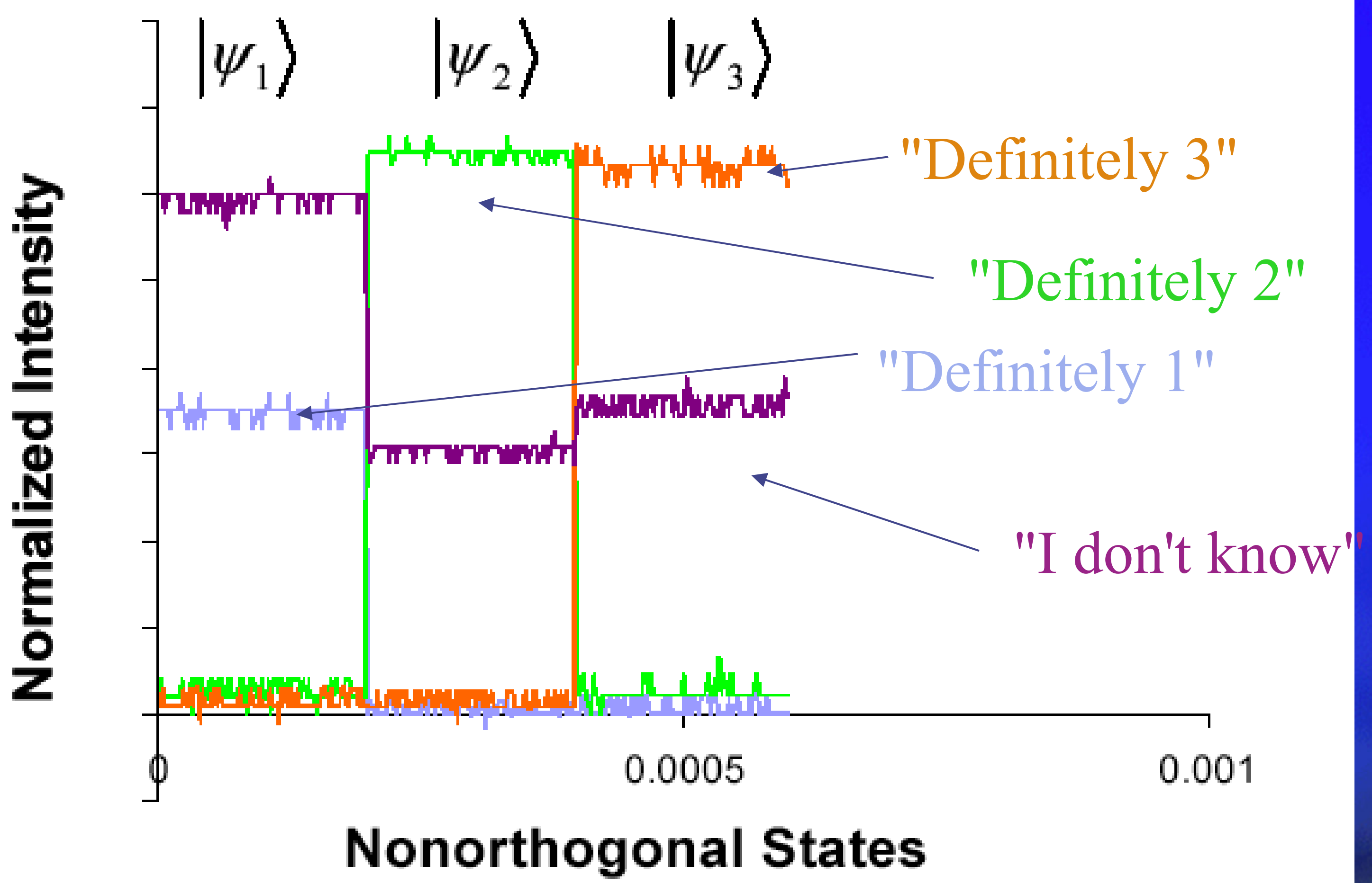


A 14-path interferometer for arbitrary 2-qubit unitaries...



From Steinberg

Success!



From Steinberg

The correct state was identified 55% of the time--
Much better than the 33% maximum for standard measurements.

M. Mohseni, A.M. Steinberg, and J. Bergou, Phys. Rev. Lett. 93, 200403 (2004)

参考QCQI §2.2, M.A. Nielsen and I.L. Chuang

M. Mohseni, A.M. Steinberg, and J. Bergou, Phys. Rev. Lett. 93, 200403 (2004)

谢谢