

Exercise 1 of PHYS5251P:
“Introduction to Quantum Information”
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1. Consider an experiment in which we prepare the state $|0\rangle$ with the probability $|C_0|^2$, and the state $|1\rangle$ with the probability $|C_1|^2$. How to describe this type of quantum state? Compare the differences and similarities between it with the state $C_0|0\rangle + C_1e^{i\theta}|1\rangle$.
2. In the context of optical experiments using polarized quantum states, consider preparing the state $C_0|0\rangle + C_1e^{i\theta}|1\rangle$ from the initial $|0\rangle$ state using a combination of half-wave plates and quarter-wave plates. To achieve arbitrary single-qubit unitary transformations, how can we determine the minimum number of such wave plates required and what is the corresponding operational procedure?
3. (1) Let $f(\cdot)$ be any function from complex numbers to complex numbers. Let $\vec{n}$ be a normalized vector in three dimensions, and let θ be real. Show that

$$f(\theta\vec{n} \cdot \vec{\sigma}) = \frac{f(\theta) + f(-\theta)}{2}I + \frac{f(\theta) - f(-\theta)}{2}\vec{n} \cdot \vec{\sigma}.$$

- (2) Let $\vec{v}$ be a normalized vector in three dimensions, and let θ be real. Show that

$$\exp(i\theta\vec{v} \cdot \vec{\sigma}) = \cos(\theta)I + i\sin(\theta)\vec{v} \cdot \vec{\sigma},$$

where $\vec{v} \cdot \vec{\sigma} = \sum_{i=1}^3 v_i\sigma_i$ and σ_i are Pauli matrices.

4. $\rho_A = \frac{I + \vec{n}_A \cdot \vec{\sigma}}{2}, \rho_B = \frac{I + \vec{n}_B \cdot \vec{\sigma}}{2}$, prove that $\text{tr}(\rho_A \rho_B) = \frac{1 + \vec{n}_A \cdot \vec{n}_B}{2}$.

5. Prove that for any 2-dimension linear operator A ,

$$A = \frac{1}{2} \text{Tr}(A) I + \frac{1}{2} \sum_{k=1}^3 \text{Tr}(A \sigma_k) \sigma_k,$$

in which σ_k are Pauli matrices.

6. Let ρ be a density operator.

(1) Show that ρ can be written as

$$\rho = \frac{I + \mathbf{r} \cdot \boldsymbol{\sigma}}{2}$$

where $\mathbf{r}$ is a real three-dimensional vector and $\|\mathbf{r}\| \leq 1$.

(2) Show that $\text{Tr}(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

(3) Show that a state ρ is a pure state if and only if $\|\mathbf{r}\| = 1$.

7. Describe the no-cloning theorem and prove that if quantum cloning is allowed, then superluminal communication is allowed too.

8. Suppose a 2-qubit pure state is of the form $|\Phi\rangle = \sum_{ij} a_{ij} |i\rangle |j\rangle$. By defining $A_{ij} = a_{ij}$ where A_{ij} are elements of a matrix A , calculate the reduced density matrices ρ_A and ρ_B .

9. Suppose a 2-qubit pure state is of the form $|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}|0\rangle_A \left(\frac{\sqrt{3}}{2}|0\rangle_B + \frac{1}{2}|1\rangle_B \right) + \frac{1}{\sqrt{2}}|1\rangle_A \left(\frac{1}{2}|0\rangle_B - \frac{\sqrt{3}}{2}|1\rangle_B \right)$.

(1) Calculate the reduced density matrices ρ_A and ρ_B .

(2) Perform Schmidt decomposition of $|\Psi\rangle_{AB}$.

10. Suppose $|\psi\rangle$ and $|\phi\rangle$ are two pure states of a composite quantum system AB , with identical Schmidt coefficients. Show that there are unitary transformations U on system A and V on system B such that $|\psi\rangle = (U \otimes V) |\phi\rangle$.

11. Suppose ABC is a three component quantum system. Show by example that there are pure quantum states ψ of such systems which can not be written in the form

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle$$

where λ_i are real numbers, and $|i_A\rangle, |i_B\rangle, |i_C\rangle$ are orthonormal bases of the respective systems.

12. Consider a single qubit and unit vectors $\vec{n}_k, k \in 1, 2, \dots, N$ such that $\sum_k \lambda_k \vec{n}_k = 0$ for $\lambda_k \in (0, 1)$ and $\sum_k \lambda_k = 1$. Show that a measurement on a qubit defined by

$$F_k = 2\lambda_k |\uparrow_{\vec{n}_k}\rangle \langle \uparrow_{\vec{n}_k}|$$

is a POVM.

13. A symmetrical informationally-complete POVM (also known as a SIC POVM) on $\mathbb{C}^d$ is a set of d^2 rank-1 projections

$$\{|\psi_1\rangle\langle\psi_1|, \dots, |\psi_{d^2}\rangle\langle\psi_{d^2}|\} \subset Proj(\mathbb{C}^d)$$

such that

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{d\delta_{ij} + 1}{d + 1}$$

for any $i, j \in \{1, \dots, d^2\}$. Construct a SIC POVM on $\mathbb{C}^2$. (Consider the vertices of a regular tetrahedron in the Bloch sphere.)

14. Suppose $\{|\psi_i\rangle\}, \{|\tilde{\psi}_i\rangle\}$ are two sets of normalized states in space H and they satisfy the conditions that $\langle\psi_i|\psi_j\rangle = \langle\tilde{\psi}_i|\tilde{\psi}_j\rangle$ for $\forall i, j$, prove that there exist a transformation U , such that $U|\psi_i\rangle = |\tilde{\psi}_i\rangle$, and construct the transformation U .