Exercise 2 of PHYS5251P:
"Introduction to Quantum Information"
Fall 2026, USTC

Lei-Yu Cheng and Kai Chen

Hefei National Research Center for Physical Sciences at the Microscale and School of Physical Sciences, University of Science and Technology of China, Hefei 230026, China

1. Suppose Alice and Bob share the two-qubit state $|\psi_{AB}\rangle = \frac{\sqrt{3}}{2}|00\rangle + \frac{1}{2}|11\rangle$. Recall that a quantum measurement is specified by a set of operators $\{M_0, M_1\}$ such that $\sum_k M_k^\dagger M_k = I$.

   (1) Give a and b such that the quantum measurement outcome from operator $M_0 = a|0\rangle\langle 0| + b|1\rangle\langle 1|$ acting on $|\psi_{AB}\rangle$ produces the post-measurement result $(|00\rangle + |11\rangle)/\sqrt{2}$ with probability 1/4.

   (2) Give an operator M_1 such that $M_0^\dagger M_0 + M_1^\dagger M_1 = I$. With what probability does the corresponding outcome occur, acting on $|\psi_{AB}\rangle$, and what is the post-measurement result?

2. Consider a particle with initial state $|0\rangle$. We perform N sequential measurements $\sigma_k \equiv \vec{n}_k \cdot \vec{\sigma}$ with $\vec{n}_k = \left(\sin\left(\frac{k\pi}{2N}\right), 0, \cos\left(\frac{k\pi}{2N}\right)\right)$ and $k = 1, 2, \cdots, N$. What's the probability that all outcomes are $+1$? What if $N \to \infty$?

3. The four Bell states have the following mathematical expressions on the basis

$\{|0\rangle, |1\rangle\}$ (the eigenstates of σ_z),

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle),$$
$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).$$

(1) Prove that the four Bell states can be transformed to each other using single qubit rotations $\{I, \sigma_x, \sigma_y, \sigma_z\}$.

(2) Show that each of the four Bell states is an eigenstate of the observables $\{\sigma_{1x}\sigma_{2x}, \sigma_{1y}\sigma_{2y}, \sigma_{1z}\sigma_{2z}\}$ and write down the corresponding eigenvalues.

4. (1) What conditions should a good entanglement measures meet?

(2) Describe the definition of distillable entanglement and entanglement cost and their relationship.

(3) Write down the monogamy of entanglement and describe its physical meanings.

5. PPT (Positive Partial Transposition) criterion is a strong separability criterion for quantum states, which is very convenient and practical for entanglement detection.

(1) Describe the PPT criterion and the realignment criterion.

(2) For the 2-qubit state $\rho = p|\psi^-\rangle\langle\psi^-| + (1-p)\frac{\mathbb{I}}{4}$, where, $0 \leq p \leq 1$, $|\psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$, calculate the lower bound of p for ρ to be an entangled state using PPT criterion and realignment criterion respectively.

6. (1) For the 3-qubit W state $|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, if one particle is lost, what's the reduced density matrix of the remaining two particles?

(2) For the n-qubit W state $|W_n\rangle = \frac{1}{\sqrt{n}}(|10\cdots0\rangle + |01\cdots0\rangle + \cdots + |00\cdots1\rangle)$, if $(n-2)$ particles are lost, what's the reduced density matrix of the remaining two particles? Use the PPT criterion to find out whether the remaining two particles are entangled or not.

7. (1) Calculate the amount of entanglement of the state $\rho = \lambda |\phi^+\rangle \langle\phi^+| + (1-\lambda) |\psi^+\rangle \langle\psi^+|, (0 \leq \lambda \leq 1)$ with negativity measure, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle, |\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

   (2) Derive the value range for λ when the state ρ is entangled by using negativity measure.

8. An entanglement witness(EW) is a functional which distinguishes a specific entangled state from separable ones.

   (1) Describe the definition of the Entanglement Witness (EW).

   (2) For the mixed state $\rho = p\frac{\mathbf{I}}{8} + (1-p)|GHZ\rangle\langle GHZ|$ $(0 \leq p \leq 1)$, calculate p's upper bound when ρ is an entangled state using the entanglement witness $\mathcal{W} = \frac{1}{2}\mathbf{I} - |GHZ\rangle\langle GHZ|$.

9. Consider the density matrix $\rho_w = r|\phi^+\rangle\langle\phi^+| + \frac{1-r}{4} I_4$, where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is Bell state and $0 \leq r \leq 1$. Calculate the concurrence of ρ_w.

10. For the 2-qubit state $\rho = p|\Psi^-\rangle\langle\Psi^-| + (1-p)\frac{\mathbb{I}}{4}$, where $0 \leq p \leq 1$, $|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$, calculate the EOF (Entanglement of Formation) of ρ.

11. (1) Prove that $0 \leq S(\rho) \leq \log D$, where D is the number of non-zero eigenvalues of ρ.

    (2) Prove that for any two positive definite matrices A and B we have $\log(A \otimes B) = \log(A) \otimes I + I \otimes \log(B)$.

12. The entropy of quantum state, expressed as a density matrix ρ, is $S(\rho) = -Tr(\rho \log_2 \rho)$; in terms of its eigenvalues λ_k, this is $S(\rho) = -\Sigma_k \lambda_k \log_2 \lambda_k$. A state ρ is a pure state if and only if $Tr(\rho^2) = 1$. Prove that this is equivalent to $S(\rho) = 0$. You may use the fact ρ is a valid density matrix if and only if $Tr(\rho) = 1$ and ρ is a positive operator.

13. (1) Prove the subadditivity of the von Neumann entropy

$$|S(A) - S(B)| \leq S(A, B) \leq S(A) + S(B).$$

(2) Prove the concavity of the von Neumann entropy

$$S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i).$$

(3) Suppose ABC is a composite quantum system. Prove that

$$S(A|B,C) \leq S(A|B).$$

14. (1) Calculate the von Neumann entropy of the following density matrix,
(a) $\rho_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, (b) $\rho_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, (c) $\rho_3 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, (d) $\rho_4 = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

(2) Consider the states

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B),$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} |0\rangle (\frac{1}{2} |0\rangle + \frac{\sqrt{3}}{2} |1\rangle) + \frac{1}{\sqrt{2}} |1\rangle (\frac{\sqrt{3}}{2} |0\rangle + \frac{1}{2} |1\rangle).$$

Calculate the Von Neumann entropy of ρ_{1_A} and ρ_{2_A}.

15. Let ρ be a single qubit density matrix, and recall the Bloch sphere representation

$$\rho = \frac{I + \boldsymbol{r} \cdot \boldsymbol{\sigma}}{2},$$

where $\boldsymbol{r}$ is a real three-dimensional vector and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Express the von Neumann entropy of ρ in terms of $\boldsymbol{r}$ and the binary entropy function $H(p) = -p \log p - (1-p) \log(1-p)$.

16. Suppose $|AB\rangle$ is a pure state of a composite system belonging to Alice and Bob. Show that $|AB\rangle$ is entangled if and only if $S(B|A) < 0$.

17. Suppose $\{P_i\}$ is a complete set of orthogonal projectors and ρ is a density operator. Prove that the entropy of the state $\rho' \equiv \sum_i P_i \rho P_i$ of the system after the measurement is at least as great as the original entropy, $S(\rho') \geq S(\rho)$, with equality if and only if $\rho = \rho'$.

18. Consider a composed system $A \otimes B \otimes C$ with a shared state ρ_{ABC}, we can define a conditional version of the mutual information between A and B as

$$I(A:B|C) = S(A|C) + S(B|C) - S(AB|C) = S(A|C) - S(A|BC).$$

Consider the so-called cat state shared by four qubits, that is defined as $|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$. Calculate the mutual information between qubits A and B changes with the knowledge of the remaining qubits, namely: $I(A:B), I(A:B|C), I(A:B|CD)$.