

# Fidelity-Guaranteed Entanglement Routing in Quantum Networks

Jian Li<sup>1</sup>, Member, IEEE, Mingjun Wang, Kaiping Xue<sup>1</sup>, Senior Member, IEEE, Ruidong Li<sup>1</sup>, Senior Member, IEEE, Nenghai Yu<sup>1</sup>, Qibin Sun, Fellow, IEEE, and Jun Lu

**Abstract**—Entanglement routing establishes remote entanglement connection between two arbitrary nodes, which is one of the most important functions in quantum networks. The existing routing mechanisms mainly improve the robustness and throughput facing the failure of entanglement generations, which, however, rarely include the considerations on the most important metric to evaluate the quality of connection, entanglement fidelity. To solve this problem, we propose purification-enabled entanglement routing designs to provide fidelity guarantee for multiple Source-Destination (S-D) pairs in quantum networks. In our proposal, we first consider the single S-D pair scenario and design an iterative routing algorithm, Q-PATH, to find the optimal purification decisions along the routing path with minimum entangled pair cost. Further, a low-complexity routing algorithm using an extended Dijkstra algorithm, Q-LEAP, is designed to reduce the computational complexity by using a simple but effective purification decision method. Finally, we consider the common scenario with multiple S-D pairs and design a greedy-based algorithm considering resource allocation and re-routing process for multiple routing requests. Simulation results show that the proposed algorithms not only can provide fidelity-guaranteed routing solutions, but also has superior performance in terms of throughput, fidelity of end-to-end entanglement connection, and resource utilization ratio, compared with the existing routing scheme.

**Index Terms**—Quantum networks, fidelity-guaranteed, entanglement purification, entanglement routing, resource allocation.

## I. INTRODUCTION

IN RECENT years, quantum information technologies have been widely developed and achieved remarkable breakthroughs especially in secure communications [1]. Along with

Manuscript received 12 January 2022; revised 19 June 2022 and 7 August 2022; accepted 7 August 2022. Date of publication 18 August 2022; date of current version 18 October 2022. This work is supported in part by Anhui Initiative in Quantum Information Technologies under grant No. AHY150300, China Postdoctoral Science Foundation Funded Project under Grant No. 2021M703118 and Youth Innovation Promotion Association of Chinese Academy of Sciences (CAS) under Grant No. Y202093. The associate editor coordinating the review of this article and approving it for publication was A. S. Cacciapuoti. (Corresponding author: Kaiping Xue.)

Jian Li, Mingjun Wang, Kaiping Xue, Nenghai Yu, and Qibin Sun are with the School of Cyber Science and Technology, University of Science and Technology of China, Hefei, Anhui 230027, China (e-mail: kpxue@ustc.edu.cn).

Ruidong Li is with the College of Science and Engineering, Kanazawa University, Kakuma, Kanazawa 920-1192, Japan.

Jun Lu is with the Department of Electronic Engineering and Information Science, and the School of Cyber Science and Technology, University of Science and Technology of China, Hefei, Anhui 230027, China.

Color versions of one or more figures in this article are available at <https://doi.org/10.1109/TCOMM.2022.3200115>.

Digital Object Identifier 10.1109/TCOMM.2022.3200115

0090-6778 © 2022 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See <https://www.ieee.org/publications/rights/index.html> for more information.

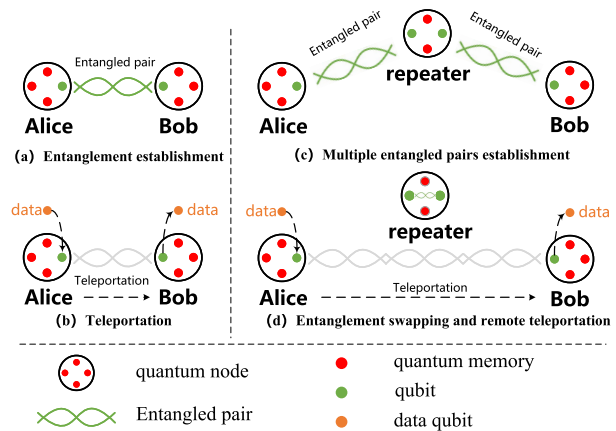


Fig. 1. Illustration of information transmission in a quantum network, data qubit represents the state that Alice wishes to teleport [8, chapter 4]. (a)(b) an arbitrary single qubit can be sent using short-distance quantum teleportation. (c)(d) establishment of remote entanglement connection through entanglement swapping, data transmission using long-distance quantum teleportation.

the concept validation of quantum repeater and long-distance quantum communications [2], [3], the quantum network, which is foreseen to be a “game-changer” to the classic network, is being developed at a rapid pace.

In a quantum network, quantum nodes (including quantum processors and repeaters) are interconnected via optical links, and they can generate, store, exchange, and process quantum information [4], [5]. When two faraway quantum nodes, serving as source and destination, attempt to exchange information, the quantum network first establishes the entanglement connection between them, and then information is transmitted in the form of quantum bits (called qubits) over entanglement connection to the destination. As shown in Fig. 1, to establish such end-to-end entanglement connection, entangled pairs between adjacent nodes are first generated. After that, quantum repeater connects quantum nodes over longer distances by performing entanglement swapping, i.e., joint Bell state measurements at the local repeater aided by classical communication [6].

To build up a large-scale functional quantum network with satisfying the dynamic requests from Source-Destination (S-D) pairs, the critical problem we have to face firstly is how to select routing path and utilize network resources efficiently (such as limited entangled pairs on each edge). Recently, some existing studies are dedicated to solve such problem [7], and

propose multiple entanglement routing designs to improve the robustness and throughput facing the failure of entanglement generations. Although the development of quantum network is currently at the primitive stage, these routing designs bring a good start to facilitate the process of quantum networks' construction in the future.

However, an important metric to evaluate the quality of remote entanglement connection, entanglement fidelity, is rarely considered in the existing entanglement routing designs. In practice, due to the noise in the system, quantum repeaters sometimes might not generate entangled pairs with a certain desired fidelity, which brings negative effects on various quantum applications [9]. For example, in quantum cryptography protocols (e.g., BB84 protocol), an entanglement fidelity lower than the quantum bit error rate can reduce the security of key distribution [10]. To improve the fidelity of entanglement connection and satisfy the requirement of quantum applications, a technique called entanglement purification can be used to increase the fidelity of entangled pairs [11]. It consumes shared lower-fidelity entangled pairs along the link between adjacent nodes to obtain one higher-fidelity entangled pair. By adopting purification technique, the entanglement routing can provide fidelity guarantee for end-to-end entanglement connection. Nevertheless, due to the nonlinear relationship between fidelity improvement and resource consumption in purification operation, the additional purification decision makes the entanglement routing problem more complicated. Thus, how to design such fidelity-guaranteed entanglement routing remains an unsolved problem.

Based on such considerations, in this paper, we focus on purification-enabled entanglement routing design under the fidelity constraint in general quantum networks. To address the complicated entanglement routing problem, we first study the entanglement routing problem in single S-D pair scenarios, and respectively propose an iterative routing algorithm to obtain the optimal solution and a low-complexity routing algorithm to obtain near-optimal but efficient solution. To obtain the optimal purification decisions, we also analyze the characteristic of purification operations and propose an optimal decision approach. After that, we further study the entanglement routing problem in multiple S-D pairs scenarios, and propose a greedy-based routing algorithm considering two resource allocation methods. We also conduct extensive simulations to show the superiority of the proposed algorithms compared with the existing ones. Although the existing work [12] has already proposed an entanglement distribution design and imposed a minimum end-to-end fidelity as a requirement, it does not take purification into consideration and then the fidelity of each Bell pair cannot be further improved. Thus, to the best of our knowledge, this is the first work that provides end-to-end fidelity-guaranteed entanglement routing with purification decision, which can fully leverage the advantages of purification operation and significantly improve the end-to-end fidelity with abundant low-fidelity entangled pairs.

The contributions of this paper can be summarized as follows:

- For the requirement of high-quality entanglement connections from various quantum applications, we propose

the first entanglement routing and purification design that provides end-to-end fidelity-guaranteed connections for S-D pairs in “advance generation” model based quantum networks. For single S-D pair scenarios, we devise two novel entanglement routing algorithms, i.e., Q-PATH and Q-LEAP, respectively. The former one can obtain multiple routing paths for satisfying single S-D pair and provide the optimal routing solution with minimum entangled pair cost, and the latter one can efficiently provide the routing solution with minimum fidelity degradation and has the advantage of low computational complexity.

- Based on the routing solutions provided by algorithms designed for single S-D pair, we further consider the routing problem in multiple S-D pairs scenarios as a resource allocation problem, and propose a greedy-based routing design, which leverages two important factors of a given routing solution, i.e., resource consumption and degree of freedom, to globally allocate entanglement resources for various routing solutions and improve the efficiency of resource utilization.
- To verify the effectiveness of the proposed algorithms, extensive simulations are conducted. Compared to the existing routing scheme with purification decisions, the proposed algorithms not only provide fidelity-guaranteed routing solutions, but also show the significant superiority in terms of throughput, the average fidelity of the end-to-end connections, and network resource utilization.

The rest of this paper is organized as follows. Firstly, related work is discussed in Section II. Then, the motivation, the network model and the routing problem considered in this paper are given in Section III. After that, the entanglement routing designs for single S-D pair and multiple S-D pairs are given in Section IV and Section V, respectively. Finally, the performance evaluation is conducted in Section VI and conclusions are drawn in Section VII.

## II. RELATED WORK

The interconnection of quantum devices forms a quantum network by enabling quantum communications among remote quantum nodes, and academic community believes the ultimate objective of the development of quantum networks is to build a global system, called quantum internet, with interconnected networks around the world that uses quantum internet protocol [13], which is similar to the Internet. To achieve this ambition, Cacciapuoti [9], Caleffi [14] and others thoroughly survey the theoretical and practical problems of networking, and significantly push forward the development of quantum internet in terms of entanglement distribution, protocol design, optimization of physical devices and so on. In quantum networks, long-distance entanglement connection is required by various quantum applications, such as distributed quantum computing, sensing and metrology and clock synchronization [15]. To establish a multi-hop quantum entanglement connection via quantum repeaters for multiple S-D pairs, an efficient entanglement routing solution is required. In general, two kinds of quantum network models, i.e., “*advance generation*” model (entanglement generation before routing

decision) and “*on-demand generation*” model (entanglement generation after routing decision), are concerned in existing work [16]. The former basically isolates the functions between link layer and network layer, and only resource allocation and path selection should be considered in the routing design. The latter, however, tightly couples link layer and network layer, and not only path selection but also entanglement generation and potential failures should be considered in the routing design. Thus, in this paper, we focus on the entanglement routing problem based on “*advance generation*” model.

Most of the existing studies on quantum networks focus on a specific network topology, such as diamond [17], star [18], and square grid [6], [19]. Pirandola [17], Vardoyan *et al.* [18], and Pant *et al.* [19] considered the physical characteristics of quantum networks, such as quantum memory and decoherence time, and developed routing protocols and theoretical analysis about end-to-end capacities and expected number of stored *qubits* for homogeneous systems. Considering the routing problem with quantum memory failures, Gyongyosi. *et al.* [20] proposed an efficient adaptive routing based on base-graph. Thus, when the failure happens, some entanglement connections can be destroyed but a seamless network transmission can still be provided since shortest replacement paths can be found by using the adaptive routing. Caleffi [21] considered stochastic framework that jointly accounts several physical-mechanisms such as decoherence time, atom–photon/photon–photon entanglement generation and entanglement swapping, and derived the closed-form expression of the end-to-end entanglement rate. Based on that, the authors further proposed an optimal routing protocol when using the proposed entanglement rate as routing metric. After that, Hahn *et al.* [22] utilized a graph state and proposed a general routing method in arbitrary networks. To be noticed, the existing studies rarely consider fidelity as one of the metric in entanglement routing. One representative study was proposed by Li *et al.* [6], who considered a lattice topology and proposed an effective routing scheme to enable automatic responses for multiple requests of S-D pairs. The authors considered the purification operation to ensure the fidelity of entanglement connection, however, the purification operation is performed before routing decision to satisfy the fidelity constraint. Due to the fidelity degradation of entanglement swapping, this simple purification decision cannot provide end-to-end fidelity guarantee. Thus, the lack of existing routing design involving fidelity encourages us to design a novel fidelity-guaranteed entanglement routing scheme in future quantum network.

### III. NETWORK MODEL AND PROBLEM DESCRIPTION

In this section, we first provide the motivation of our work, and then introduce the network model. Further, we define the entanglement routing problem in quantum networks and analyze its property. The notations used in this paper are summarized in Table I.

#### A. Motivation

The work proposed in this paper is motivated by the desire to provide fidelity guarantee for various quantum

TABLE I  
SYMBOLS USED IN THE PAPER

Symbol	Notation
$R_i$	The number of routing requests from $i$ -th S-D pair, and the unit is one end-to-end entanglement connection.
$c(u, v)$	Remaining capacity on edge $(u, v)$ .
$F_i^{th}$	Fidelity threshold for $i$ -th S-D pair.
$\mathcal{N}(u)$	Number of neighbors of node $u$ .
$F_{i,j}(s, d)$	Fidelity of end-to-end entanglement for $j$ -th routing path of $i$ -th S-D pair.
$P_{i,j}(s, d)$	Routing path $\{(s_i, u_1), \dots, (u_n, d_i)\}$ for $j$ -th routing path of $i$ -th S-D pair.
$N_{i,j}^{Pur}(u, v)$	Number of purification rounds on edge $(u, v)$ for $j$ -th routing path of $i$ -th S-D pair.
$D_{i,j}^{pur}$	Purification decisions for $j$ -th routing path of $i$ -th S-D pair, $D_{i,j}^{pur} = [N_{i,j}^{pur}(s_i, u_1), \dots, N_{i,j}^{pur}(u_n, d_i)]$ .
$W_{i,j}^{min}$	The minimum width, i.e., the maximally available number of establishing end-to-end entanglement connections, along the routing path $P_{i,j}(s_i, d_i)$ .
$t_{i,j}^{EXT}(u, v)$	Expected throughput on edge $(u, v)$ for $j$ -th path of $i$ -th S-D pair after purification.
$T_{i,j}^{EXT}(s_i, d_i)$	Expected throughput on routing path $P_{i,j}(s_i, d_i)$ .

applications. Although several routing protocols and algorithms have already been proposed to handle the requests from multiple S-D pairs in existing work, the fidelity degradation during the entanglement swapping has not been considered and an end-to-end fidelity guarantee of entanglement connection<sup>1</sup> cannot be provided. Thus, we consider an entanglement generation before routing decision model, i.e., “*advance generation*” model, and focus on satisfying end-to-end fidelity constraint with minimum resource consumption through optimizing path selection with considerations on purification operations.

#### B. Network Model

A general quantum network is described by a graph  $G = (V, E, C)$ , where  $V$  is the set of  $|V|$  quantum nodes,  $E$  is the set of  $|E|$  edges, and  $C$  is the set of edge capacity. An edge  $(u, v)$  between two nodes means that two nodes share one or more quantum channels, and the capacity  $c(u, v)$  determines the maximum number of the entangled pairs that can be provided. For the quantum node  $v \in V$ , the quantum channel on edge  $e \in E$ , and the S-D pair of a routing request, we give the definition as follows:

1) *Quantum Node*: Each quantum node holds the complete function of a quantum repeater.<sup>2</sup> Arbitrary quantum nodes are

<sup>1</sup>In this paper, we consider a entanglement connection as an end-to-end shared entangled pair between source node and destination node.

<sup>2</sup>In this paper, we consider first-generation quantum repeaters, hence a finite number of qubit memories is considered, and entanglement generation and purification are applied on the repeaters, and quantum error correction is not available [23], [24].

equipped with quantum processors and quantum applications can be deployed.

2) *Quantum Channel*: A quantum channel is established between adjacent quantum nodes to support the transmission of *qubits* via physical links (such as optical fibers [4] and free-space [25]) with shared entanglement pairs and support qubit transmission. The encoding of *qubits* may have multiple choices, e.g., polarization-encoded *qubits*, phase-encoded *qubits*, etc. Here, a constant capacity of each quantum channel is considered, which means a constant number of the entangled pairs between two adjacent nodes are generated at the start of each time slot. The process of entanglement generation can also be considered as a deterministic black box (Nitrogen-Vacancy platform is considered), and the fidelity of a generated entanglement pair on each quantum channel can be approximately calculated by a deterministic formula without consideration of noise [14], [26].

3) *Source-Destination Pair*: Due to the requirement of quantum applications, a quantum node may intend to establish entanglement connection with the other node. Herein, we name such pair of quantum nodes with the intention of entanglement connection establishment as a Source-Destination (S-D) pair.

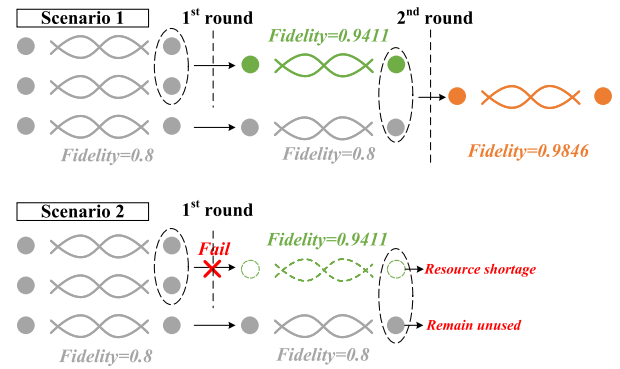
Based on the above definitions on quantum node, channel, and S-D pairs, we introduce the network management method. In quantum networks, all quantum nodes are connected via classical networks, and each node has a certain level of classical computing and storage capacity. Similar to the existing studies [4], [7], we assume a time-synchronous network operating in time slots.<sup>3</sup> To manage a quantum network, all quantum nodes are controlled by a centralized controller via classical networks. The controller holds all the basic information of the network, such as network topology and resources, which can be reported and updated by the quantum nodes. For an entanglement routing process, it consists of three phases. At the beginning of each time slot, adjacent quantum nodes start to generate the entangled pairs, and the controller collects routing requests from quantum nodes. Then, the controller executes routing algorithm to determine the routing path of each S-D pair and resource allocation in the network. Note that part of routing requests might be denied due to the connectivity or resource limitation. Finally, according to the instructions from the controller, all quantum nodes perform purification and swapping to concatenate single-hop entangled pair and establish multi-hop entanglement connections for S-D pairs.

To establish end-to-end entanglement connection in quantum networks, three unique operations, i.e., entanglement generation, purification and swapping, which have no analogue in classical networking, should be considered:

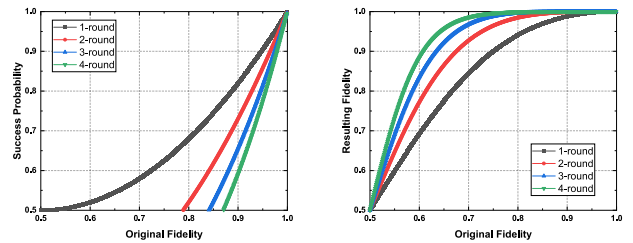
4) *Entanglement Generation*: Physical entanglement generation can be performed between two controllable quantum nodes, which connect to an intermediate station, called the heralding station, over optical fibers by using various hardware platform, such as nitrogen-vacancy centers in diamond [15]. After one success generation attempt, the entangled pair<sup>4</sup> can be stored in the memory of quantum nodes as the available

<sup>3</sup>The network is synchronized to a clock where each timestep is no longer than the memory decoherence time.

<sup>4</sup>In this paper, only Bell entangled state is considered.



(a) The process of 2-round purification operations



(b) success probability

(c) resulting fidelity

Fig. 2. Illustration of purification operations and performance for multi-round purification operations.

resource to establish entanglement connection and transmit an arbitrary single *qubit* state using teleportation.

5) *Entanglement Purification*: Entanglement purification enables two low-fidelity Bell pairs to be merged into a single higher-fidelity one, which can be implemented using CNOT gates or optically using polarizing beamsplitters [27]. By considering bit flip errors, the resulting fidelity after purification operation can be calculated by [8]:

$$f(x_1, x_2) = \frac{x_1 x_2}{x_1 x_2 + (1 - x_1)(1 - x_2)}. \quad (1)$$

where  $x_1, x_2$  is the fidelity of two Bell pairs in the purification operation. If we consider the fidelity of Bell pairs on the same edge are the same, then  $x_1 = x_2$ , and the formula can be simplified as  $f(x) = \frac{x^2}{2x-1}$ . This process can be applied recursively so as to in principle achieve arbitrarily high fidelities. An example of multi-round purification operation on the same edge is shown in Fig. 2(a). Note that the dashed oval in the figure represents a purification operation, and the resulting fidelity is obtained by using Eq. (1). Meanwhile, all entangled pairs before 1st round purification are generated on the same edge with the same fidelity (i.e., 0.8), a *pumping* purification scheme [28] is considered, which means that each round of purification operation consumes an extra entangled pair. In this example, we consider two scenarios, the first one is that all purification operations are implemented successfully, and thus the final fidelity after second round purification is 0.9846. The second one is that 1st purification operation is failed, and two low-fidelity entangled pairs are thus broken. In the next, second purification operation cannot be implemented, and the bottom entangled pair in scenario 2 remains unused and it can be used for other purification operations in the

future. The specific performance for multi-round purification operation is shown in Figs. 2(b)-2(c). One challenge problem for designing a fidelity-guaranteed routing is to determine whether purification will be performed or not and the number of purification rounds for the intermediate nodes on the path between source and destination if necessary.

6) *Entanglement Swapping*: To connect quantum nodes and establish long-distance entanglement connection, entanglement swapping can be regarded as an attractive approach. As shown in Fig. 1, a quantum repeater that carries entangled pairs shared by both Alice and Bob can turn the two one-hop entanglements into one direct entanglement between Alice and Bob [13], [28]. By repeating swapping operations, multi-hop entanglement connection along the path of repeaters carrying entangled pairs can be established. To be noticed, due to the imperfect measurement (i.e., noisy operation [29], [30]) on the repeater, the fidelity of multi-hop entanglement would degrade during entanglement swapping. Meanwhile, considering fidelity variance of entangled pairs on different quantum channel, different routing paths can lead to distinct fidelity results of end-to-end entanglement connection after swapping. It is another challenge issue for designing fidelity-guaranteed entanglement routing.

### C. Entanglement Routing Problem

A fidelity-guaranteed entanglement routing problem can be described as follows: Given a quantum network with topology and edge capacity as  $G = (V, E, C)$ , finding routing solutions, including purification decision  $D_{i,j}^{pur}$  and path selection  $P_{i,j}(s_i, d_i)$ , to enable the entanglement establishment between S-D pairs to satisfy fidelity constraint  $F_{i,j}(s_i, d_i) > F_i^{th}, \forall i$ .

The routing problem in traditional networks, as a classic one, has been studied for decades [31], [32], [33]. However, the routing problem for multiple S-D pairs, which belongs to multi-commodity flow problem [34], [35], has been proven as a NP-hard problem. In quantum networks, due to the unique characteristics such as entanglement purification, entanglement routing problem becomes knotty since the special characteristics and constraints have to be considered. In specific, for the entanglement routing problem with fidelity guarantee, additional purification decision problem are coupled with path searching problem, which makes the entanglement routing problem more complicated.

## IV. ROUTING DESIGN FOR SINGLE S-D PAIR

In this section, we focus on the routing problem for single S-D pair. At first, we propose Q-PATH, a Purification-enabled iterative routing algorithm, to obtain the optimal routing path and purification decisions with minimum entangled pair cost. To further reduce the high computational complexity, we propose Q-LEAP, a Low-complexity routing Algorithm from the perspective of “multiplicative” routing metric of the fidelity degradation.

### A. Problem Definition and Design Overview

1) *Problem Definition*: Given a routing request from single S-D pair and a quantum network with topology and edge

---

### Algorithm 1 Q-PATH: Iterative Routing

---

**Input:**  $G = (V, E, C)$ ,  $F_i^{th}$ , request  $R_i$  and  $\langle s_i, d_i \rangle$ ;

**Output:**  $P_{i,j}(s_i, d_i)$ ,  $D_{i,j}^{pur}$ ,  $F_j(s_i, d_i)$ ,  $T_{i,j}^{EXT}$ ;

```

1 Step 1 Initialization:
2 Calculate Purification Cost Table for  $(u, v) \in E$ ;
3 Delete all edges  $(u, v)$  from  $G$ , if  $F_{max}^{pur}(u, v) < F_i^{th}$ ;
4 Construct auxiliary graph  $G^a = (V, E^a, C^a, Cost)$ ;
5  $Q \leftarrow$  Priority queue according to value of min_cost;
6 Find shortest path on  $G$  with  $H^{min}$  by using BFS;
7 for  $min\_cost = H^{min} : |E|C_{max}$  do
8   Step 2 Path Selection Procedure:
9   Multiple shortest paths set  $P_{min\_cost}^{SPF}$  with the same
   cost  $min\_cost \leftarrow K$ -shortest path algorithm;
10  if no available path for  $\langle s_i, d_i \rangle$  then
11    break;
12  end
13  Step 3 Edge Cost Update:
14  for  $P_{i,j}(s_i, d_i) \in P_{min\_cost}^{SPF}$  with minimum hops do
15    while  $F_{i,j}(s_i, d_i) < F_i^{th}$  do
16      Find  $(u, v) \in P_{i,j}(s_i, d_i)$  with maximum fidelity
      improvement;
17       $N_{i,j}^{pur}(u, v) = N_{i,j}^{pur}(u, v) + 1$ ;
18    end
19  end
20   $Q \leftarrow P_{i,j}(s_i, d_i)$ ,  $cost(P_{i,j}(s_i, d_i))$ ,  $D_{i,j}^{pur}$ ;
21  Step 4 Throughput Update:
22  while  $cost(Q.pop) \leq min\_cost + 1$  do
23    Find  $W_{i,j}^{min}$  along the path  $P_{i,j}(s_i, d_i)$  in  $G^a$ ;
24    if  $W_{i,j}^{min} \geq 1$  then
25      Subtract  $\min\{W_{i,j}, R_i\} \times (N_{i,j}^{pur}(u, v) + 1)$  on
      each  $(u, v) \in P_{i,j}(s_i, d_i)$  from  $C^a$  in  $G^a$ ;
26    end
27     $T_{i,j}^{EXT}(s_i, d_i) \leftarrow$  Calculate expected throughput of
      each edge  $(u, v) \in P_{i,j}(s_i, d_i)$ ;
28    Output  $P_{i,j}(s_i, d_i)$ ,  $D_{i,j}^{pur}$ ,  $\min\{W_{i,j}, R_i\}$ 
       $F_{i,j}(s_i, d_i)$  and delete this solution from  $Q$ ;
29    if  $\sum_j T_{i,j}^{EXT}(s_i, d_i) \geq R_i$  then
30      terminate;
31    end
32  end
33 end

```

---

capacity as  $G = (V, E, C)$ , finding routing solutions, including purification decision  $D_{i,j}^{pur}$  and path selection  $P_{i,j}(s_i, d_i)$ , to enable the entanglement establishment between S-D pairs to satisfy fidelity constraint  $F_{i,j}(s_i, d_i) > F_i^{th}, \forall i$ .

2) *Design Overview*: In order to solve the entanglement routing problem in quantum networks, we start with the investigation on single S-D pair scenario, and first design a routing algorithm that can provide the optimal routing solution with both path selection and purification decisions. By utilizing such algorithm, we can obtain the upper bound of the routing performance and provide guidance for the optimal purification decisions. After that, due to the relatively high computational complexity of the optimal routing algorithm, we further design a heuristic routing algorithm which can efficiently find

near-optimal routing solution with “best quality” path and fidelity guarantee. Here, “best quality” path represents the end-to-end routing path with the minimum fidelity degradation.

### B. Iterative Routing Design for Single S-D Pair

For a single S-D pair, the goal of routing design is to find the routing path satisfying fidelity constraint with minimum entangled pair cost. To reach such goal, the proposed Q-PATH algorithm is described in Algorithm 1. The basic idea of Q-PATH can be described as follows. It searches all possible solutions (including routing path and purification decision) from the lowest routing cost (i.e., the number of consumed entangled pairs) and update purification decision in an iterative manner. In each iteration, the algorithm sets a certain “expected cost value” (i.e.,  $min\_cost$ ), and checks all possible routing solutions if they equals to “expected cost value”. Multiple routing solutions may be found in each iteration by using Step 2 and Step 3, but all routing solutions output in the same iteration must have the same entangled pair cost, i.e.,  $min\_cost$ . Once the actual entangled cost of the routing solution meets the “expected cost value”, the routing solution can be returned by the algorithm as the one with minimum cost.

It should be clarified that Q-PATH considers minimum entangled pair cost (i.e., minimum cost) as the metric to find the optimal routing solution rather than minimum number of hops in classic routing algorithm such as Dijkstra. For a given routing request  $R_i$ , the entangled pair cost means the total cumulation of resource consumption to establish one entanglement connection that satisfies corresponding fidelity threshold  $F_i^{th}$  for  $i$ -th S-D pair, it includes the consumption for basic path establishment (consumes one entangled pair on each edge along the routing path  $P_{i,j}(s,d)$ ) and extra purification operation (consumes  $N_{i,j}^{pur}(u,v)$  entangled pairs on edge  $(u,v)$ ).

In specific, Q-PATH contains 4 steps as follows:

1) *Initialization*: Q-PATH first calculates a “Purification Cost Table” for each edge  $(u,v) \in E$ . Since the fidelity constraint  $F_{i,j}(s_i, d_i) \geq F^{th}$  must be satisfied for the routing path, if an edge  $(u,v)$  cannot provide entangled pairs satisfying  $F_{max}^{pur}(u,v) \geq F^{th}$  even after purifications, it should be deleted on graph  $G$  for complexity reduction. After that, Q-PATH constructs an update graph  $G^a$  to record purification decision. Finally, Q-PATH finds the shortest path on graph  $G$  by using Breadth-First-Search (BFS) to ensure the possible minimum cost  $H^{min}$  used in step 2, and also constructs a priority queue to save potential routing paths during iterations.

2) *Path Selection Procedure*: To obtain the optimal routing path with minimum cost, we design an iterative method that utilizes the cost as the metric to continue each loop, which provides an indicator during the path searching. Here,  $k$ -shortest path algorithm [31] is used to obtain multiple shortest paths with the same cost  $min\_cost$ .

To establish the end-to-end entanglement connection, entanglement swappings are required after the generation of entangled pairs. Considering the imperfect measurement on quantum repeaters, each swapping operation brings fidelity

TABLE II  
PURIFICATION COST TABLE

Round	Fidelity	Fidelity Improvement
0	$F_{min}^{pur}(u,v)$	0
1	$F_1^{pur}(u,v)$	$F_1^{pur}(u,v) - F_{min}^{pur}(u,v)$
2	$F_2^{pur}(u,v)$	$F_2^{pur}(u,v) - F_1^{pur}(u,v)$
...	...	...
$n$	$F_{max}^{pur}(u,v)$	$F_{max}^{pur}(u,v) - F_{n-1}^{pur}(u,v)$

degradation. By adopting extra purification operations, the expected fidelity of the entanglement connection following routing path  $P_{i,j}(s_i, d_i)$  can be calculated as:

$$F_{i,j}(s_i, d_i) = \prod_{(u,v) \in P_{i,j}(s_i, d_i)} F_{i,j}^{pur}(u,v, N_{i,j}^{pur}), \quad (2)$$

where  $F_{i,j}^{pur}(u,v, N_{i,j}^{pur})$  denotes the fidelity of quantum channel on edge  $(u,v)$  after  $N_{i,j}^{pur}$  round purification operations allocated for  $j$ -th routing path of  $i$ -th S-D pair. Given that the purification protocol as shown in Fig. 2(a), the total number of entangled pairs consumed after  $N^{pur}(u,v)$  round purification should be  $N^{pur}(u,v)$ . Thus, fidelity  $F_{i,j}^{pur}(u,v, N_{i,j}^{pur})$  can be calculated by:

$$F_{i,j}^{pur}(u,v, N_{i,j}^{pur}) = \begin{cases} F^0(u,v), & N_{i,j}^{pur} = 0, \\ f(F^0(u,v), F_{i,j}^{pur}(u,v, N_{i,j}^{pur} - 1)), & N_{i,j}^{pur} \geq 1, \end{cases} \quad (3)$$

where  $F^0(u,v)$  denotes the original fidelity of the generated entangled pair on  $(u,v)$ , and  $f(\cdot)$  represents the resulting fidelity of quantum channel after purification in Eq. (1).

3) *Edge Cost Update*: For a given path  $P_{i,j}(s_i, d_i)$ , edge cost (i.e., number of consumed entangled pairs) equals to the throughput without purification. However, due to the fidelity constraint  $F_i^{th}$ , purification might be required on some edges along the path. To guarantee the end-to-end fidelity with minimum entangled pair cost, the purification decision process in line 15-17 is designed to check  $F_{i,j}(s_i, d_i)$  and add the number of purification round  $N_i^{pur}$  once a time. Note that  $N_{i,j}^{pur}(u,v) = 0$  for all  $(u,v) \in E$  at the start of the algorithm.

To provide all possible purification options and corresponding cost, we design a “Purification Cost Table” for each edge  $(u,v) \in E$  as described in Table II. “Purification Cost Table” gives the guidance for the resource consumption of purification operation, and also gives the maximum and the minimum fidelity that each quantum channel can provide. For example, for an edge  $(u,v)$  with capacity  $c(u,v) = 5$ , the original fidelity  $F_{min}^{pur}(u,v) = 0.75$ , then the maximum purification round is 4. According to Eq. (1), the resulting fidelity after 1st round can be easily calculated as  $F_1^{pur}(u,v) = 0.9$ , and the fidelity improvement is  $F_1^{pur}(u,v) - F_{min}^{pur}(u,v) = 0.15$ . Similarly, the resulting fidelity after second round can be easily calculated as  $F_2^{pur}(u,v) = 0.9642$ , and the fidelity improvement is  $F_2^{pur}(u,v) - F_1^{pur}(u,v) = 0.0642$  and the following table entries are so on in the same manner. Note

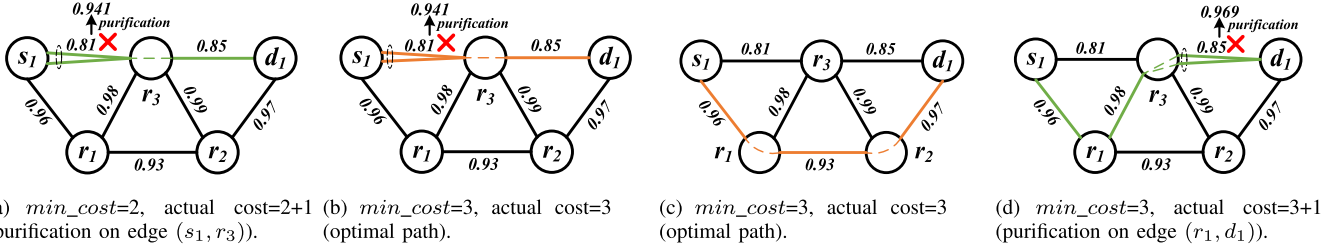


Fig. 3. A routing example for single S-D pair, and the fidelity threshold is  $F_1^{th} = 0.8$ . (a)-(d) show the iterative searching procedure of Q-PATH.

that the maximum fidelity and improvement are  $F_{max}^{pur}(u, v) = 0.9959$  and  $F_{max}^{pur}(u, v) - F_3^{pur}(u, v) = 0.0081$  after 4-round purification, thus we can tell that the fidelity improvement is decreasing along with the increase of purification round on the same edge.

In the following, we prove that the greedy approach used in **Step 3** can find the optimal purification decision. The basic idea behind such greedy approach is that, on the same edge  $(u, v)$ , when the original fidelity of the entangled pair is low, e.g.,  $F^0(u, v) = 0.75$ , the fidelity improvement is high after one purification operation, i.e., 0.15. However, when the original fidelity is high, e.g.,  $F^0(u, v) = 0.95$ , the fidelity improvement obtained after one purification operation significantly decreases, i.e., 0.0472. In other words, the fidelity improvement brought by purification operation has monotonicity. Hence, the greedy approach<sup>5</sup> becomes useful by leveraging the monotonicity property of purification operation. The specific conclusion and proof procedure are given in Theorem 1.

**Theorem 1:** *The greedy approach used in Step 3 of Q-PATH can find the optimal purification decision with minimum entangled pair cost when the original fidelity is above  $x^*$ .*

*Proof:* Please see Appendix A for details. ■

Due to the imperfect measurement on quantum repeaters, the purification is inherently probabilistic in nature [8], [36], [37], [38]. The probability of the successful one-round purification can be calculated by  $P^{pur}(x_1, x_2, 1) = x_1 x_2 + (1 - x_1)(1 - x_2)$ . For multi-round purification operation, the cumulative successful purification operation can be further calculated by:

$$\begin{aligned} & P^{pur}(F^0, N_{i,j}^{pur}(u, v)) \\ & N_{i,j}^{pur}(u, v) \\ &= \prod_{n=1}^{N_{i,j}^{pur}(u, v)} P^{pur}(F^0, F_{i,j}^{pur}(u, v, n), n). \end{aligned}$$

To verify the optimal routing solution including path selection and purification decision obtained by Q-PATH, a performance evaluation would be conducted in Section VI-B compared with brute-force method, and the results show that the routing solution between the one proposed in Q-PATH and the optimal one are the same.

4) **Throughput Update:** The algorithm calculates the maximum achievable throughput  $W_{i,j}$  based on purification

decision  $D_{i,j} = [N_{i,j}^{pur}(s, u_1), \dots, N_{i,j}^{pur}(u_n, d)]$  for given path  $P_{i,j}(s_i, d_i)$ , and judges if the request of  $i$ -th S-D pair is satisfied. Note that available width  $W_{i,j}$  means available number of end-to-end entanglement establishment for  $j$ -th routing path of  $i$ -th S-D pair based on given purification decision. If the request is satisfied, the algorithm terminates and produces the routing path  $P_{i,j}(s_i, d_i)$ , purification decision  $D_{i,j}$ , and expected fidelity  $F_{i,j}(s_i, d_i)$ . Due to the probability of failure situation of purification, the maximum throughput on a given path  $P_{i,j}(s, d)$  would be lower than the one without purification operation. Similar to [4], we define a metric called expected throughput (i.e., expected number of *qubits*) to quantify an arbitrary end-to-end path  $P_{i,j}(s, d)$ :

$$t_{i,j}^{EXT}(u, v) = P^{pur}(F^0(u, v), N_{i,j}^{pur}(u, v)) \times W_{i,j}^{min}.$$

where  $W_{i,j}^{min} = \min_{(u,v) \in P_{i,j}(s_i, d_i)} \lfloor \frac{c(u,v)}{N_{i,j}^{pur}(u,v)+1} \rfloor$  represents the minimum width, i.e., the maximally available number of establishing end-to-end entanglement connections, along the routing path  $P_{i,j}(s_i, d_i)$ . Furthermore, the expected throughput on path  $P_{i,j}(s_i, d_i)$  can be further calculated by:

$$T_{i,j}^{EXT}(s_i, d_i) = \min(t_{i,j}^{EXT}(u, v) | (u, v) \in E).$$

### C. Discussion and Complexity Analysis of Q-PATH

To clearly explain the searching process of Q-PATH, a routing example is given in Fig. 3. The fidelity value of entangled pairs is shown on each edge. Since  $H^{min} = 2$  (at least 2 hops are required to reach the destination) in this example, in the first iteration, the algorithm attempts to find the routing solution with  $\min\_cost = 2$ . Only one potential path with length  $l = 2$  can be found as shown in Fig. 3(a). However, according to line 15, purification is required on edge  $(s_1, r_1)$  to satisfy fidelity constraint. Thus, this routing solution with  $cost = 3$  is enqueued into  $Q$ , and the cost condition in line 22 cannot be satisfied. Similarly, in the second iteration, the algorithm attempts to find the routing solution with  $\min\_cost = 3$ . Two potential paths with length  $l = 3$  can be found as shown in Figs. 3(c)-3(d). After checking the fidelity constraint in line 15, these two routing solutions with  $cost = 3$  and  $cost = 4$  is enqueued into  $Q$ . Then, two routing solutions (illustrated in Figs. 3(b)-3(c) that satisfy the cost condition in line 22 can be found from  $Q$  and they will be outputted. If request  $R_i$  is satisfied with the expected throughput  $t^{EXT}$  provided by the optimal paths, then the algorithm ends. Otherwise, the process repeats as above.

<sup>5</sup>Note that the greedy approach used in Q-PATH requires sufficient entangled pairs on edges to execute the optimal purification operations, otherwise it cannot find any effective solutions.

Based on Theorem 1, the following theorem proves that the proposed algorithm can find the routing path with minimum cost. The basic idea can be explained as follows. For a given routing path, Theorem 1 can guarantee that Q-PATH finds the optimal purification decision with minimum entangled pair cost. Meanwhile, the minimum entangled pair cost for a given routing path is decided by its path length. Thus, Q-PATH searches routing path according to its path length (i.e., minimum entangled pair cost), compares potential routing paths with their actually entangled pair cost after purification operations. If there exists an optimal routing path, it must be close to its minimum entangled pair cost, and it can be found before other routing paths with larger minimum entangled pair cost. The specific conclusion and proof procedure are given in Theorem 2.

**Theorem 2:** *The Q-PATH can find the fidelity-guaranteed routing path with minimum entangled pair cost for arbitrary S-D pair in quantum networks.*

*Proof:* Please see Appendix B for details. ■

The computational complexity of Q-PATH can be analyzed as follows. Let  $|E|$  denote the number of edges in set  $E$ ,  $|V|$  denote the number of nodes in set  $V$ .  $C_{max}$  denotes the maximum number of entangled pairs on an edge  $(u, v) \in E$ . At **Step 1**, the worst complexity of *purification cost table* calculation and BFS can be calculated by  $\mathcal{O}(|E|C_{max})$  and  $\mathcal{O}(|V| + |E|)$ , respectively. At **Step 2**,  $K$  paths would be obtained at most from  $K$ -shortest path algorithm, thus the worst complexity can be calculated by  $\mathcal{O}(K|V|(|E| + |V|\log_2|V|))$ . At **Step 3**, to satisfy the fidelity threshold and make purification decisions, the worst complexity can be calculated by  $\mathcal{O}(K|E|C_{max})$ , where  $|E|C_{max}$  is the judgement times of line 15 in the worst case. At **Step 4**, in the worst case, the throughput update procedure for each path obtained from **Step 2** would be executed  $R_i$  times in total, and the complexity of each throughput update procedure from line 22 to line 31 is  $\mathcal{O}(|E|)$ . In the worst case, the number of iterations in line 7 is  $|E|C_{max}$ . Thus, the worst computational complexity of Q-PATH is  $\mathcal{O}(|V| + |E| + |E|C_{max}(K|V||E| + |V|^2\log_2|V| + K|E|C_{max} + R_i|E|))$ .

As a comparison, the complexity of brute-force approach is  $\mathcal{O}((C_{max})^{|E|})$ . Due to the existence of  $|E|C_{max}|V|^2\log_2V$  and  $K(|E|C_{max})^2$ , the complexity of Q-PATH can rise quickly with the increase of edges and edge capacity. Hence, in the next, we further propose a low-complexity routing algorithm to overcome such problem.

#### D. Low-Complexity Routing Design for Single S-D Pair

Although Q-PATH provides the upper bound of the routing problem for single S-D pair, considering the high computational complexity, we further design a low-complexity routing algorithm, i.e., Q-LEAP, as described in Algorithm 2. Similarly, Q-LEAP contains four steps, i.e., initialization, the path selection procedure, purification decision, and throughput update. The basic idea of Q-LEAP is that, for each iteration in Q-LEAP (line 3 in Algorithm 2), it searches one routing path with “best quality” (Step 2), and makes purification decision to let it satisfy fidelity threshold (Step 3). The process repeats

---

#### Algorithm 2 Q-LEAP: Low-Complexity Routing

---

**Input:**  $G = (V, E, C)$ ,  $F_i^{th}$ , request  $R_i$  and S-D pair;

**Output:**  $P_{i,j}(s_i, d_i)$ ,  $D_{i,j}^{pur}$ ,  $F_j(s_i, d_i)$ , and  $T_{i,j}^{EXT}$ ;

```

1 Step 1 Initialization:
2 Same as line 2-5 in Q-PATH;
3 for  $j = 1 : R_i$  do
4   Step 2 Best Quality Path Searching:
5    $P_{i,j}(s_i, d_i), U(P_{i,j}(s_i, d_i)) \leftarrow$  Using extended Dijkstra
   algorithm to search the path with minimum fidelity
   degradation according to fidelity multiplicative
   Eq. (2);
6   if no available path for  $\langle s_i, d_i \rangle$  then
7     break;
8   end
9   Step 3 Purification Decision:
10  Calculate path length  $l$  and  $F_{i,j}^{avg} = (F_i^{th})^{1/l}$ ;
11  for  $(u, v) \in P_{i,j}(s_i, d_i)$  do
12    if  $F(u, v) < F_{i,j}^{avg}$  then
13       $N_{i,j}^{pur}(u, v) = \arg \min_{N_{i,j}^{pur}(u, v)} F_{i,j}^{pur}(u, v) \geq F_{i,j}^{avg}$ 
      according to Purification Cost Table;
14    end
15  end
16   $Q \leftarrow P_{i,j}(s_i, d_i), \text{cost}(P_{i,j}(s_i, d_i)), D_{i,j}^{pur}$ ;
17  Step 4 Throughput Update:
18  while  $Q.pop! = null$  do
19    Find  $W_{i,j}^{min}$  along the path  $P_{i,j}(s_i, d_i)$  in  $G^a$ ;
20    if  $W_{i,j} \geq 1$  then
21      Subtract  $\min\{W_{i,j}^{min}, R_i\} \times (N_{i,j}^{pur}(u, v) + 1)$  on
      each  $(u, v) \in P_{i,j}(s_i, d_i)$  from  $C^a$  in  $G^a$ ;
22    end
23     $T_{i,j}^{EXT}(s_i, d_i) \leftarrow$  Calculate expected throughput of
    each edge  $(u, v) \in P_{i,j}(s_i, d_i)$ ;
24    Output  $P_{i,j}(s_i, d_i)$ ,  $D_{i,j}^{pur}$ ,  $\min\{W_{i,j}, R_i\}$ 
     $F_{i,j}(s_i, d_i)$  and delete this solution from  $Q$ ;
25    if  $\sum_j T_{i,j}^{EXT}(s_i, d_i) \geq R_i$  then
26      terminate;
27    end
28  end
29 end

```

---

until the request  $R_i$  is satisfied (see terminal condition in line 26). In the following, we discuss two main different parts between the low-complexity routing design in Q-LEAP and the iterative design in Q-PATH.

1) *Best Quality Path Searching:* Unlike the path searching process in Q-PATH, which searches all potential paths with the same cost, Q-LEAP attempts to find “the optimal one” with minimum fidelity degradation via an extended Dijkstra algorithm. As shown in Fig. 4, Q-LEAP finds the path with the highest end-to-end fidelity. Since the classical routing algorithm calculates the sum of the costs of all edges with “additive” routing metric, the original Dijkstra algorithm, which is based on greedy approach, cannot find the shortest path in quantum networks considering “multiplicative” routing metric for fidelity degradation in Eq. (2). Hence, we adopt



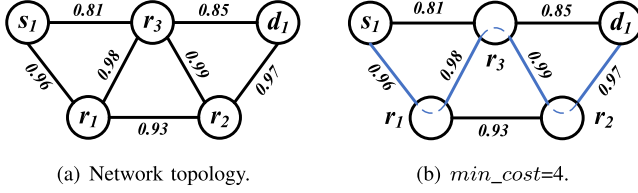


Fig. 4. A routing example for single S-D pair, and the fidelity threshold is  $F_1^{th} = 0.8$ . (a) illustrates the network topology, (b) shows the path with the highest end-to-end fidelity found by extended Dijkstra in Q-LEAP.

non-additive but monotonic routing metric [4], and design an extended Dijkstra algorithm to find the shortest path with minimum fidelity degradation between a given S-D pair.

2) *Purification Decision*: To make purification decisions  $D_{i,j}^{pur}$  with lower computational complexity, we adopt the average fidelity to satisfy the requirement of fidelity threshold  $F_i^{th}$ . For a given path  $P_{i,j}(s_i, d_i)$ , each hop on the path should satisfy  $F_i^{avg}(l) = (F_i^{th})^{\frac{1}{l}}$ , where  $l$  is the length of routing path. By doing this, the worst complexity of purification decision can be significantly reduced to  $\mathcal{O}(|E|)$ .

#### E. Discussion and Complexity Analysis of Q-LEAP

Q-LEAP contains four main steps. At **Step 1**, the complexity of *purification cost table* can be calculated by  $\mathcal{O}(|E|C_{max})$ . At **Step 2**, the worst complexity for path selection can be calculated by  $\mathcal{O}(|V|\log_2|V| + |E|)$ . At **Step 3**, the worst complexity can be calculated by  $\mathcal{O}(|E|)$ . For the loop in line 3, it should be executed  $R_i$  times. The complexity of **Step 4** is  $\mathcal{O}(|E|)$ . Thus, the overall complexity is  $\mathcal{O}(|E|C_{max} + R_i(|V|\log_2|V| + |E|))$ .

### V. ROUTING DESIGN FOR MULTIPLE S-D PAIRS

In this section, based on aforementioned routing designs for single S-D pair, we further propose a greedy-based entanglement routing design to obtain fidelity-guaranteed routing paths for multiple S-D pairs, and design a utility metric with two important factors for resource allocation.

#### A. Problem Definition and Design Overview

1) *Problem Definition*: Given multiple routing requests from multiple S-D pairs and a quantum network with topology and edge capacity as  $G = (V, E, C)$ , finding routing solutions, including purification decision  $D_{i,j}^{pur}$  and path selection  $P_{i,j}(s_i, d_i)$ , to enable the entanglement establishment between S-D pairs to satisfy fidelity constraint  $F_{i,j}(s_i, d_i) > F_i^{th}, \forall i$ .

2) *Design Overview*: Based on the routing solutions obtained by the proposed two algorithms for single S-D pair, the routing problem in multiple S-D pairs scenario can be further regarded as a resource allocation problem. When sufficient resource (i.e., entangled pairs) is provided in the network, the final solution is the combination of the optimal/near-optimal routing solution for single S-D pair. Otherwise, the requests which cannot be satisfied should be denied. Thus, we consider

#### Algorithm 3 Greedy-Based Routing for Multiple S-D Pairs

---

**Input:**  $G = (V, E, C)$ ,  $F_i^{th}$ , requests  $R_i$  and S-D pairs;  
**Output:**  $P_{i,j}(s_i, d_i)$ ,  $D_{i,j}$ ,  $F_j(s_i, d_i)$ ,  $T_{i,j}^{EXT}$ ;

- 1 **Step 1 Initialization:**
- 2 Same as line 2-5 in Q-PATH;
- 3 Construct residual graph  $G^r = (V, E^r, C^r)$ ;
- 4 **Step 2 Routing Path Predetermination:**
- 5 **for all S-D pairs do**
- 6   Find routing solutions  $\leftarrow$  Q-PATH or Q-LEAP;
- 7    $Q \leftarrow P_{i,j}(s_i, d_i)$  according to utility  $U_{i,j}$ ;
- 8 **end**
- 9 **while  $Q$  is not empty do**
- 10   **Step 3 Resource Allocation:**
- 11    $P_{i,j}(s_i, d_i) \leftarrow Q$  and let  $W_{i,j}^{min} = \text{INT\_MAX}$ ;
- 12   **for  $(u, v) \in P_{i,j}(s_i, d_i)$  do**
- 13     **if  $\lfloor \frac{c(u,v)}{N_{i,j}^{pur}(u,v)+1} \rfloor \leq W_{i,j}^{min}$  then**
- 14        $W_{i,j}^{min} = \lfloor \frac{c(u,v)}{N_{i,j}^{pur}(u,v)+1} \rfloor$ ;
- 15     **end**
- 16   **end**
- 17   **if  $W_{i,j}^{min} \geq 1$  then**
- 18     Subtract  $\min\{W_{i,j}^{min}, R_i\} \times (N_{i,j}^{pur}(u, v) + 1)$  on each  $(u, v) \in P_{i,j}(s_i, d_i)$  from  $C^r$  in  $G^r$ ;
- 19     Output  $P_{i,j}(s_i, d_i)$ ,  $D_{i,j}^{pur}$ ,  $F_j(s_i, d_i)$ ,  $T_{i,j}^{EXT}$ ;
- 20     **if  $\sum_j T_{i,j}^{EXT}(s_i, d_i) \geq R_i$  then**
- 21       Remove all paths  $P_{i,j}(s_i, d_i), \forall j$  from  $Q$ ;
- 22     **end**
- 23   **end**
- 24   **Step 4 Re-routing Process:**
- 25   **else**
- 26     Find routing solutions  $\leftarrow$  Q-PATH or Q-LEAP;
- 27     **if  $P_{i,j+1}(s_i, d_i) \neq P_{i,j}(s_i, d_i)$  then**
- 28        $Q \leftarrow P_{i,j+1}(s_i, d_i)$  according to utility  $U_{i,j}$ ;
- 29     **end**
- 30     Delete  $P_{i,j}(s_i, d_i)$  from  $Q$ ;
- 31   **end**
- 32 **end**

---

two important allocation metrics, degree of freedom and resource consumption, to evaluate the performance of each routing solution for single S-D pair, and design a greedy-based routing algorithm to achieve efficient resource allocation for requests from multiple S-D pairs.

#### B. Algorithm Procedure

The greedy-based routing design is described in Algorithm 3. The basic idea of the algorithm can be described as follows: it first calculates the routing path for single S-D pair by using the algorithms proposed in Section IV. To satisfy the requests from multiple S-D pairs as many as possible, we then allocate the network resource for each obtained routing solutions (path selection and purification decision) one by one. For each loop in line 9, the corresponding resource  $\min\{W_{i,j}^{min}, R_i\} \times (N_{i,j}^{pur}(u, v) + 1)$  consumed on each edge  $(u, v) \in P_{i,j}(s_i, d_i)$  would be deleted on the

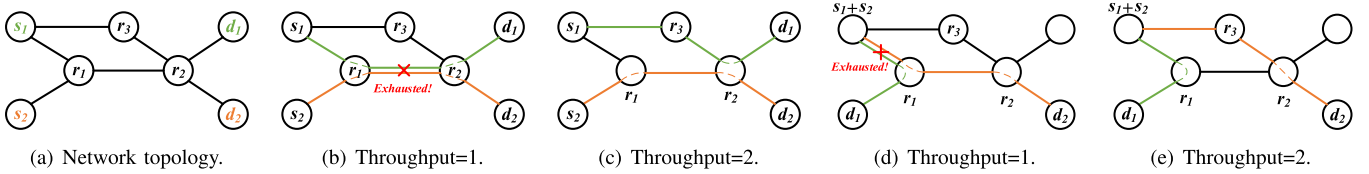


Fig. 5. Routing example for multiple S-D pairs, only one entangled pair is considered on each edge.

residual graph  $G^r$ . If the resource of some edges on the routing path  $P_{i,j}(s_i, d_i)$  has been exhausted, leading to an invalid path on  $G^r$ , a re-routing process will be performed on the residual graph  $G^r$ , and the re-routing solution for  $i$ -th S-D pair will be enqueued into priority queue  $Q$  again and continue loop in line 9.

In Algorithm 3, the most important procedure, which has the most significant impact on the performance, is the order that which routing solution in  $Q$  should be allocated first in resource allocation process. For example, in Figs. 5(a)-5(b), there are two requests for  $\langle s_1, d_1 \rangle$  and  $\langle s_2, d_2 \rangle$ . By executing Q-PATH or Q-LEAP, we obtain two routing paths  $P_{1,1}(s_1, d_1) = \{(s_1, r_1), (r_1, r_2), (r_2, d_1)\}$  and  $P_{2,1}(s_2, d_2) = \{(s_2, r_1), (r_1, r_2), (r_2, d_2)\}$ . If we first allocate the network resource for  $P_{1,1}(s_1, d_1)$ , then the resource on edge  $(r_1, r_2)$  would be exhausted. After that, the request for  $\langle s_2, d_2 \rangle$  has to be denied since  $P_{2,1}(s_2, d_2)$  becomes invalid and there is no other available routing solution for  $\langle s_2, d_2 \rangle$ , which leads to a poor performance in terms of throughput.

### C. Resource Allocation

To address the resource allocation problem, we introduce a utility metric to evaluate the degree of difficulty when the algorithm attempts to satisfy a given request, two important factors of a given routing solution is considered in the utility, i.e., resource consumption and degree of freedom:

$$U_{i,j} = \alpha \cdot G(P_{i,j}(s_i, d_i)) + \beta \cdot S(P_{i,j}(s_i, d_i), D_{i,j}^{pur}), \quad (4)$$

where  $\alpha$  and  $\beta$  are weight coefficient,  $S(\cdot)$  and  $G(\cdot)$  denotes the total resource consumption and the degree of freedom of a given routing path, respectively.

1) For degree of freedom, it is defined as the sum of routing options on each hop, which can be calculated by:

$$G(P_{i,j}(s_i, d_i)) = \sum_{\text{nodes on } P_{i,j}(s_i, d_i)} \mathcal{N}(u).$$

The degree of freedom describes the successful re-routing possibility. For example, Figs. 5(b)-5(c) show the advantage by considering degree of freedom as a factor when the algorithm decides the order of resource allocation, routing paths  $P_{1,1}(s_1, d_1)$  and  $P_{2,1}(s_2, d_2)$  are found by the Q-PATH or Q-LEAP as shown in Fig. 5(b), and  $G(P_{1,1}(s_1, d_1)) = 10$  and  $G(P_{2,1}(s_2, d_2)) = 9$  in Fig. 5(b). If we check  $P_{1,1}(s_1, d_1)$  first, the resource on edge  $(r_1, r_2)$  would be exhausted. Then  $P_{2,1}(s_2, d_2)$  becomes invalid and there is no other available routing path for  $\langle s_2, d_2 \rangle$ . If we check all routing solutions according to the value of degree of freedom, then the resource would be allocated to  $P_{1,1}(s_1, d_1)$  first since

$G(P_{2,1}(s_2, d_2)) < G(P_{1,1}(s_1, d_1))$ . In this case, maximum throughput is reached as shown in Fig. 5(c).

2) For resource consumption, it is defined as the sum of consumed entangled pairs for a given routing solution, which can be calculated by:

$$S(P_{i,j}(s_i, d_i), D_{i,j}^{pur}) = \sum_{(u,v) \in P_{i,j}(s_i, d_i)} N_{i,j}^{pur}(u, v).$$

Considering the limited resource of entangled pairs on edges, a routing path with lower resource consumption should be allocated earlier, in case the routing path with higher resource consumption consumes too much entangled pairs and makes the requests from other S-D pairs be denied. An example is shown in Figs. 5(d)-5(e), which show the advantage by considering resource consumption as a factor when the algorithm decides the order of resource allocation.  $P_{1,1}(s_1, d_1)$  is a shorter path and consumes less entangled pairs, and  $S(P_{1,1}(s_1, d_1), D_{1,1}^{pur}) = 2$  and  $S(P_{2,1}(s_2, d_2), D_{2,1}^{pur}) = 3$  in 5(d). If we check  $P_{2,1}(s_2, d_2)$  first, the resource on edge  $(s_2, r_1)$  would be exhausted, then  $P_{1,1}(s_1, d_1)$  becomes invalid and there is no other available routing solutions for  $\langle s_2, d_2 \rangle$ . In this case, as shown in Fig. 5(e), if we check all routing solutions according to the value of resource consumption  $S(\cdot)$ , maximum throughput can be achieved.

### D. Computational Complexity Analysis

The computational complexity of Algorithm 3 can be analyzed as follows. At **Step 2**, the number of iterations is decided by the number of S-D pairs  $R_i$ , and the complexity of each path selection is the same as Q-PATH or Q-LEAP. At **Step 3**,  $K$  paths would be obtained at most from Q-PATH or Q-LEAP, the complexity can be calculated by  $\mathcal{O}(K|E|)$ . At **Step 4**, re-routing procedure can be executed  $\sum_i R_i$  times at most.

As a comparison, the complexity of brute-force searching approach is  $\mathcal{O}\left(\sum_i R_i (C_{max})^{|E|}\right)$ .

## VI. PERFORMANCE ANALYSIS

### A. Evaluation Setup

To implement the proposed routing algorithms in quantum networks, we conduct a series of numerical evaluations. The simulation code is available in [39]. The software and hardware configuration of simulation platform are: AMD Ryzen 7 3700X 3.6GHz CPU, 32GB RAM, O.S. Windows 10 64bits. Along with the development of quantum networks, it can be foreseen that quantum networks will become a critical infrastructure for security communications and quantum applications in the future, which is similar to the current Internet

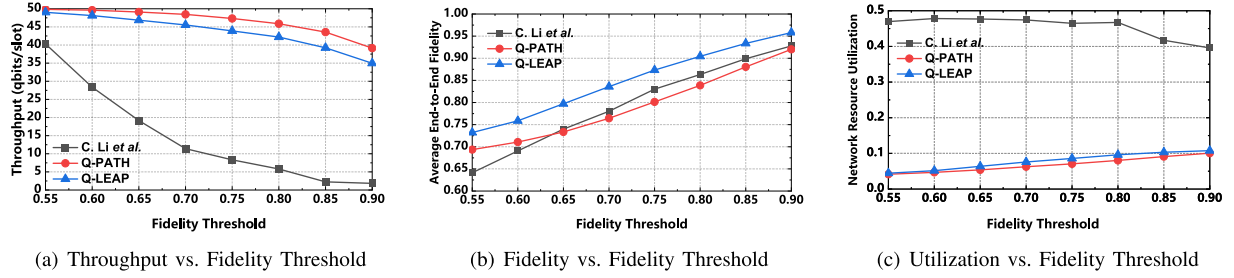


Fig. 6. Performance comparison for single S-D pair in terms of throughput, average fidelity, and network resource utilization (channel capacity = 50).

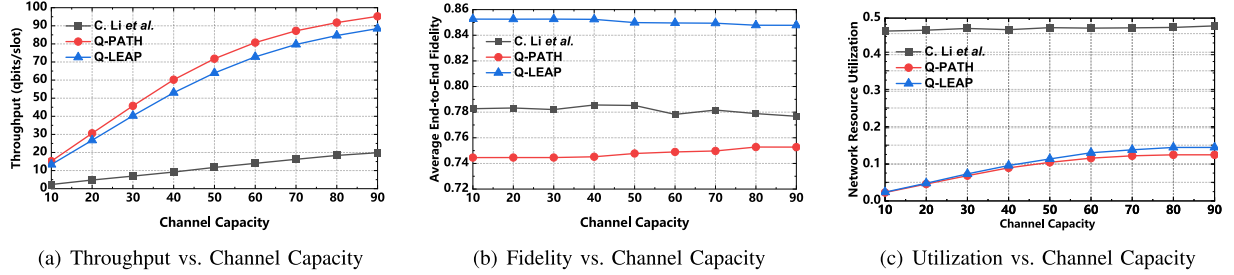


Fig. 7. Performance comparison for single S-D pair in terms of throughput, average fidelity, and network resource utilization (fidelity threshold = 0.7).

TABLE III  
ALGORITHM RUNNING TIME (FIDELITY THRESHOLD = 0.6,  
CHANNEL CAPACITY = 10)

Network Scale	Q-PATH	Q-LEAP	Baseline
100	282.25ms	0.47ms	46.56ms
200	307.81ms	1.4ms	105.78ms
300	381.52ms	2.55ms	180.78ms
400	1056.11ms	3.87ms	275.46ms
500	3644.32ms	6.76ms	376.56ms

backbone. Thus, the US backbone network [40] is adopted as the examined network topology in our simulation. For each set of parameter settings, simulations are run 1000 trials and the averaged results are given, the original fidelity of entangled pairs follows  $\mathcal{N}[0.8, 0.1]$ . According to [15], the typical qubit lifetime is 1.46s, thus we adopt the synchronization timestep as 500ms. **The performance of routing scheme proposed in [6] is adopted as the baseline** since it is the only existing routing design that considers purification operation, and proportional share is adopted as its resource allocation scheme. In specific, US backbone network topology is used in Figs. 6-11.

To evaluate the efficiency of the proposed algorithms, random network topologies are also generated in the simulations following the Waxman model [4], [7], [41], i.e., the probability that there is an edge between node  $u$  and  $v$  can be formulated as  $p(u, v) = \kappa \exp \frac{-d(u, v)}{L^\gamma}$  [41, Eq. (4)], where  $d(u, v)$  is the Euclidean distance (measured in kilometers) between node  $u$  and  $v$ ,  $L$  is the largest distance between two arbitrary nodes. In specific, random network topologies are used in Table III.

### B. Results Under Single S-D Pair Scenarios

1) *Computational Complexity Comparison*: The running time of three routing algorithms is shown in Table III. As we

analyzed before, although Q-PATH can achieve the best routing performance, the complexity of Q-PATH is relatively high, which makes it time-consuming. As the results in Table III, with the increase of network scale, the running time of Q-PATH is 5x-10x higher than the baseline, but the running time of Q-LEAP is 50x lower than the baseline.

2) *Performance Comparison vs. Fidelity Threshold*: As shown in Fig. 6, the performance of the proposed Q-PATH and Q-LEAP is evaluated compared with the baseline. In Fig. 6(a), Q-PATH obtains the highest throughput, and the gap between Q-PATH and Q-LEAP reaches the peak at fidelity threshold of 0.85. The reason can be explained as follows. With the increase of fidelity threshold, multi-round purification operations are required to satisfy end-to-end fidelity requirement, then available entangled pairs on each quantum channel are reducing from default 50 pairs to several pairs after necessary purification. Thus, when the fidelity threshold is small, the gap between Q-PATH and Q-LEAP is negligible since multi-round purification operations are unnecessary. Once the number of available entangled pairs is limited, the solution space is reducing, which leads to similar routing solution obtained by Q-PATH and Q-LEAP. For the baseline, due to the purification is performed before path selection, end-to-end fidelity can hardly be guaranteed. Thus, a poor performance in terms of throughput is obtained.

In Fig. 6(b), Q-PATH also achieves the minimum fidelity but above fidelity threshold. This phenomenon shows that the purification decision and path selection obtained from Q-PATH can achieve the minimum cost to provide end-to-end fidelity guarantee. In Fig. 6(c), the network resource utilization is calculated as the ratio of the consumed entanglement pairs and the total entanglement pairs in the network. Although Q-PATH can achieve the highest throughput, the resource utilization of Q-PATH is lower than Q-LEAP.

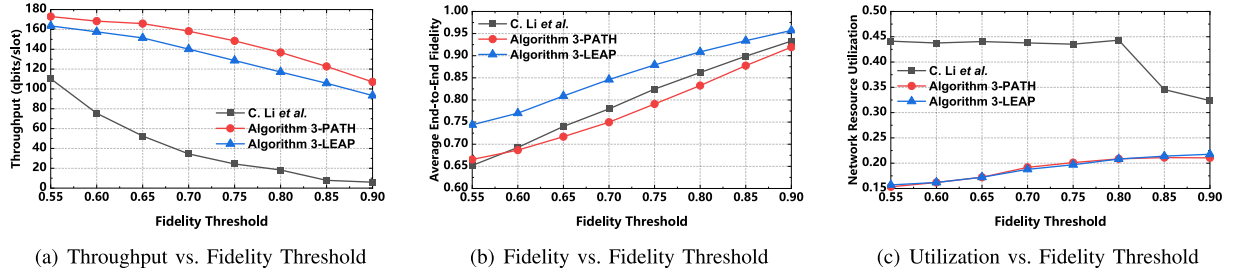


Fig. 8. Performance comparison for multiple S-D pairs versus fidelity threshold (channel capacity = 50, SD pairs = 4).

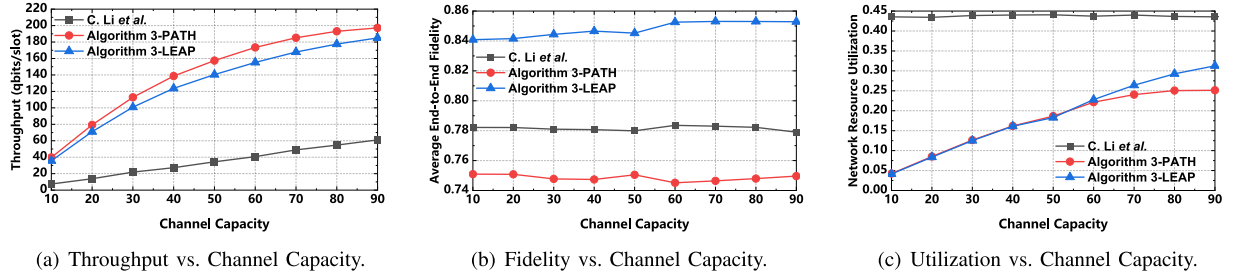


Fig. 9. Performance comparison for multiple S-D pairs versus channel capacity (fidelity threshold = 0.7, SD pairs = 4).

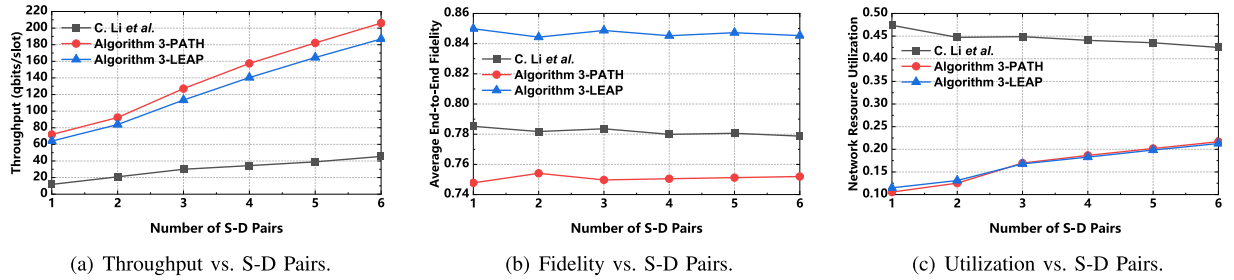


Fig. 10. Performance comparison for multiple S-D pairs versus the number of S-D pairs (fidelity threshold = 0.7, channel capacity = 50).

3) *Performance Comparison vs. Channel Capacity*: As shown in Fig. 7, the performance of the proposed Q-PATH and Q-LEAP is evaluated compared with the baseline. In Fig. 7(a), similarly, *Algorithm* Q-PATH obtain the highest throughput, but the gap between Q-PATH and the other results is expanding with the increase of channel capacity. This phenomenon shows the superiority of the proposed algorithm, and Q-PATH can achieve better performance with sufficient entanglement resource in the network.

In Fig. 7(b), the established end-to-end entanglement connection obtained from Q-PATH always has the lowest fidelity to fidelity threshold. Q-LEAP and the baseline has similar fidelity. In Fig. 7(c), due to the advance purification operations of the baseline, it has the highest resource utilization ratio. Similar to the results in Fig. 7(c), Q-PATH can also achieve the highest throughput but similar resource utilization as Q-LEAP.

### C. Results Under Multiple S-D Pairs Scenarios

As shown in Figs. 8-10, the performance of the proposed *Algorithm 3*<sup>6</sup> is evaluated compared with the baseline, the normalization is adopted for weight coefficient  $\alpha$  and  $\beta$ ,

<sup>6</sup>In the simulation, *Algorithm 3-PATH* represents Q-PATH is used in the algorithm, *Algorithm 3-LEAP* represents Q-LEAP is used in the algorithm.

i.e.,  $\alpha = \frac{\alpha^*}{2|E|}$  and  $\beta = \frac{\beta^*}{|E|C_{channel}^{el}}$ , where  $C_{channel}$  represents channel capacity, and  $\alpha^*$  and  $\beta^*$  both takes 0.5 in the utility metric. For fair comparison, we allocate 50 requests for each S-D pair in the simulation.

1) *Performance Comparison vs. Fidelity Threshold*: In Fig. 8(a), *Algorithm 3-PATH* obtains the highest throughput, but the gap between *Algorithm 3* and the gap between *Algorithm 3-PATH* and *Algorithm 3-LEAP* is relatively stable, which is inherently caused by the performance gap from Q-PATH and Q-LEAP. For the baseline, due to the purification is performed before path selection, end-to-end fidelity can hardly be guaranteed. Thus, a poor performance in terms of throughput is obtained.

In Fig. 8(b), since *Algorithm 3-PATH* always finds the routing solution with minimum entangled pair cost, the fidelity of the obtained solution is also minimum but above the fidelity threshold. Compared to *Algorithm 3-PATH* and the baseline, *Algorithm 3-LEAP* only selects one path with “best quality”, which provides a higher fidelity for each obtained routing solution in nature. Thus, *Algorithm 3-LEAP* obtains the routing solutions with the highest fidelity. In Fig. 8(c), *Algorithm 3-PATH* and *Algorithm 3-LEAP* utilizes similar entangled pair resource to build end-to-end connection. The reason why the resource utilization of the baseline drops when fidelity threshold larger than 0.8 is that, the higher fidelity

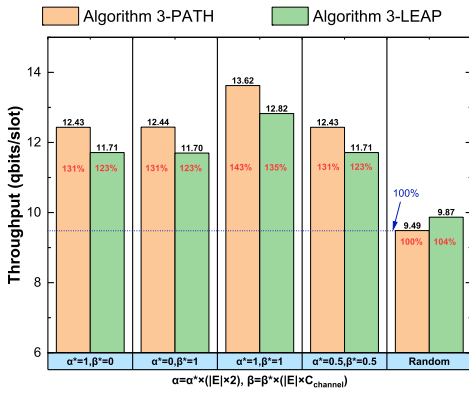


Fig. 11. Performance comparison for multiple S-D pairs versus weight coefficient (fidelity threshold = 0.7, channel capacity = 2, number of S-D pairs = 10).

constraint prevents the successful connection establishments for requests and most of the requests will be denied. Thus, the resource utilization of the baseline significantly drops when fidelity threshold becomes larger.

2) *Performance Comparison vs. Channel Capacity*: In Fig. 9(a), along with the increase of channel capacity, three routing schemes obtain significant improvement in terms of throughput, *Algorithm 3-PATH* always obtains the highest throughput, and the gap between *Algorithm 3-PATH* and *Algorithm 3-LEAP* reaches the peak at channel capacity of 50. This expected phenomenon can be explained as follows. Since the solution space of purification operation is related to channel capacity, *Algorithm 3-PATH* can find superior solutions with higher channel capacity compared with *Algorithm 3-LEAP* and the baseline. Due to the limited requests of each S-D pair, i.e., 50 requests, the gap between two methods of *Algorithm 3* becomes smaller when the channel capacity further increases. The gap between *Algorithm 3-PATH* and *Algorithm 3-LEAP* is 11.4%-6.5% with the increase of channel capacity.

In Fig. 9(b), due to the same setting of fidelity threshold, the fidelity of established end-to-end entanglement connections obtained by three routing schemes is relatively stable. *Algorithm 3-PATH* obtains the lowest fidelity that satisfies threshold, and has the lowest resource consumption as shown in Fig. 9(c), which indicates that it can obtain better routing decisions and purification decisions with minimum entangled pair cost. Along with the increase of channel capacity, the gap between *Algorithm 3-PATH* and *Algorithm 3-LEAP* is also expanding. This phenomenon is similar to the one in single S-D pair scenario, but significantly magnified by multiple S-D pairs.

3) *Performance Comparison vs. S-D Pairs*: In Fig. 10(a), the throughput of three routing schemes is rising along with the increase of the number of S-D pairs. As the routing scheme with best performance, *Algorithm 3-PATH* obtain 12.3%-10.2% and 510.7%-353.2% improvement compared with *Algorithm 3-LEAP* and the baseline, respectively.

In Fig. 10(b), due to the same setting of fidelity threshold, the fidelity of established end-to-end entanglement connec-

tions obtained by three routing schemes is relatively stable. In Fig. 10(c), since the purification decisions of the baseline are made in advance, most of the resource on each quantum channel has been exhausted to generate entangled pairs with higher fidelity. Hence, along with the increase of the number of S-D pairs, the baseline can hardly satisfy the requests from multiple S-D pairs, and lots of requests are denied, which causes the slight decrease of resource utilization ratio. For the proposed routing schemes, the purification decisions are made according to the requests of various S-D pairs, thus the resource utilization ratio of *Algorithm 3-PATH* and *3-LEAP* is rising with the increase of the number of S-D pairs.

4) *Performance comparison vs. Weight coefficient*: In Fig. 11, the relationship between the setting of weight coefficient and the performance of the proposed algorithms is evaluated. The normalization is adopted for weight coefficient  $\alpha$  and  $\beta$ , and  $C_{channel}$  and  $|E| = 122$  represent channel capacity and the number of edges, respectively. At first, to prove the effectiveness of the proposed resource allocation method based on the utility metric in Eq. (4), we adopt “Random” resource allocation method and set it as performance benchmark (100% in the figure). As a comparison, the proposed resource allocation method has significant superiority with 23% – 43% performance improvement. Second, the performance comparison under different weight coefficient settings is also conducted. For the given network topology, i.e., US backbone network, both factors considered in the utility function, i.e., degree of freedom and resource consumption, can provide performance improvement compared with “Random” resource allocation method. If we only consider one factor by setting  $\alpha^* = 1, \beta^* = 0, \alpha^* = 0.5, \beta^* = 0.5$  or  $\alpha^* = 0, \beta^* = 1$ , the performance can be improved 31% and 23% for *Algorithm 3-PATH* and *Algorithm 3-LEAP*, respectively. Furthermore, if we both consider two factors by setting  $\alpha^* = 1, \beta^* = 1$ , the performance can be further improved 43% and 35%, respectively. Due to the significant influence of the proposed two factors in Eq. (4), it can lead to better routing performance by jointly considering degree of freedom and resource consumption.

## VII. CONCLUSION

In this paper, we studied purification-enabled entanglement routing designs to provide end-to-end fidelity guarantee for various quantum applications. Considering difficulty of entanglement routing designs, we started with single S-D pair scenario, and proposed Q-PATH, an iterative entanglement routing algorithm, and proved the optimality of the algorithm. To further reduce the high computational complexity, we proposed Q-LEAP, a low-complexity routing algorithm which considers “the shortest path” with minimum fidelity degradation and a simple but effective purification decision method. Based on the routing designs for single S-D pair scenarios, the design of multiple S-D pair scenarios could be regarded as the resource allocation problem among multiple single S-D routing solutions, and a greedy-based routing algorithm was further proposed. To efficiently allocate resource

of entangled pairs for different requests and corresponding routing solutions, two allocation metrics were considered, i.e., degree of freedom and resource consumption.

The superiority of the proposed routing designs was proved by the extensive simulations. For the proposed routing algorithms, Q-PATH achieves the optimal performance with relatively high computational complexity, and Q-LEAP achieves near-optimal performance but highly efficiency, and both algorithms provide significant performance improvement compared to the existing purification-enabled routing scheme. In practice, the performance superior of Q-PATH makes it suitable for network optimization and the efficiency of Q-LEAP makes it suitable for a large-scale quantum networks. In the future, we will study the routing problem under “*on-demand generation*” model with fidelity constraint, and also explore the inherent relationship between the value of fidelity and the possibility of *qubit* error.

#### APPENDIX A PROOF OF THEOREM 1

*Proof:* At first, we prove that the purification operation to the entangled pair close to  $x^*$  can bring the highest improvement. Let the original fidelity of two entangled pairs in the first-round purification as  $x_1, x_1$ , and the original fidelity of two entangled pairs in the second-round purification as  $x_1, x_2$ , where  $x_2 = f(x_1, x_1)$ . Since  $x_2 > x_1$ , we can have  $x_1 x_2 + (1 - x_1)(1 - x_2) > x_1^2 + (1 - x_1)^2$ . Thus, we can have:

$$f(x_1, x_2) - f(x_1, x_1) = \frac{x_1^2 + x_2 - x_1 x_2 - x_1}{A} \stackrel{(a)}{=} \frac{2x_1^2 - 3x_1 + 1}{x_1^2 + (1 - x_1)^2} \cdot A \stackrel{(b)}{<} 0.$$

Here,  $A = (x_1 x_2 + (1 - x_1)(1 - x_2)) (x_1^2 + (1 - x_1)^2)$  in (a) and we utilize the following conclusion in (b): When  $x_1, x_2 \in [0.5, 1]$ , we can obtain that  $A > 0$ ,  $x_1^2 + (1 - x_1)^2 > 0$  and  $2x_1^2 - 3x_1 + 1 < 0$ , and thus  $f(x_1, x_2) - f(x_1, x_1) < 0$ . Similar proof can be done when multi-round purification is considered. In this case, the improvement of any multi-round purification cannot be higher than the first-round purification. Thus, let  $x_1 = x_2$ , the resulting fidelity after purification can be calculated by Eq. (1), and the derivative of Eq. (1) can be obtained as:

$$\frac{df(x)}{dx} = \frac{-2x^2 + 2x}{4x^4 - 8x^3 + 8x^2 - 4x + 1}.$$

Let  $\frac{df(x)}{dx} = 1$ , we can obtain  $x^* \approx 0.743$  when  $x \in [0.5, 1]$ , and  $\frac{df(x)}{dx} > 0$  and  $\frac{d^2f(x)}{dx^2} < 0$  remain when  $x \in [0.5, 1]$ . In this case, for one-round purification, if the original fidelity equals to  $x^*$ , the highest fidelity improvement can be obtained. In other words, among multiple entangled pairs on different edges, the purification operation to the entangled pair with the lowest fidelity can bring the highest improvement when the original fidelity is above  $x^*$ .

Next, we prove that the greedy approach can lead to the optimal purification decision with minimum entangled pair cost. For a given path  $P_{i,j}(s_i, d_i)$ , we assume the original fidelity of the entanglement connection following routing path  $P_{i,j}(s_i, d_i)$  is lower than the fidelity threshold but higher

than  $x^*$ , i.e.,  $x^* < F_{i,j}(s_i, d_i) < F_i^{th}$ . We assume that  $\widehat{D}_{i,j}^{pur}$  is the optimal purification decision with minimum  $N^* = \sum_{(u_1, u_2) \in P_{i,j}(s_i, d_i)} N_{i,j}^{pur}(u_1, u_2)$  purification operations. When  $N^* = 1$ , if the fidelity of entangled pairs on edge  $(v_1, v_2)$  is the minimum fidelity, then we have purification decision  $D_{i,j}^{pur}$  with  $N_{i,j}^{pur}(v_1, v_2) = 1$ . According to the monotonicity obtained from the first and second derivative when the original fidelity belongs to  $[x^*, 1]$ , we can obtain that:

$$\begin{aligned} & \widehat{F}_{i,j}(s_i, d_i) - F_{i,j}(s_i, d_i) \\ &= (y_1 + \delta_1) y_2 - y_1 (y_2 + \delta_2) = \delta_1 y_2 - \delta_2 y_1 \stackrel{(c)}{\leq} 0, \end{aligned} \quad (5)$$

where  $\widehat{F}_{i,j}(s_i, d_i)$  is the fidelity of entanglement connection under the optimal purification decision  $\widehat{D}_{i,j}^{pur}$ ,  $y_1 = F_{i,j}^{pur}(u_1, u_2, 0)$ ,  $y_2 = F_{i,j}^{pur}(v_1, v_2, 0)$ ,  $\delta_1 = F_{i,j}^{pur}(u_1, u_2, 1) - F_{i,j}^{pur}(u_1, u_2, 0)$ , and  $\delta_2 = F_{i,j}^{pur}(v_1, v_2, 1) - F_{i,j}^{pur}(v_1, v_2, 0)$ . (c) holds due to  $y_1 > y_2$  and  $\delta_1 < \delta_2$ . Note that  $F_{i,j}^{pur}(u_1, u_2, N^*)$  is the resulting fidelity after  $N^*$  round purification on edge  $(u_1, u_2)$  in Eq. (3). Thus, the equality in Ineq. (5) can be only obtained when  $\widehat{D}_{i,j}^{pur} = D_{i,j}^{pur}$ , which implies the purification decision  $D_{i,j}^{pur}$  based on greedy approach is the optimal one when  $N^* = 1$ .

After that, when  $N^* = 2$ , we can assume  $\widehat{F}_{i,j}(s_i, d_i) = F_{i,j}(s_i, d_i)$  after the first purification operation. Then the situation goes back to the one when  $N^* = 1$ . Thus, the rest proof when  $N^* > 2$  can be done in the same manner. This completes the proof. ■

#### APPENDIX B PROOF OF THEOREM 2

*Proof:* Here, we prove this theorem by contradiction. For a S-D pair  $\langle s, d \rangle$ , we assume  $P^*(s, d)$  is the routing path that satisfies fidelity constraint  $F^{th}$  with minimum entangled pair cost  $cost^*$  for  $\langle s, d \rangle$ , and  $P'(s, d)$  is the routing path found by Q-PATH with  $cost' > cost^*$ . Then, the mathematical induction is used in the following:

1. At the beginning, the path with minimum hops  $H^{min}$  on graph  $G$  is found by using Breadth-First-Search (BFS). Thus,  $cost^* \geq H^{min}$  should be satisfied. For the first iteration in line 7, the algorithm would search the shortest paths  $P_{H^{min}}^{SPF}$ . To satisfy the fidelity constraint  $F^{th}$ , the cost of each path  $P_{i,j}(s, d) \in P_{H^{min}}^{SPF}$  is not less than  $H^{min}$ . If the cost of the path  $P'(s, d) \in P_{H^{min}}^{SPF}$  equals to  $H^{min}$  after **Step 3**, and  $cost^* = H^{min}$ , then  $P^*(s, d)$  must be a shorter path than the one found by BFS, it reaches a contradiction from the assumption that  $H^{min}$  is minimum hops for  $\langle s, d \rangle$ .

2. For the second iteration in line 7, we have  $min\_cost = H^{min} + 1$ . We assume that the path with  $min\_cost = H^{min}$  is not found in the first iteration. If the algorithm finds a path that satisfies condition in line 22, and  $cost^* = H^{min} + 1$ , then  $cost^*$  must equal  $H^{min}$ . Path  $P^*(s, d)$  and  $P'(s, d)$  must have relationship:

$$\sum_{(u,v) \in P^*(s,d)} cost(u,v) = \sum_{(u',v') \in P'(s,d)} cost(u',v') - 1.$$

Since the shortest paths  $P_{H^{min}}^{SPF}$  have already been checked in the first iteration. If any path  $P_{i,j}(s, d) \in P_{H^{min}}^{SPF}$  satisfies

the fidelity constraint, condition in line 22 should be satisfied. Thus, we obtain a contradiction from the assumption that a path with  $\min\_cost = H^{min}$  is not found in the first iteration.

3. After  $k$  iterations in line 7, we have  $\min\_cost = H^{min} + k$ . We assume that the algorithm first finds a path that satisfies condition in line 22. If  $cost^* < H^{min} + k$ , there must exist at least one routing path that is not stored in priority queue  $Q$ , then path  $P^*(s, d)$  and  $P'(s, d)$  must have relationship:

$$\sum_{(u,v) \in P^*(s,d)} cost(u,v) < \sum_{(u',v') \in P'(s,d)} cost(u',v').$$

Similarly, we obtain a contradiction that a path with  $\min\_cost < H^{min} + k$  is not found in the first  $k - 1$  iterations. This completes the proof. ■

#### ACKNOWLEDGMENT

The authors sincerely thank the editor, Dr. Angela Sara Cacciapuoti and the anonymous referees for their valuable suggestions that have led to the present improved version of the original manuscript.

#### REFERENCES

- [1] J. Yin *et al.*, "Satellite-based entanglement distribution over 1200 kilometers," *Science*, vol. 356, no. 6343, pp. 1140–1144, 2017.
- [2] H. J. Kimble, "The quantum internet," *Nature*, vol. 453, no. 7198, pp. 1023–1030, Jun. 2008.
- [3] Y.-F. Pu, N. Jiang, W. Chang, H.-X. Yang, C. Li, and L.-M. Duan, "Experimental realization of a multiplexed quantum memory with 225 individually accessible memory cells," *Nature Commun.*, vol. 8, no. 1, pp. 1–6, Aug. 2017.
- [4] S. Shi and C. Qian, "Concurrent entanglement routing for quantum networks: Model and designs," in *Proc. ACM SIGCOMM Conf. Appl., Technol., Architectures, Protocols*, 2020, pp. 62–75.
- [5] Z. Li *et al.*, "Building a large-scale and wide-area quantum internet based on an OSI-alike model," *China Commun.*, vol. 18, no. 10, pp. 1–14, Oct. 2021.
- [6] C. Li, T. Li, Y.-X. Liu, and P. Cappellaro, "Effective routing design for remote entanglement generation on quantum networks," *NPJ Quantum Inf.*, vol. 7, no. 1, pp. 1–12, Dec. 2021.
- [7] Y. Zhao and C. Qiao, "Redundant entanglement provisioning and selection for throughput maximization in quantum networks," in *Proc. IEEE Conf. Comput. Commun. (INFOCOM)*, May 2021, pp. 1–10.
- [8] R. Van Meter, *Quantum Networking*. Hoboken, NJ, USA: Wiley, 2014.
- [9] A. S. Cacciapuoti, M. Caleffi, F. Tafuri, F. S. Cataliotti, S. Gherardini, and G. Bianchi, "Quantum internet: Networking challenges in distributed quantum computing," *IEEE Netw.*, vol. 34, no. 1, pp. 137–143, Jan. 2020.
- [10] Q. Jia, K. Xue, Z. Li, M. Zheng, D. S. L. Wei, and N. Yu, "An improved QKD protocol without public announcement basis using periodically derived basis," *Quantum Inf. Process.*, vol. 20, no. 2, pp. 1–11, Feb. 2021.
- [11] A. S. Cacciapuoti, M. Caleffi, R. Van Meter, and L. Hanzo, "When entanglement meets classical communications: Quantum teleportation for the quantum internet," *IEEE Trans. Commun.*, vol. 68, no. 6, pp. 3808–3833, Jun. 2020.
- [12] K. Chakraborty, D. Elkouss, B. Rijsman, and S. Wehner, "Entanglement distribution in a quantum network: A multicommodity flow-based approach," *IEEE Trans. Quantum Eng.*, vol. 1, pp. 1–21, 2020.
- [13] J. Illiano, M. Caleffi, A. Manzalini, and A. S. Cacciapuoti, "Quantum internet protocol stack: A comprehensive survey," 2022, *arXiv:2202.10894*.
- [14] M. Caleffi and A. S. Cacciapuoti, "Quantum switch for the quantum internet: Noiseless communications through noisy channels," *IEEE J. Sel. Areas Commun.*, vol. 38, no. 3, pp. 575–588, Mar. 2020.
- [15] A. Dahlberg *et al.*, "A link layer protocol for quantum networks," in *Proc. ACM SIGCOMM Conf. Appl., Technol., Architectures, Protocols*, 2019, pp. 159–173.
- [16] Z. Li, K. Xue, J. Li, N. Yu, D. S. L. Wei, and R. Li, "Connection-oriented and connectionless remote entanglement distribution strategies in quantum networks," *IEEE Netw.*, early access, Jul. 25, 2022, doi: 10.1109/MNET.107.2100483.
- [17] S. Pirandola, "End-to-end capacities of a quantum communication network," *Commun. Phys.*, vol. 2, no. 1, pp. 1–10, Dec. 2019.
- [18] G. Vardoyan, S. Guha, P. Nain, and D. Towsley, "On the stochastic analysis of a quantum entanglement switch," *ACM SIGMETRICS Perform. Eval. Rev.*, vol. 47, no. 2, pp. 27–29, Dec. 2019.
- [19] M. Pant *et al.*, "Routing entanglement in the quantum internet," *NPJ Quantum Inf.*, vol. 5, no. 1, pp. 1–9, Mar. 2019.
- [20] L. Gyongyosi and S. Imre, "Adaptive routing for quantum memory failures in the quantum internet," *Quantum Inf. Process.*, vol. 18, no. 2, pp. 1–21, Feb. 2019.
- [21] M. Caleffi, "Optimal routing for quantum networks," *IEEE Access*, vol. 5, pp. 22299–22312, 2017.
- [22] F. Hahn, A. Pappa, and J. Eisert, "Quantum network routing and local complementation," *NPJ Quantum Inf.*, vol. 5, no. 1, pp. 1–7, Dec. 2019.
- [23] A. Singh, K. Dev, H. Siljak, H. D. Joshi, and M. Magarini, "Quantum internet—Applications, functionalities, enabling technologies, challenges, and research directions," *IEEE Commun. Surveys Tuts.*, vol. 23, no. 4, pp. 2218–2247, 4th Quart., 2021.
- [24] S. Muralidharan, L. Li, J. Kim, N. Lütkenhaus, M. D. Lukin, and L. Jiang, "Optimal architectures for long distance quantum communication," *Sci. Rep.*, vol. 6, no. 1, pp. 1–10, Apr. 2016.
- [25] J.-G. Ren *et al.*, "Ground-to-satellite quantum teleportation," *Nature*, vol. 549, no. 7670, pp. 70–73, Sep. 2017.
- [26] P. C. Humphreys *et al.*, "Deterministic delivery of remote entanglement on a quantum network," *Nature*, vol. 558, no. 7709, pp. 268–273, Jun. 2018.
- [27] X.-M. Hu *et al.*, "Long-distance entanglement purification for quantum communication," *Phys. Rev. Lett.*, vol. 126, no. 1, Jan. 2021, Art. no. 010503.
- [28] R. Van Meter, T. D. Ladd, W. J. Munro, and K. Nemoto, "System design for a long-line quantum repeater," *IEEE/ACM Trans. Netw.*, vol. 17, no. 3, pp. 1002–1013, Jun. 2009.
- [29] L. Childress, J. M. Taylor, A. S. Sørensen, and M. D. Lukin, "Fault-tolerant quantum repeaters with minimal physical resources and implementations based on single-photon emitters," *Phys. Rev. A, Gen. Phys.*, vol. 72, no. 5, Nov. 2005, Art. no. 052330.
- [30] S. Perseguers, G. J. Lapeyre, D. Cavalcanti, M. Lewenstein, and A. Acín, "Distribution of entanglement in large-scale quantum networks," *Rep. Prog. Phys.*, vol. 76, no. 9, Sep. 2013, Art. no. 096001.
- [31] J. Y. Yen, "Finding the K shortest loopless paths in a network," *Manage. Sci.*, vol. 17, no. 11, pp. 712–716, Jul. 1971.
- [32] T. H. Szymanski, "Max-flow min-cost routing in a future-internet with improved QoS guarantees," *IEEE Trans. Commun.*, vol. 61, no. 4, pp. 1485–1497, Apr. 2013.
- [33] J. Li, H. Lu, K. Xue, and Y. Zhang, "Temporal netgrid model-based dynamic routing in large-scale small satellite networks," *IEEE Trans. Veh. Technol.*, vol. 68, no. 6, pp. 6009–6021, Jun. 2019.
- [34] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network Flows: Theory, Algorithms, and Applications*. Upper Saddle River, NJ, USA: Prentice-Hall, 1995.
- [35] T. Zhang, H. Li, J. Li, S. Zhang, and H. Shen, "A dynamic combined flow algorithm for the two-commodity max-flow problem over delay-tolerant networks," *IEEE Trans. Wireless Commun.*, vol. 17, no. 23, pp. 7879–7893, Dec. 2018.
- [36] W. J. Munro, K. Azuma, K. Tamaki, and K. Nemoto, "Inside quantum repeaters," *IEEE J. Sel. Topics Quantum Electron.*, vol. 21, no. 3, pp. 78–90, May/Jun. 2015.
- [37] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, "Purification of noisy entanglement and faithful teleportation via noisy channels," *Phys. Rev. Lett.*, vol. 76, no. 5, p. 722, Jan. 1996.
- [38] W. Dür and H. J. Briegel, "Entanglement purification and quantum error correction," *Rep. Prog. Phys.*, vol. 70, no. 8, p. 1381, 2007.

- [39] *Simulation Code for Routing Algorithms*. Accessed: Aug. 2022. [Online]. Available: <https://github.com/infonetlijian/Fidelity-Guaranteed-Entanglement-Routing>
- [40] S. Orłowski, R. Wessály, M. Pióro, and A. Tomaszewski, "SNDlib 1.0-survivable network design library," *Netw., Int. J.*, vol. 55, no. 3, pp. 276–286, May 2010.
- [41] B. M. Waxman, "Routing of multipoint connections," *IEEE J. Sel. Areas Commun.*, vol. SAC-6, no. 9, pp. 1617–1622, Dec. 1988.



**Jian Li** (Member, IEEE) received the B.S. degree from the Department of Electronics and Information Engineering, Anhui University, in 2015, and the Ph.D. degree from the Department of Electronic Engineering and Information Science (EEIS), University of Science and Technology of China (USTC), in 2020. From November 2019 to November 2020, he was a Visiting Scholar with the Department of Electronic and Computer Engineering, University of Florida. He is currently a Post-Doctoral Fellow with the School of Cyber Science and Technology, USTC.

His research interests include wireless communications, satellite networks, next-generation internet architecture, and quantum networking.

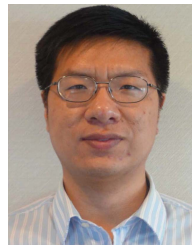


**Mingjun Wang** received the bachelor's degree from the Department of Information Security, University of Science and Technology of China (USTC), in 2015. He is currently a Graduate Student with the School of Cyber Science and Technology, USTC. His research interests include quantum internet architecture and quantum networking.



**Kaiping Xue** (Senior Member, IEEE) received the bachelor's degree from the Department of Information Security, University of Science and Technology of China (USTC), in 2003, and the Ph.D. degree from the Department of Electronic Engineering and Information Science (EEIS), USTC, in 2007. From May 2012 to May 2013, he was a Post-Doctoral Researcher with the Department of Electrical and Computer Engineering, University of Florida. He is currently a Professor with the School of Cyber Science and Technology, USTC. His research interests

include next-generation internet architecture design, transmission optimization, and network security. He is an IET Fellow. He serves on the Editorial Board of several journals, including the *IEEE TRANSACTIONS ON DEPENDABLE AND SECURE COMPUTING*, the *IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS*, and the *IEEE TRANSACTIONS ON NETWORK AND SERVICE MANAGEMENT*. He has also served as a (Lead) Guest Editor for many reputed journals/magazines, including *IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS*, *IEEE Communications Magazine*, and *IEEE Network*.



**Ruidong Li** (Senior Member, IEEE) received the bachelor's degree in engineering from Zhejiang University, China, in 2001, and the Ph.D. degree in engineering from the University of Tsukuba in 2008. He is currently an Associate Professor with the College of Science and Engineering, Kanazawa University, Japan. Before joining Kanazawa University, he was a Senior Researcher with the Network System Research Institute, National Institute of Information and Communications Technology (NICT). His current research interests include future networks, big data networking, blockchain, information-centric networks, the Internet of Things, network security, wireless networks, and quantum internet. He is a member of IEICE. He is the Founder and the Chair of the IEEE SIG on big data intelligent networking and the IEEE SIG on intelligent internet edge, and the Secretary of the IEEE Internet Technical Committee. He also serves as the chairs for conferences and workshops, such as IWQoS 2021, MSN 2020, BRAINS 2020, ICC 2021 NMIC Symposium, ICCN 2019/2020, and NMIC 2019/2020, and organized the Special Issues for the leading magazines and journals, such as *IEEE Communications Magazine*, *IEEE Network*, and *IEEE TRANSACTIONS ON NETWORK SCIENCE AND ENGINEERING*.



**Nenghai Yu** received the B.S. degree from the Nanjing University of Posts and Telecommunications, Nanjing, China, in 1987, the M.E. degree from Tsinghua University, Beijing, China, in 1992, and the Ph.D. degree from the Department of Electronic Engineering and Information Science (EEIS), University of Science and Technology of China (USTC), Hefei, China, in 2004. He is currently a Professor with the School of Cyber Security and the School of Information Science and Technology, USTC. He is the Executive Dean of the School of Cyber Science and Technology, USTC, and the Director of the Information Processing Center, USTC. He has authored or coauthored more than 130 papers in journals and international conferences. His research interests include multimedia security, information hiding, and quantum networking.



**Qibin Sun** (Fellow, IEEE) received the Ph.D. degree from the Department of Electronic Engineering and Information Science (EEIS), University of Science and Technology of China (USTC), in 1997. From 1996 to 2007, he was with the Institute for Infocomm Research, Singapore, where he was responsible for industrial as well as academic research projects in the area of multimedia security, and image and video analysis. He was the Head of Delegates of Singapore in ISO/IEC SC29 WG1(JPEG). From 2000 to 2001, he worked as a Research Scientist at Columbia University. He is currently a Professor with the School of Cyber Science and Technology, USTC. He has published more than 120 papers in international journals and conferences. His research interests include multimedia security, network intelligence, and security.



**Jun Lu** received the bachelor's degree from Southeast University in 1985 and the master's degree from the Department of Electronic Engineering and Information Science (EEIS), University of Science and Technology of China (USTC), in 1988. He is currently a Professor with the School of Cyber Science and Technology, and the Department of EEIS, USTC. His research interests include theoretical research and system development in the field of integrated electronic information systems. He is an Academician of the Chinese Academy of Engineering (CAE).