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# Online Supplement to "Computing Near-Optimal Stable Cost Allocations for Cooperative Games by Lagrangian Relaxation"

Lindong Liu, Xiangtong Qi
Department of Industrial Engineering and Logistics Management
The Hong Kong University of Science and Technology
{ldliu@ust.hk, ieemqi@ust.hk}

Zhou Xu
Department of Logistics and Maritime Studies
Faculty of Business
The Hong Kong Polytechnic University
lgtzx@polyu.edu.hk

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## Appendix A: Column and Row Generation Approaches for IM Games

We summarize the column generation approach and the row generation approach (LPB algorithm) discussed by Caprara and Letchford (2010) in this section.

For the column generation approach, consider the definition of OCAP by LP (1). Its dual problem is:

$$\min_{\beta} \Big\{ \sum_{s\in S} c(s)\beta_s : \sum_{s\ni k} \beta_s = 1, \forall k \in V, \beta_s \geq 0, s \in S \Big\}. \tag{A.1}$$

Denote by $Q^{x\gamma}$ the overall set of feasible solutions of ILP (3), i.e.,

$$Q^{x\gamma} = \big\{ x \in \{0,1\}^{t\times 1}, \gamma \in \{0,1\}^{v\times 1} : Ax \geq B\gamma + D, A'x \geq B'\gamma + D', \gamma = \gamma^s, \forall s \in S \big\}.$$

Then LP (A.1) can be re-formulated, for the purpose of doing column generation, by enumerating all values in $Q^{x\gamma}$. Specifically, for each $(\bar{x},\bar{\gamma}) \in Q^{x\gamma}$, define variable $\beta_{\bar{x},\bar{\gamma}}$ with cost $C\bar{x}$. We will have a master LP

$$\min_{\beta} \Big\{ \sum_{(\bar{x},\bar{\gamma})\in Q^{x\gamma}} (C\bar{x})\beta_{(\bar{x},\bar{\gamma})} : \sum_{(\bar{x},\bar{\gamma})\in Q^{x\gamma}} \bar{\gamma}_k \beta_{(\bar{x},\bar{\gamma})} = 1, \forall k \in V, \beta_{\bar{x},\bar{\gamma}} \geq 0, (\bar{x},\bar{\gamma}) \in Q^{x\gamma} \Big\}. \tag{A.2}$$

Though the formulation is straightforward, the above column generation is difficult to solve because the pricing problem amounts to optimizing over $Q^{x\gamma}$, which is usually NP-hard in the strong sense.

As to the row generation approach, it needs to identify a set of so-called assignable constraints, analogous to the cutting-plane method for solving IP where tight valid constraints are added to sharpen the LP bound. We let $P_I^x = conv\big\{x \in \{0,1\}^{t\times 1} : Ax \geq B\mathbf{1} + D, A'x \geq B'\mathbf{1} + D'\big\}$ and $P_I^{x\gamma} = conv\ Q^{x\gamma}$, where function $conv\{\cdot\}$ represents the convex hull of a vector. Note that $P_I^x$ is the convex hull of the integer solutions to (3).

Definition 1. A valid inequality $ax \geq b$ for $P_I^x$ is said to be *assignable* if there exists a valid inequality $ax \geq b'\gamma$ for $P_I^{x\gamma}$ such that $\sum_{k\in V} b'_k = b$.

Theorem 1. *For an IM game $(V,c)$, if there exists an LP $\min_x\{Cx : Ex \geq F\gamma\}$ that gives a lower bound to ILP (3), where all constraints $Ex \geq F\gamma$ are assignable, then vector $\alpha_{LP}^{EF}$ given by*

$$\alpha_{LP}^{EF}(k) = \sum_{l=1}^{m_E} f_{lk}\mu_l^*, \ \forall k \in V,$$

*is a stable cost allocation for the IM game, where $\mu^*$ is the LP dual variable value for an optimal solution to $\min_x\{Cx : Ex \geq F\mathbf{1}\}$, and $m_E$ is the number of rows of matrix $E$. In addition, the total shared cost $\sum_{k\in V} \alpha_{LP}^{EF}(k) = \min_x\{Cx : Ex \geq F\mathbf{1}\}$.*

In fact, Theorem 1 stands true if $\mu^*$ is relaxed to be an optimal solution to the dual of $\min_x\{Cx : Ex \geq F\mathbf{1}\}$. The proof is straightforward based on the results in Caprara and Letchford (2010). We will apply the extended results in our analysis.

Note that $c_{LP}^{EF}(V) = \min_x\{Cx : Ex \geq F\mathbf{1}\}$ gives an LP lower bound of the grand coalition cost $c(V)$. According to Theorem 1, the quality of the LPB cost allocation $\alpha_{LP}^{EF}$ greatly depends on the tightness of constraints set $\{Ex \geq F\mathbf{1}\}$, i.e., the tighter the constraints set is, the better the resulting LPB cost allocation $\alpha_{LP}^{EF}$ is. In addition, if given a basic optimal solution of $\min_x\{Cx : Ex \geq F\mathbf{1}\}$, then the resulting $\mu^*$ can be regarded as the shadow prices of constraints $Ex \geq F\mathbf{1}$, and therefore this leads to some LPB cost allocations with strong business insights. Such examples can be seen in the UFL LPB cost allocations.

Four IM games are investigated in Caprara and Letchford (2010), namely, the Uncapacitated Facility Location game, the Rooted and Unrooted Travelling Salesman games and the Vehicle Routing game. For each game, they give a tight constraint set such that the total shared cost $c_{LP}^{EF}(V)$ is no smaller than $c_{LP}(V)$, the LP lower bound of $c(V)$ from the original ILP formulation.

## References

Caprara A, Letchford AN (2010) New techniques for cost sharing in combinatorial optimization games. *Math. Programming* 124(1-2):93–118.