Profit Allocation in Investment-Based Crowdfunding

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Abstract

Even distribution is a normal profit allocation mechanism for investment-based crowdfunding projects on many platforms. In other words, the investors with the same pledging funds will be paid evenly when the investment ends. The even allocation mechanism works well under the assumption that the investors arrive at the platform simultaneously. However, in practice, the investors are sequential, therefore, the stories are different. In this paper, we study ways to design appropriate profit allocation mechanisms to enhance the success rate of an investment-based crowdfunding project. The basic model focuses on the two-investor case, where only two sequential investors are considered. The profit allocation mechanism is shown to have great impacts on the pledging probabilities of investors, as well as the success rate of a project. After that, we shift our focus to the two-cohort case, where investors are assumed to arrive at the platform as two sequential cohorts. By taking the sizes of each cohort into consideration, we are able to analyze the success rate of a project under various practical situations.

Keywords: decision analysis, profit allocation, success rate, investment-based crowdfunding

1. Introduction

It is well recognized that small start-ups and entrepreneurs encounter great difficulties while seeking finance from banks or venture capitalists (Cassar 2004; Cosh et al. 2009), especially during their initial stages. Complementing traditional financing options, crowdfunding emerged as an innovative form of seeking finance from people and networks, with a low-barrier (Mollick & Nanda 2015; Bouncken et al. 2015).

Among various types of crowdfunding options, investment-based crowdfunding, through which investors can receive financial returns (e.g., equity, interest, revenue, and loyalty) rather than appreciation or specific products, has experienced rapid growth since the Jumpstart Our Business Start-ups (JOBS) Act was passed in the USA in 2012. As reported in Massolution (2013), the average funding size in investment-based crowdfunding is more than 100 times larger than the size in donation-based crowd-

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funding. In addition, the World Bank has also estimated that the total funding size of investment-based crowdfunding would reach \$90 billion by 2020 and surpass the size of venture capital (Barnett 2015).

Crowdfunding platforms make it possible for small firms and entrepreneurs to simplify and decentralize their funding processes. By communicating with potential investors directly through the internet, entrepreneurs can introduce their proposals in a better manner and raise funds from a large number of individuals (Schwienbacher & Larralde 2010).

On an investment-based crowdfunding platform, a typical crowdfunding project will announce a funding target, along with a unit pledging price, a funding deadline, a proposal that specifies how the funds will be used, and a profit allocation mechanism. The funding part succeeds only when the total amount of investment exceeds the target within the given period. If the project fails, all the funds raised will be returned to the investors. After raising enough funds, the entrepreneur will execute the proposal and final earnings will be allocated to investors, according to the profit allocation mechanism, in return. During the period of crowdfunding, investors make their decisions based on their pledges to the project and their valuations of the financial return from the proposal.

It is clear that successful crowdfunding projects can benefit all participants: entrepreneurs can get enough funds to start their businesses; investors can make use of spare cash for promising investments; and the platform can earn commission fees from the organization. However, because of uncertainty and asymmetric information, about two-thirds of the total number of projects have failed at the crowdfunding stage[1]. This indicates the urgent necessity of investigations on enhancing success rates of investment-based crowdfunding projects.

It is shown that the success rate of a project is significantly affected by its performance in the early stage. On the one hand, lesser investment in the early stage not only puts more funding pressure on the later stages, but also weakens the investing willingness of later investors. Many existing studies(see, e.g., Li & Duan 2016; Belleflamme et al. 2015) have suggested the existence of positive network externality and negative time effect in crowdfunding, that is, the portion of the target already reached has a positive influence, while the time remaining has a negative influence on later investors. On the other hand, investors arriving in the early stages are usually less willing to participate since they incur higher waiting costs. Du et al. (2017) concludes that, among all the failed projects, 88.34% ended up raising lesser than 20% of their original targets. Similarly, Mollick & Kuppuswamy (2014) observes that the crowdfunding projects either succeed or fail by large margins, and the average percentage of raised funds is only 8% among all the failed projects.

[1]Source:https://www.entrepreneur.com/article/269663

In the past, to motivate early investors to improve success rates of crowdfunding projects, entrepreneurs were encouraged to make some sacrifice, including offering free gifts and lowering pledging prices. However, first, due to the lack of initial capital, entrepreneurs usually cannot afford to give free gifts. Second, the competition in investment-based crowdfunding is so intense that each entrepreneur prefers to set the pledging price at the lowest level. Once the initial pledging price is lowered further, the total amount of funds raised decreases, and the proposal is more likely to fail.

In this paper, instead of sacrificing the entrepreneur's profits, we are interested in motivating early investors by reallocating final profits earned from the proposal. Intuitively, we assign more profits to early investors so that their waiting costs are balanced out and the resulting pledging probabilities are raised. Note that more profits allocated to (higher pledging probabilities of) early investors means less profits remain for (lower pledging probabilities of) the late ones. To enhance the overall success rate of a crowdfunding project, it is of utmost importance to provide the entrepreneur with appropriate profit allocation mechanisms. Our main contributions are summarized as follows.

First, to the best of our knowledge, this paper is the first attempt to analytically study the profit allocation mechanism to enhance the success rates of investment-based crowdfunding projects. Most literature on crowdfunding, especially investment-based crowdfunding, is empirical and existing efforts on motivating investors focus on offering additional benefits and price discounts. Our study helps the entrepreneur design an optimal profit allocation mechanism to maximize the success rate without sacrificing the profits of the entrepreneur.

Second, we develop static models to analyze the pledging behavior of investors, and we characterize the "waiting cost" to explain the inequity between investors at different stages in crowdfunding projects. The main results show that because of the waiting cost, investors who arrive early are less willing to pledge money. It also shows that the entrepreneur should motivate early investors to enhance the success rate of the project. In addition, the extra return given to early investors as an incentive should increase with the waiting cost.

Third, as a generalization, we consider the difference in the number of investors who group as cohorts, arriving at different points in time. We find that investors in different-sized cohorts are not equally sensitive with changes in profit allocation, and the entrepreneur should motivate investors in smaller cohorts to enhance the success rate of his crowdfunding project. This property, together with the effect of the waiting cost, decides the profit allocation strategy of the entrepreneur. In addition, we also provide managerial guidance on how the entrepreneur should adjust the optimal profit allocation mechanism when other factors in the market change.

The rest of this paper is structured as follows. The following section reviews relevant literature. We describe the basic problem in Section 3. In Section 4, we analyze the profit allocation mechanism using a primary model where there are only two potential investors. Section 5 generalizes the results of Section 4 by studying a two-cohort model where there are two cohorts of investors. The conclusions are shown in Section 6.

2. Literature Review

Although crowdfunding is a relatively new phenomenon with nascent related research, the rapid growth of all kinds of crowdfunding platforms, as well as enormous economic benefits brought by them every year, have intrigued more and more researchers.

Most of the existing research studies focus on the empirical side. Researchers have studied many characteristics of crowdfunding mechanisms that might influence the success rate, including geographic distance among investors (Mollick 2014; Agrawal et al. 2015, 2011), types of projects (Belleflamme et al. 2013), choices of return offered in projects (Wang et al. 2016), dynamic process of investing behavior (Kuppuswamy & Bayus 2015; Chung & Lee 2015), choices of market mechanisms (Wei & Lin 2016), the existence of home bias (Lin & Viswanathan 2015), long-term benefits for entrepreneurs after successfully launching a project (Mollick & Kuppuswamy 2014), comparison between "Keep-It-All" and "All-Or-Nothing" (Cumming et al. 2014), perverse incentives in crowdfunding (Hildebrand et al. 2016), social capital (Zheng et al. 2014; Colombo et al. 2015) and network externalities (Li & Duan 2016; Belleflamme et al. 2015).

On the analytical side, Ellman & Hurkens (2015) and Strausz (2016) analyze how crowdfunding projects ameliorate the uncertainty of demand and deal with moral hazards. Belleflamme et al. (2014) gives instructions on choosing between pre-order crowdfunding and equity crowdfunding under different conditions, but they focus on the condition that the entrepreneur is tapping into a certain crowd with known valuations, and the equity crowdfunding works as an alternative to finance for a certain product. Different from our study, there is no uncertainty of success and the project will either definitely fail or definitely succeed, depending on the price and target. Chakraborty & Swinney (2016) reveals that entrepreneurs may behave differently under the objectives of maximizing success rates or the expected return. Chen et al. (2017) investigates whether entrepreneurs who essentially need to convince angel investors for venture capital should launch a crowdfunding project in advance to prove the market size and customer valuations, or not. Du et al. (2017) studies the optimal time in a reward-based crowdfunding project, and finds that the entrepreneur should contingently add a stimulus, such as offering free samples or updating project features, for success. Hu et al. (2015) develops a two-period model to study how pricing and product design strategies in crowdfunding differ from traditional financing. Our work studies investment-based crowdfunding that has seldom been studied analytically. It is well recognized that a good success rate lies at the core of crowdfunding. We focus on enhancing the success rate by designing a profit allocation mechanism without reducing profits in crowdfunding proposals.

As a supplement, crowdfunding is related to many fields of literature. For example, the “All-Or-Nothing” mechanism, in which money is refunded when the entrepreneur fails to collect enough within a certain period, is similar to the common provision-point mechanism used by researchers to study private provisions of public goods (see, e.g., Palfrey & Rosenthal 1988; Bagnoli & Lipman 1989). However, everyone can benefit from the provision of public goods once a project is built, while in crowdfunding, people must invest in the project to receive their return, thereby making the free-riding effect in the provision of public goods less essential.

Another stream of research similar to crowdfunding is group buying, wherein a qualified number of committed purchasers can get special discount on products. Anand & Aron (2003) compares the group buying mechanism with conventional-posted price mechanism. Liang et al. (2014) shows that an improvement in information quality has positive effects on customer surplus and the success rate. Tran & Desiraju (2017) and Yan et al. (2017) study the impact of asymmetric information on group buying from the perspective of the manufacturer and the retailer. Hu et al. (2013) suggests that sellers disclose the cumulative sign-up information to later customers to increase success rates. Moreover, Wu et al. (2015) reveals the threshold effect that the sign-up behavior of customers accumulates right before and after the target is reached. This is consistent with the discovery that we have underlined, namely, that pledging probabilities of investors are higher in the later stages, where the threshold is about to be reached and the risk is much lower. A study on group buying that is similar to ours is Kauffman et al. (2010). They introduced demand externalities and concluded that motivating early consumers to join in on group buying efficiently improves the performance of projects. However, they explored the incentive mechanisms based on offering an extra and attractive discount to the first few participants or those who arrived within a short period of time, as soon as the project began. Group buying shares more similarities with reward-based crowdfunding than with investment-based crowdfunding. Group buying projects are often offered by well-established companies that launch these projects to advertise their brands and expand market share. It is easy for these large companies to give up profit to attract customers. But investment-based crowdfunding projects are always associated with new ventures and small start-ups that are in urgent need of initial funds and therefore, offering discounts and samples may not be feasible for them.

3. Problem Description

On an investment-based crowdfunding platform, an entrepreneur will launch a project with a detailed proposal, a target amount of funds, an unit pledging price for each investor, and a specified profit allocation mechanism when the proposal is implemented. Then, the investors will arrive at the platform sequentially, and decide whether to pledge or not by maximizing their own expected utilities. After that, the project closes. If the project succeeds (i.e., the target is achieved), the entrepreneur will implement his proposal, and the investors will get paid according to the preset profit allocation mechanism after the implementation. Otherwise, the entrepreneur will return the pledged money to the investors.

Owing to the refunding policy, the objective of the entrepreneur is to increase the success rate of the crowdfunding project as far as possible. In particular, once the target amount of funds and the unit pledging price are predetermined, the profit allocation mechanism would be the remaining key factor that would affect the success rate of a project. This is the main focus of our paper.

As a first attempt to tackle the profit allocation mechanism in investment-based crowdfunding, this paper will restrict itself to the two-cohort situation, that is, the investors group as two cohorts, arriving in two specific periods. This two-period assumption is widely used to study the crowdfunding process (see, e.g., Hu et al. 2015; Jing & Xie 2011; Liang et al. 2014). In fact, many of our results can be generalized to the case of multiple cohorts. For example, in subsection 4.3 we conclude that the entrepreneur should motivate investors in the early cohort, and the return given to this cohort increases with the waiting cost. This conclusion can be generalized to multiple-cohort cases that the return given to each cohort decreases with its waiting cost, that is, the later this cohort of investors arrives, the less return they receive. In the basic model that is presented in Section 4, we focus on the two-investor case, where each cohort contains only one investor. In Section 5, we generalize our results to the two-cohort model.

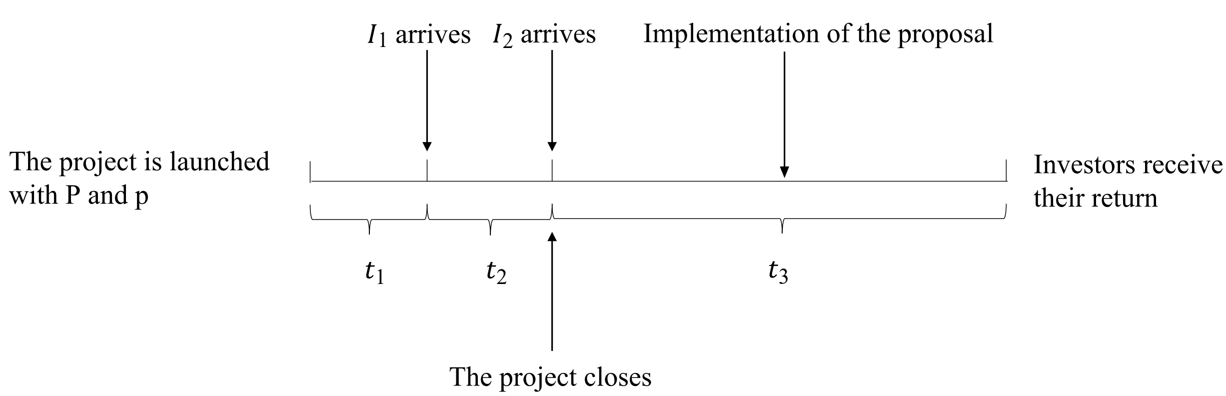


Figure 1: Procedures of the two-investor case

Figure 1 shows the basic procedures involved in two-investor crowdfunding. To be specific, the unit pledging price is p, the target amount of funds is $P = 2p$, and there are two potential investors I_1 and I_2. In each period t_i $(i = 1, 2)$, investor I_i arrives and makes his pledging decision. At the end of period t_2, the project closes. If either I_1 or I_2 chooses not to pledge, the project fails. Otherwise, the project succeeds and the entrepreneur implements the proposal during the period t_3. After the implementation of the proposal, the investors get their return at the end of period t_3. Note that t_3 is usually much longer than t_1 and t_2.

While making pledging decisions, each investor would maximize his own utility by comparing the expected return from pledging (ERP) with the expected return from not pledging (ERNP). To measure the ERP, we denote the valuation of I_i $(i = 1, 2)$ on the proposal as $V_i \times P$, where V_i can be regarded as the valuation rate of return of the proposal estimated by I_i. Then, the ERP of I_i is simply his share of $V_i \times P$ under some given profit allocation mechanism. For the valuation rate V_i, we assume that V_i $(i = 1, 2)$ are i.i.d., with a uniform distribution over interval $\left[0, A\right]$ to tackle the heterogeneity of different investors. Furthermore, the valuation rates of the investors are assumed to be private, while their distributions are known to each other and the entrepreneur. Such assumptions are widely used in crowdfunding studies (see, e.g., Belleflamme et al. 2014).

To measure the ERNP, by denoting the risk-free rate of return of the market during period t_3 as R, each investor can get a risk-free return of $R \times p$ during period t_3 with fixed investment p. Besides, note that I_1 pledges earlier and waits t_2 longer than I_2 until the project closes. Let $\Delta = 1 + \delta$ be the risk-free rate of return of the market during period t_2, where δ can be viewed as the rate of waiting cost for I_1. Thus, the risk-free return of I_1 would be $(1 + \delta) \times R \times p$ during periods t_2 and t_3 if he chooses not to pledge. By comparing the ERP with ERNP, an investor can make his own pledging decision. We now formally summarize the notations described above in Table 1.

Table 1: Notations used in the problem description

P	The target amount of funds in the project
p	The unit pledging price for each investor
t_i	The pledging period of the crowdfunding project, $i \in \{1, 2\}$
I_i	The investor arriving at period t_i, $i \in \{1, 2\}$
t_3	The implementing period of the proposal in the crowdfunding project
V_i	The rate of return from this proposal estimated by investor I_i, $i \in \{1, 2\}$
Δ	The risk-free rate of return of the market during period t_2
R	The risk-free rate of return of the market during period t_3

4. Analyses of the Profit Allocation Mechanism

It is clear that different profit allocation mechanisms lead to different pledging strategies for investors, and in turn, decide the success rates of crowdfunding projects. In this section, we will focus on the two-investor case where there are only two potential investors.

In most existing research, the profit allocation mechanism is simply even distribution among all investors, which is referred to as an even allocation mechanism in our paper. We will generalize the results by allocating the profits among the investors unevenly. To be formal, for a given profit allocation mechanism $(\alpha, 1-\alpha)$, we let the share of return allocated to I_1 be α $(0 < \alpha < 1)$, and consequently, the share of return allocated to I_2 can be written as $1-\alpha$.

4.1. Pledging Strategies of the Investors

We first study the impacts of the profit allocation mechanism on the pledging strategies of investors by backward induction. The details are shown as follows.

When I_2 arrives during period t_2, he can observe the pledging decision made by I_1. If I_1 did not pledge, I_2 will walk away directly, since the target P cannot be met and the project will definitely fail. Otherwise, the project will succeed as long as I_2 pledges. On the one hand, since the valuation rate of return of I_2 on the proposal is V_2, the resulting ERP is given by $(1-\alpha) \times V_2 \times P = 2p \times (1-\alpha) \times V_2$. On the other hand, the ERNP of I_2 with investment p is simply $R \times p$ during period t_3. In this case, I_2 will pledge only when his ERP surpasses ERNP, that is,

$$2p \times (1-\alpha) \times V_2 > R \times p, \text{ which is equivalent to } V_2 > R/2(1-\alpha).$$

By noting that V_2 is uniformly distributed over interval $[0, A]$, we can claim that when I_1 pledged, the pledging probability of I_2, denoted as q_2, is $1 - R/2A(1-\alpha)$.

When I_1 arrives during period t_1, although he has no information on the pledging decision of I_2, he can speculate the pledging strategy of I_2 due to the awareness of the distribution of V_2. To be specific, the pre-condition for I_2 to pledge is that I_1 pledges and the pledging probability is q_2. In this case, on one hand, the ERP of I_1 can be written as $q_2 \times \alpha \times V_1 \times P + (1-q_2) \times R \times p = q_2 \times 2\alpha \times V_1 \times p + (1-q_2) \times R \times p$, where the former part is the expected return when I_2 pledges, and the latter part is the expected return when I_2 does not pledge and I_1 is refunded. On the other hand, the ENRP of I_1 with investment p is $R \times (1+\delta) \times p$, which includes risk-free returns during both periods t_2 and t_3. Thus, I_1 will pledge only when

$$2\alpha \times V_1 \times p \times q_2 + (1-q_2) \times R \times p > R \times (1+\delta) \times p,$$

which is equivalent to

$$V_1 > (\delta + q_2) \times R/(2\alpha \times q_2).$$

Therefore, we can claim that the pledging probability of I_1, denoted as q_1, is $1-(\delta+q_2)\times R/(2\alpha\times q_2\times A)$.

Since the (crowdfunding) project succeeds only when both investors pledge, the success rate of the project, denoted as S, is $q_1 \times q_2$. By letting $r = R/A$, we can express the pledging probabilities of the investors and the success rate of the project as

$$q_1 = 1 - \frac{\delta r(1-\alpha)}{2\alpha(1-\alpha) - \alpha r} - \frac{r}{2\alpha},\ q_2 = 1 - \frac{r}{2(1-\alpha)},\ \text{and } S = q_1 \times q_2, \text{respectively.}$$

The ratio $r = R/A$ can be regarded as a factor reflecting the competitiveness of the risk-free market over the proposal provided by the entrepreneur. When r is high, the risk-free market is so competitive that the investors are not interested in the proposal in the crowdfunding project, and when r is low, the results reverse.

4.2. Feasibility of a Project

One of the most important steps for an entrepreneur before starting a crowdfunding project on a platform is to check the feasibility of his crowdfunding project, that is, the positivity of the success rate of a project. From the expressions of q_1 and q_2, we can see that the success rate is decided by r, δ, and α, where r and δ are exogenous, while α can be adjusted by the entrepreneur.

It is important to remember that $r = R/A$ reflects the competitiveness of the risk-free market over the proposal in the crowdfunding project. We now study the feasibility of a project from the perspective of r. Lemma 1 shows that there exists a tolerance bound on r, above which the project is destined for failure with given δ and α.

Lemma 1. *Under a given profit allocation mechanism* $(\alpha, 1-\alpha)$, *the project is feasible only when* $r < \bar{r}(\alpha, \delta)$, *where* $\bar{r}_1(\alpha, \delta) = 1 + (1-\alpha)\delta - [1 + (1-\alpha)^2\delta^2 + 2(1-\alpha)(\delta - 2\alpha)]^{1/2}$.

Lemma 1 indicates that the entrepreneur will start a crowdfunding project only when $r < \bar{r}(\alpha, \delta)$.

To enhance this tolerance bound, it is desired to study the monotonicity of $\bar{r}(\alpha, \delta)$ in δ and α, respectively. For the sake of simplicity, we will write $\bar{r}(\alpha, \delta)$ as $\bar{r}$ in short when the context is not confusing, and the same operations are applied to all other functions throughout this paper.

Proposition 1. *For any given* α, *function* $\bar{r}$ *decreases in* δ.

The intuition behind the decreasing of $\bar{r}$ in δ is that a higher waiting cost rate δ results in higher pledging unwillingness of I_1, and in turn, reduces the tolerance bound of r.

Although the tolerance bound is monotonic in δ, it is hard to enhance $\bar{r}$ by simply decreasing δ, since the length of the pledging period t_2 (i.e., the value of δ) is hard to reduce in practice. In this case, we turn to study how α will affect $\bar{r}$, and the results are shown in Proposition 2.

Proposition 2. *For given δ, function $\bar{r}$ is unimodal in α and reaches its maximum at $\bar{\alpha}$, where $\bar{\alpha}$ is equal to* $(2+\delta(1+\delta-\sqrt{\delta}))/(4+\delta^2)$.

The unimodality of $\bar{r}$ in α can be interpreted as follows. Regardless of the dependence of the pledging decisions, the pledging probabilities of I_1 and I_2 are increasing in α and $1-\alpha$, respectively. However, since the feasibility (positivity of the success rate) of a project is decided by the product of the two pledging probabilities, a straightforward result is that the monotonicity of $\bar{r}$ coincides with the monotonicity of $\alpha(1-\alpha)$ in α, that is, $\bar{r}$ is an unimodal function of α. Apparently, the tolerance bound $\bar{r}$ reaches its maximum when $\alpha = \bar{\alpha}$.

Corollary 1. *For a given δ, the maximum tolerance bound, denoted as $\bar{r}^*$, is equal to $\frac{2(\delta+2-2\sqrt{\delta})}{4+\delta^2}$.*

Corollary 1 shows that, for any given δ, if $r > \bar{r}_1^*$, crowdfunding is infeasible, no matter how the entrepreneur will allocate the profits to the investors. In particular, when $\delta = 0$, the maximum tolerance bound is equal to 1. This indicates that when period t_2 is so short that the waiting cost of I_1 is close to 0, the necessary condition for a positive success rate is simply $R < A$ ($r < 1$), that is, the return rate of the proposal has a chance to surpass the return rate of the risk-free market.

4.3. Success Rate of a Project

The previous subsection provides a necessary condition (a tolerance bound $\bar{r}_1^*$ on r) under which a project has a chance to succeed. In this part, we will focus on the case where $r < \bar{r}_1^*$, that is, the project is feasible under some allocation mechanism, and study how the success rate of a project will change with different profit allocation mechanisms.

It is important to remember that in Section 4.1 we have shown that the pledging probabilities of the two investors and the success rate of the project are

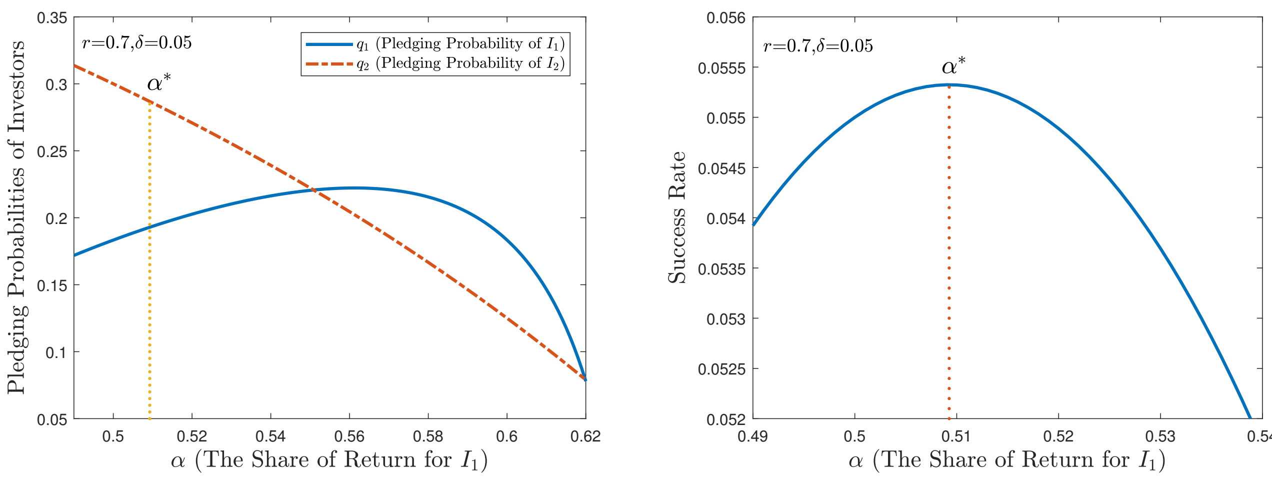
$$q_1 = 1 - \frac{\delta r(1-\alpha)}{2\alpha(1-\alpha) - \alpha r} - \frac{r}{2\alpha}, \;\; q_2 = 1 - \frac{r}{2(1-\alpha)}, \;\text{and}\; S = q_1 \times q_2, \;\text{respectively.}$$

From the expressions of q_1 and q_2, we can find that q_2 decreases in α while the monotonicity of q_1, as well as S, in α is unknown. To this end, we have Theorem 1 showing the monotonicity of S in α.

Theorem 1. *The success rate S is unimodal in α and reaches its maximum at α^*, where α^* is equal to* $(2+2\delta-r-[(2-r)(2+2\delta-r)]^{1/2})/2\delta$ *and larger than* $1/2$.

The unimodality of S is expected. We can interpret this in a manner similar to what we did after Proposition 2. Suffice to say that the monotonicity of S is consistent with the monotonicity of $\alpha(1-\alpha)$ in α. For any given pair of δ and r, the entrepreneur is able to maximize the success rate of his crowdfunding project by letting α equal α^*. In addition, the intuition behind $\alpha^* > 1/2$ is that the entrepreneur should compensate I_1 for his waiting cost during period 2. Compared with $\alpha = 1/2$, which maximizes $\alpha(1-\alpha)$, the entrepreneur should motivate investor I_1 with a greater return. Therefore, we can claim that the entrepreneur should always take sides with the first investor to maximize S.

We now use a numerical example to illustrate how α affects the pledging probabilities of the investors and the success rate of the project. The results are shown in Figure 2, where $\delta = 0.05$, $r = 0.7$, the horizontal axes represent α, and the vertical axes represent the pledging probability and the success rate, respectively.



(a) Pledging Probability of Investors

(b) Success rate

Figure 2: Pledging probability of investors and success rate in the profit allocation mechanism

As we can see from Theorem 1, the optimal α^* to maximize the success rate S is decided by both, r and δ. We now show the monotonicity of α^* in r and δ in Proposition 3.

Proposition 3. *The optimal α^* for S increases in both, δ and r.*

It is important to bear in mind that the risk-free return of I_1 and I_2 are $\big(1+\delta\big) \times R \times p$ and $R \times p$, respectively. Compared with I_2, investor I_1 suffers an additional waiting cost of $\delta \times R \times p$, therefore, the entrepreneur is suggested to allocate more return to I_1 when δ or r increases. We refer to the increasing of α^* in δ as the effect of waiting cost, and the δ-effect for short.

5. Two-Cohort Model

In Section 4, we studied the basic case, where there are only two potential investors arriving at the platform sequentially. In this section, we will extend our investigations to a general case where there are two cohorts of potential investors.

The main changes of the two-cohort model can be concluded as follows. We denote the two sequential cohorts arriving at the platform during periods t_1 and t_2 as C_1 and C_2, respectively. Let $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$ be the respective shares of return allocated to C_1 and C_2 by the entrepreneur. For each cohort C_i $(i = 1, 2)$, there are N_i identical investors: each of whom (1) has the same valuation rate of V_i^N on the proposal, which is uniformly distributed over $\left[0,\text{A}\right]$; (see, e.g. Hu et al. 2015) and (2) expects an average share of return of $\alpha_i \times V_i^N \times P/N_i$.

It is expected that the two-cohort model shares some similar results with the two-investor model. For example, the δ-effect still holds, that is, when δ increases, the entrepreneur needs to compensate the first cohort by allocating them more shares of return. However, the optimal profit allocation mechanism might change because of the emergence of the scale-effect of the cohorts.

We can interpret the scale-effect in the two-cohort model as follows. Suppose that there are two cohorts C_1 and C_2 containing N_1 and N_2 investors, respectively. When the entrepreneur decides to motivate C_1 by allocating them an extra return of x, the average return allocated to each investor in C_1 is increased by x/N_1, while the average return of each investor in C_2 is decreased by x/N_2. Thus, the investors in different cohorts are not equally sensitive with the same change of α. To take advantage of such unequal sensitivity, the scale-effect suggests that the entrepreneur should take sides with the smaller cohort while maximizing the success rate of his crowdfunding project. The scale-effect, together with the δ-effect, decides the incentive strategy of the entrepreneur in the two-cohort case.

From the problem setting, it is clear that the pledging strategies of different investors within the same cohort are identical. Similar to the two-investor model, to investigate the optimal profit allocation mechanism in the two-cohort case, we first analyze the pledging strategies of each cohort by backward induction.

When C_2 arrives, the investors in this cohort only pledge if C_1 has pledged. One the one hand, if C_1 pledged, since the valuation rate of return of C_2 on the proposal is V_2^N, the ERP for each investor in C_2 is given by $(N_1 + N_2) \times p \times (1 - \alpha) \times V_2^N / N_2$. On the other hand, the ERNP of each investor in C_2 with investment p is $R \times p$ during period t_3. In this case, investors in C_2 will pledge only when the ERP surpasses ERNP, that is,

$$V_2^N > N_2 \times R / [(N_1 + N_2)(1 - \alpha)].$$

To conclude, when C_1 pledged, the pledging probability of C_2, denoted as q_2^N, is equal to $1 - N_2 \times R/[(N_1 + N_2)(1 - \alpha)A]$.

When C_1 arrives in period t_1, investors in C_1 know that the pre-condition for C_2 to pledge is that C_1 pledges and the pledging probability is q_2^N. On the one hand, the ERP of each investor in C_1 can be written as $q_2^N \times p \times (N_1 + N_2)\alpha \times V_1^N/N_1 + (1 - q_2^N) \times R \times p$, where the former part is the expected return when C_2 pledges, and the latter part is the expected return when C_2 does not pledge. On the other hand, the ERNP of each investor in C_1 with investment p is $R \times (1 + \delta) \times p$, which includes the risk-free returns in both periods t_2 and t_3. Thus, investors in C_1 will pledge only when the ERP is larger than the ERNP, that is,

$$V_1^N > N_1 \times (\delta + q_2^N)R/[(N_1 + N_2) \times q_2^N \times \alpha]$$

To conclude, the pledging probability of C_1, denoted as q_1^N, is equal to $1 - N_1 \times (\delta + q_2^N)R/[(N_1 + N_2) \times q_2^N \times \alpha \times A]$.

Let $\rho = N_1/(N_1 + N_2)$ and denote the success rate of the project in the two-cohort situation as S_N. Then, we have

$$q_1^N = 1 - \frac{(1-\alpha)\delta\rho r}{\alpha((1-\alpha) - (1-\rho)r)} - \frac{\rho r}{\alpha}, \; q_2^N = 1 - \frac{(1-\rho)r}{(1-\alpha)}, \text{ and } S_N = q_1^N \times q_2^N.$$

Note that the two-investor model is a special case of the two-cohort model where $\rho = 1/2$. The results are consistent with what we derived in the basic model.

There also exists a tolerance bound $\bar{r}_N$ on r, above which the crowdfunding project is infeasible. It is clear that $\bar{r}_N$ is decided by r, δ, ρ and α. By changing the value of α, we are able to adjust the tolerance bound. In addition, we can still show that function $\bar{r}_N$ is unimodal in α. The detailed explanations are omitted for the sake of simplicity. We present Corollary 2 as a conclusion.

Corollary 2. *In the two-cohort model, the tolerance bound $\bar{r}_N$ is unimodal in α, and the maximum tolerance bound is $\bar{r}_N^* = (1 + \delta \times \rho - 2\sqrt{\delta \times \rho(1-\rho)})/[(1 - \delta \times \rho)^2 + 4\delta \times \rho^2]$.*

When a crowdfunding project is feasible ($r < \bar{r}_N^*$), we can maximize its success rate by choosing an optimal profit allocation mechanism. By denoting the optimal share of return allocated to C_1 as α_N^*, we have Theorem 2 which shows the profit allocation strategy of the entrepreneur.

Theorem 2. *To maximize the success rate S_N in the two-cohort model, when $\rho \neq 1/(2+\delta)$, we have that*

$$\alpha_N^* = \frac{(1+\delta)\rho - (1-\rho)\rho r}{(2+\delta)\rho - 1} - \frac{1}{(2+\delta)\rho - 1}\left[(1 - 2\rho + \rho^2)\rho^2 r^2 - (1-\rho)(\delta\rho + 1)\rho r + (1+\delta)(1-\rho)\rho\right]^{1/2}.$$

When $\rho = 1/(2+\delta)$, $\alpha_N^* = 1/2$.

Proposition 4. *The entrepreneur should adjust the optimal profit allocation mechanism when ρ, δ and r changes:*

(i) The optimal share of return α_N^* allocated to C_1 increases in ρ.

(ii) The optimal share of return α_N^* allocated to C_1 increases in δ.

(iii) The optimal share of return α_N^* allocated to C_1 increases in r when $\rho > 1/(2+\delta)$, and decreases in r when $\rho < 1/(2+\delta)$.

As we can see from Theorem 2, the optimal α_N^* is jointly decided by ρ, δ, and r. Propositions 4 describes the monotonicity of α_N^* in δ, r and ρ. Intuitively, Proposition 4 (i) indicates that when there are more investors in C_1, the optimal share α_N^* allocated to the first cohort is increased. However, as shown later in Theorem 3, due to the scale-effect, when ρ is large, the average return of C_1 is smaller than the average return of C_2, that is, $\alpha_N^* < \rho$. The result of Proposition 4 (ii) coincides with the δ-effect. It is straightforward that the entrepreneur needs to compensate investors in the first cohort with more return when their waiting cost increases.

Unlike the basic model, where the optimal share of return allocated to the first investor is simply increasing in r, the monotonicity of α_N^* in r is complicated in the two-cohort case. We can explain the result of Proposition 4 (iii) as follows. First, when ρ is large, the cumulated δ-effect of C_1 is massive due to its large size. It is important to remember that the δ-effect results in an additional waiting cost of $\delta \times R \times p$ for each investor in the first cohort, and thus, if r increases, the entrepreneur tends to compensate the first cohort with more return to enhance the success rate of the project, and therefore, α_N^* is increased. Second, when ρ is small, the cumulated δ-effect of C_1 is minor. If r increases, since the proposal is less attractive to all the investors, the entrepreneur prefers to give more return to C_2 (the cohort with more investors) to enhance the success rate, therefore, α_N^* is decreased.

Following Proposition 4 (iii), we can investigate the detailed profit allocation strategy of the entrepreneur under different values of ρ. The results are shown in Theorem 3.

Theorem 3. *There exists a cohort ratio threshold $\rho^* = \big(1+\delta-r\big)/\big(2+\delta-2r\big) > 1/2$ such that:*

(i) *If $\rho = \rho^*$, then $\alpha_N^* = \rho$, that is, the entrepreneur will not motivate any cohort;*

(ii) *If $0 < \rho < \rho^*$, then $\alpha_N^* > \rho$, that is, the entrepreneur should motivate C_1;*

(iii) *If $\rho^* < \rho < 1$, then $\alpha_N^* < \rho$, that is, the entrepreneur should motivate C_2.*

It is important to remember that the δ-effect indicates that the entrepreneur takes sides with the first cohort. Furthermore, due to the scale-effect, the entrepreneur tends to motivate the smaller cohort. Thus, we can claim that there exists a ratio threshold ρ^* at which the effects of scale and waiting cost cancel each other out, and ρ^* is larger than $1/2$. When $\rho < \rho^*$, the entrepreneur will motivate the first cohort, while when $\rho > \rho^*$, the entrepreneur will motivate the second cohort. In particular, when $\rho = 1/2 < \rho^*$, we have that $\alpha_N^* > \rho = 1/2$, which is consistent with the result in Theorem 1.

We now illustrate the results of Proposition 4 (iii) and Theorem 3 through a numerical example in Figure 3. In the rectangular coordinates, the vertical axis represents the share of return allocated to C_1, and the horizontal axis represents the ratio of cohort C_1. The diagonal dotted line represents the straight line of $\alpha = \rho$ on which the entrepreneur motivates neither cohort, and the return is evenly distributed to each investor. The solid curve associates with the optimal α_N^* for different values of ρ. It is clear that if $\rho < \rho^*$, the solid line is above the dotted line, that is, $\alpha_N^* > \rho$, thus, the entrepreneur should motivate C_1 to maximize the success rate of the project. On the contrary, if $\rho > \rho^*$, we have that $\alpha_N^* < \rho$ and the entrepreneur should motivate C_2. According to Figure 3, one can easily decide the optimal profit allocation mechanism to maximize the success rate for a given crowdfunding project.

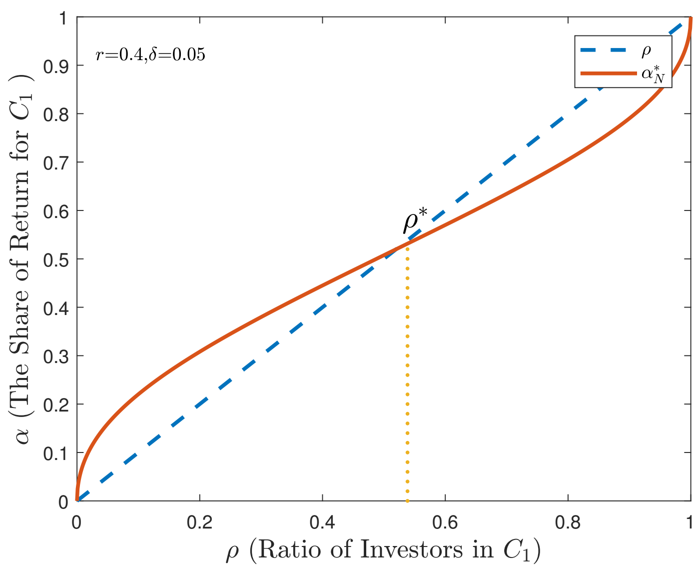


Figure 3: The optimal α to maximize the success rate with different values of ρ

As we can see from Theorems 1 and 3, the profit allocation strategies in the two-investor and two-cohort models are different due to the existence of the scale-effect. In order to eliminate the impacts of scales, we now study how the extra return received by each investor changes with ρ. The results are shown in Proposition 5. For preparation, according to Theorem 3, when $\rho < \rho^*$, the first cohort is motivated and each investor in C_1 gets an extra incentive of $\epsilon_1 = (\alpha_N^*(\rho, \delta, r) - \rho)/\rho$, while when $\rho > \rho^*$, the second cohort is motivated and each investor in C_2 gets an extra incentive of $\epsilon_2 = (\rho - \alpha_N^*(\rho, \delta, r))/\big(1 - \rho\big)$.

Proposition 5. *Let ρ^* be the ratio threshold given in Theorem 3, we have that the following:* (i) *if $\rho < \rho^*$, then $\epsilon_1 > 0$ and decreases in ρ;* (ii) *if $\rho > \rho^*$, then $\epsilon_2 > 0$ and decreases in $1 - \rho$.*

Proposition 5 indicates that in order to maximize the success rate of the project, if cohort C_i is motivated, the average-extra return received by an individual investor in C_i always decreases in the size of C_i. To be specific, it is shown that ϵ_1 is decreasing in ρ and ϵ_1 is decreasing in $1-\rho$. This is exactly the scale-effect that we introduced in the beginning of this section, that is, the entrepreneur takes sides with a cohort of smaller size. In particular, when $\rho = \rho^*$, we have that $\epsilon_1 = \epsilon_2 = 0$, which indicates that the entrepreneur will motivate neither cohort.

We still adopt the numerical example used in Figure 3 to illustrate the results of Proposition 5. In Figure 4, the horizontal axis represents the size ratio of C_1, and the vertical axis represents the average-extra incentive received by an investor. The left-hand side and right-hand side curves denotes the "$\rho \sim \epsilon_1$" and "$\rho \sim \epsilon_2$" functions, respectively. These two functions intersect at point $\left(\rho^*, 0\right)$ at which no incentive mechanism is applied and the success rate of the project is maximized.

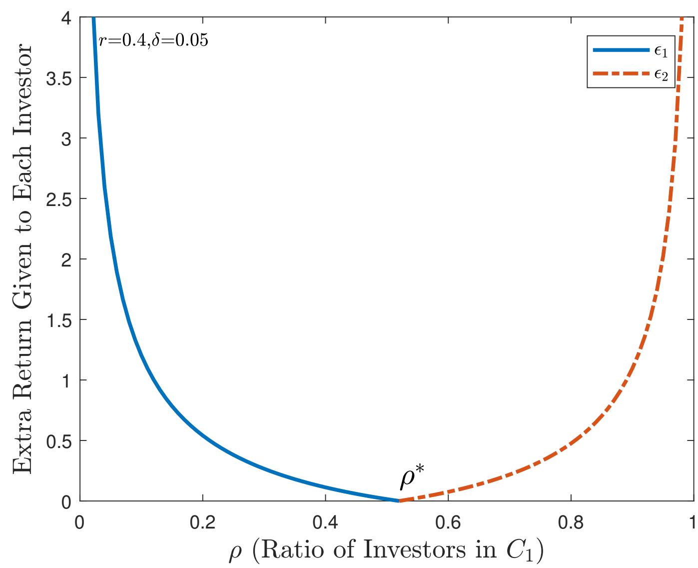


Figure 4: Additional incentive allocated to each investor with different values of ρ

6. Conclusion

Crowdfunding is emerging as an important source of finance for small start-ups and new entrepreneurs, and its market size has grown enormously in recent years. Note that success rate is the core problem in crowdfunding, especially in investment-based crowdfunding, where investors receive financial return. It is well recognized that performance in the early stage of a crowdfunding project is crucial to its success, while investors are less willing to take on the higher risk of pledging earlier. Therefore it is intuitive to offer an incentive to investors.

Instead of sacrificing the profits of the entrepreneur to motivate investors like in past literature, this paper studies how an entrepreneur should maximize the success rate with the profit allocation mechanism in investment-based crowdfunding. In our study, we stressed the need to provide the appropriate profit allocation mechanism to ensure the feasibility of the projects and enhance the success rate. Our main results show that the existence of the waiting cost, that is, the δ-effect, encourages the entrepreneur to motivate early investors in order to maximize the success rate. However, the entrepreneur also needs to take into account the difference in the sizes of cohorts arriving at different points in time, that is, the scale-effect. The smaller the cohort, the more suitable it is to be motivated. Our results suggest that the entrepreneur takes both, the scale-effect and the δ-effect into consideration while deciding which cohort to motivate. For example, different from the two-investor case, when too many investors arrive in the early stages of crowdfunding, the entrepreneur may choose to motivate the investors coming in later, instead.

Moreover, our analysis provides managerial guidance on how the entrepreneur should adjust his optimal profit allocation mechanism according to changes in the market. First, no matter which cohort is motivated, each investor in this cohort should receive more return as the incentive when this cohort becomes smaller (the scale-effect becomes stronger). Second, the entrepreneur should give early investors a greater return when their additional waiting cost increases (the δ-effect becomes stronger). Third, when the risk-free market becomes more competitive over the crowdfunding proposal than before, if the number of investors in the later cohort is very large, the entrepreneur should give them a greater return.

Crowdfunding, as an important source of finance, needs more attention in future research. One limitation of our research is that we simplify the study by assuming that the valuations of investors are distributed uniformly, while the valuations can be far more complex or even affected by the description and advertisement of entrepreneurs. Further, we did not consider the occasion that investors may strategically delay their pledges. Moreover, the arrival of investors can be stochastic, so the number of investors is uncertain in reality, and there is also the possibility of overfunding, which can be analyzed in the future.

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Appendix A. Proof

Proof of Lemma 1.

The project is feasible only when the pledging probabilities of both investors are positive. Apparently, $1 > q_2 > 0$ holds when $0 < r < 2(1-\alpha)$. In addition, we find out that $1 > q_1 > 0$ holds when $r^2 - 2[(1-\alpha)(1+\delta)+\alpha]r + 4\alpha(1-\alpha) > 0$, this quadratic polynomial of r is equal to $4\alpha(1-\alpha) > 0$ when $r = 0$; and $-4(1-\alpha)^2\delta < 0$ when $r = 2(1-\alpha)$, respectively, so there exists one root within $(0, 2(1-\alpha))$ and this root is $1+(1-\alpha)\delta-[1+(1-\alpha)^2\delta^2+2(1-\alpha)(\delta-2\alpha)]^{1/2} < 2(1-\alpha)$. Suffice to say that the pledging probabilities of both investors are positive when $r < 1+(1-\alpha)\delta - [1+(1-\alpha)^2\delta^2+2(1-\alpha)(\delta-2\alpha)]^{1/2}$. Consequently, $\bar{r}(\alpha,\delta) = 1+(1-\alpha)\delta - [1+(1-\alpha)^2\delta^2+2(1-\alpha)(\delta-2\alpha)]^{1/2}$ and the project is feasible when $r < \bar{r}(\alpha,\delta)$. □

Proof of Proposition 1.

Taking the derivative of $\bar{r}(\alpha,\delta)$ with respect to δ yields:

$$\frac{\partial \bar{r}(\alpha,\delta)}{\partial \delta} = (1-\alpha)[1-\frac{(1-\alpha)\delta+1}{\sqrt{(1-\alpha)^2\delta^2+2(1-\alpha)\delta+1-\alpha(1-\alpha)}}]$$

Since $\alpha \in [0,1]$, $(1-\alpha)^2\delta^2+2(1-\alpha)\delta+1-\alpha(1-\alpha) = [(1-\alpha)\delta+1]^2-(1-\alpha)\alpha < [(1-\alpha)\delta+1]^2$ and we have $\frac{\partial \bar{r}(\alpha,\delta)}{\partial \delta} < 0$. For a given α, $\bar{r}(\alpha,\delta)$ always decreases in δ. □

Proof of Proposition 2.

To analyze the monotonicity of $\bar{r}(\alpha,\delta)$ in α, we take the derivative of $\bar{r}(\alpha,\delta)$ with respect to α and yield:

$$\frac{\partial \bar{r}(\alpha,\delta)}{\partial \alpha} = \frac{(1-\alpha)(\delta^2+2)+(\delta-2\alpha)}{\sqrt{1+(1-\alpha)^2\delta^2+2(1-\alpha)(\delta-2\alpha)}} - \delta$$

We set $f_1(\alpha) = (1-\alpha)(\delta^2+2)+(\delta-2\alpha)-[1+(1-\alpha)^2\delta^2+2(1-\alpha)(\delta-2\alpha)]^{1/2}\delta$, then

$$\frac{\partial \bar{r}(\alpha,\delta)}{\partial \alpha} = 0 \;\Leftrightarrow\; f_1(\alpha)=0 \;\Leftrightarrow\; \alpha = (2+\delta(1+\delta-\sqrt{\delta}))/(4+\delta^2)$$

We can prove that function $f_1(\alpha)$ is strictly decreasing in α

$$\begin{aligned}\frac{df_1(\alpha)}{d\alpha} &= -\delta^2-4+\frac{(1-\alpha)(\delta^2+2)+(\delta-2\alpha)}{\sqrt{1+(1-\alpha)^2\delta^2+2(1-\alpha)(\delta-2\alpha)}} \\ &< -\delta^2-4+\frac{\delta^2+\delta+2}{1+\delta} \quad \text{(because } 0<a<1\text{)} \\ &< -\delta^2-4+4<0\end{aligned}$$

Define $\bar{\alpha} = \alpha = (2+\delta(1+\delta-\sqrt{\delta}))/(4+\delta^2)$, according to the monotonicity of $f_1(\alpha)$ in α, we can conclude that when $\alpha < \bar{\alpha}$, $f_1(\alpha) > 0$, so $\frac{\partial \bar{r}(\alpha,\delta)}{\partial \alpha} > 0$. In the same way, $\frac{\partial \bar{r}(\alpha,\delta)}{\partial \alpha} < 0$ when $\alpha > \bar{\alpha}$. Thus, for a given δ, $\bar{r}$ is unimodal in α and reached its maximum when $\alpha = \bar{\alpha}$. □

Proof of Corollary 1.

Just conclude $\bar{r}(\bar{\alpha},\delta)$ and we have the maximum tolerance bound $\bar{r}^* = \frac{1}{4}(2+\delta r) - \frac{1}{4}[(\delta^2+4)r^2-4(\delta+2)r+4]^{1/2}$. □

Proof of Theorem 1.

Taking the derivative of S with respect to α yields:

$$\frac{\partial S}{\partial \alpha} = \frac{r}{4\alpha^2(1-\alpha)^2}[2\delta\alpha^2-(4+4\delta-2r)\alpha+2+2\delta-r]$$

Define $f_2(\alpha) = 2\delta\alpha^2-(4+4\delta-2r)\alpha+2+2\delta-r$, $\frac{df_2(\alpha)}{d\alpha} = 2r-4<0$. Note that $f_2(0) = 2+2\delta-r>0$ and $f_2(1) = r-2<0$, then there exists a maximum point in $(0,1)$ and is equal to $\alpha^* = (2+2\delta-r)/2\delta-[(2-$

$r)(2+2\delta-r)]^{1/2}/2\delta$. We can conclude that function S is unimodal in α. In addition, $f_2(1/2)=\delta/2>0$, so $\alpha^*>1/2$. $\square$

Proof of Proposition 3.

Taking derivative of α^* with respect to δ and r respectively yields:

$$\frac{\partial\alpha^*}{\partial\delta}=\frac{(2-r)(2+\delta-r-\sqrt{(2-r)(2+2\delta-r)})}{2\delta^2\sqrt{(2-r)(2+2\delta-r)}},\ \frac{\partial\alpha^*}{\partial r}=\frac{2+\delta-r-\sqrt{(2-r)(2+2\delta-r)}}{2\delta\sqrt{(2-r)(2+2\delta-r)}}$$

Note that $2+\delta-r=[(2-r)+(2+2\delta-r)]/2$, so $(2+\delta-r)^2>(2-r)(2+2\delta-r)$ and $2+\delta-r-\sqrt{(2-r)(2+2\delta-r)}>0$. Apparently, $\frac{\partial\alpha^*}{\partial\delta}$ and $\frac{\partial\alpha^*}{\partial r}$ are both positive, α^* increases in δ and r. $\square$

Proof of Corollary 2.

To make the project feasible:

$$q_2^N>0 \text{ holds when } r<(1-\alpha)/(1-\rho)$$
$$q_1^N>0 \text{ holds when } f_3(r)=(1-\rho)\rho r^2-[(1-\rho)\alpha+(1-\alpha)\delta\rho+(1-\alpha)\rho]r+\alpha(1-\alpha)>0$$
$$f_3(0)=\alpha(1-\alpha)>0,\ f_3(\frac{1-\alpha}{1-\rho})=-(1-\alpha)\delta\rho r<0$$

Therefore, there must be one left root of $f_3(r)$ in $(0,(1-\alpha)/(1-\rho))$. The project is feasible when $r<\bar{r}_N=\bar{r}_N(\alpha)=((1-\alpha)(1+\delta)\rho+(1-\rho)\alpha-[((1-\alpha)(1+\delta)\rho+(1-\rho)\alpha)^2-4\alpha(1-\alpha)\rho(1-\rho)]^{1/2}/2(1-\rho)\rho$. Taking the derivative of $\bar{r}_N$ with respect to α yields:

$$\frac{\partial\bar{r}_N}{\partial\alpha}=\frac{1}{2(1-\rho)\rho}\times f_4(\alpha)$$
$$f_4(\alpha)=1-(2+\delta)\rho-\frac{\alpha\left(1+\delta^2\times\rho^2+2\rho\times\delta\,(2\rho-1)\right)-\rho\,(1-\delta+\delta\times\rho\,(3+\delta))}{\alpha^2\,(1+\delta^2\times\rho^2+2\rho\times\delta\,(2\rho-1))-2\alpha\,(1+\delta^2\times\rho+\delta\,(3\rho-1))+(1+\delta)^2\,\rho^2}$$
$$\frac{df_4(\alpha)}{d\alpha}=-\frac{4\delta\times(1-\rho)^2\times\rho^2}{[\alpha^2\,(1+\delta^2\times\rho^2+2\delta\times\rho(2\rho-1))-2\alpha\times\rho\,(1+\delta^2\rho+\delta(3\rho-1))+(1+\delta)^2\rho^2]^{3/2}}<0$$
$$f_4(0)=\frac{2(1-\rho)}{(1+\delta)}>0,\ f_4(1)=-2\rho<0$$

Note that $f_4(\alpha)$ is decreasing in α and there must exist a point satisfying $f_4(\alpha)=0$, therefore $\bar{r}_N$ is unimodal in α. Since the expression of $\bar{r}_N$ is very complex, we can conclude the maximum tolerance bound in another way. Note that the project is feasible when $f_3>0$, we transform f_3 in the form of α and $f_3(\alpha)=-\alpha^2+(1-r+2\rho r+\delta\rho r)\alpha+(1-\rho)\rho r^2-\delta\rho r-\rho r$. The project is feasible only when this function has roots, that is, the discriminant $\Delta=(1+\delta^2\rho^2-2\delta\rho+4\rho^2\delta)r^2-(2+2\delta\rho)r+1$ is positive. (Note that all the Δ in our appendix is the discriminant of a polynomial instead of the risk-free factor Δ in our model.) The discriminant is positive only when $r<(1+\delta\times\rho-2\sqrt{\delta\times\rho(1-\rho)})/[(1-\delta\times\rho)^2+4\delta\times\rho^2]$, therefore the maximum tolerance bound if $\bar{r}_N^*=(1+\delta\times\rho-2\sqrt{\delta\times\rho(1-\rho)})/[(1-\delta\times\rho)^2+4\delta\times\rho^2]$. $\square$

Proof of Theorem 2.

To maximize the success rate, we conclude S and the derivative of S with respect to α as follows:

$$
\begin{aligned}
S_N &= [\alpha^2 + \rho r(1+\delta-(1-\rho)r) - \alpha(1-(1-(2+\delta)\rho)) + (1+\delta-(1-\rho)r)\rho r]/\alpha(\alpha-1) \\
\frac{\partial S_N}{\partial \alpha} &= \frac{r}{\alpha^2(1-\alpha)^2} \times f_5(\alpha) \\
f_5(\alpha) &= \rho\left(1+\delta-(1-\rho)\,r\right) - 2\rho\left(1+\delta-(1-\rho)\,r\right)\alpha + \left((2+\delta)\,\rho-1\right)\alpha^2 \\
f_5(0) &= \rho(1+\delta-(1-\rho)r) > 0,\ f_5(1) = (1-\rho r)(\rho-1) < 0
\end{aligned}
$$

There must exist roots of $f_5(\alpha)$ in $(0,1)$ according to intermediate value theorem. When $\rho = 1/(2+\delta)$, $f_5(\alpha)$ is linear and $\alpha = 1/2$ is its only root, so $\alpha = 1/2$ is the maximum point. When $\rho < 1/(2+\delta)$, $f_5(\alpha)$ is concavely quadratic and maximize at its larger root:

$$
\alpha_N^* = \frac{(1+\delta)\rho-(1-\rho)\rho r}{(2+\delta)\rho-1} - \frac{1}{(2+\delta)\rho-1}\left[(1-2\rho+\rho^2)\rho^2r^2-(1-\rho)(\delta\rho+1)\rho r+(1+\delta)(1-\rho)\rho\right]^{1/2}
$$

When $\rho > 1/(2+\delta)$, $f_5(\alpha)$ is convexly quadratic and maximize at its smaller root, we can conclude that it is also α_N^*. □

Proof of Proposition 4.

First, we prove (i) and take the derivative of α_N^* with respect to ρ as follows:

$$
\frac{\partial \alpha_N^*}{\partial \rho} = \frac{-\left[(2+\delta)\rho^2-2\rho+1\right]r+1+\delta}{2\left[(2+\delta)\rho-1\right]^2} \times \frac{1+\delta\times\rho-2\rho(1-\rho)r-2\sqrt{\rho(1-\rho)(1-\rho\times r)\left[1+\delta-(1-\rho)r\right]}}{\sqrt{\rho(1-\rho)(1-\rho\times r)\left[1+\delta-(1-\rho)r\right]}}
$$

We first analyze the numerator of the first fraction. Note that $(2+\delta)\rho^2-2\rho+1$ is always positive for $\delta > 0$ and $\rho \in (0,1)$, so $-\left[(2+\delta)\rho^2-2\rho+1\right]r+1+\delta > -\left[(2+\delta)\rho^2-2\rho+1\right]\times 1+1+\delta = -(2+\delta)\rho^2+2\rho+\delta$. It is easy to prove that $-(2+\delta)\rho^2+2\rho+\delta > 0$, so $-\left[(2+\delta)\rho^2-2\rho+1\right]r+1+\delta > 0$.

In the same way, we can also prove that $1+\delta\times\rho-2\rho(1-\rho)r > 1+\delta\times\rho-2\rho(1-\rho) > 0$ always holds. In addition, $[1+\delta\times\rho-2\rho(1-\rho)r]^2-4\rho(1-\rho)(1-\rho\times r)\left[1+\delta-(1-\rho)r\right] = [(2+\delta)\rho-1]^2 > 0$, therefore the numerator of the second fraction $1+\delta\times\rho-2\rho(1-\rho)r-2\sqrt{\rho(1-\rho)(1-\rho\times r)\left[1+\delta-(1-\rho)r\right]} > 0$. Consequently, $\frac{\partial \alpha_N^*}{\partial \rho} > 0$, α_N^* increases in ρ.

Next, we prove (ii) and take the derivative of α_N^* with respect to δ:

$$
\frac{\partial \alpha_N^*}{\partial \delta} = \frac{\rho(1-\rho)(1-\rho r)}{2\left[(2+\delta)\rho-1\right]^2} \times \left[-2+\frac{1+\delta\times\rho-2\rho(1-\rho)r}{\sqrt{\rho(1-\rho)(1-\rho\times r)\left[1+\delta-(1-\rho)r\right]}}\right]
$$

We have proved in the proof of (i) that $1+\delta\times\rho-2\rho(1-\rho)r > 0$ and $[1+\delta\times\rho-2\rho(1-\rho)r]^2-4\rho(1-\rho)(1-\rho\times r)\left[1+\delta-(1-\rho)r\right] = [(2+\delta)\rho-1]^2 > 0$, therefore, $-2+\frac{1+\delta\times\rho-2\rho(1-\rho)r}{\sqrt{\rho(1-\rho)(1-\rho\times r)[1+\delta-(1-\rho)r]}}$ is positive and α_N^* increases in δ.

Last, we prove (iii) and take the derivative of α_N^* with respect to r:

$$
\frac{\partial \alpha_N^*}{\partial r} = \frac{\rho(1-\rho)}{2[(2+\delta)\rho-1]}[-2+\frac{1+\delta\rho-2\rho r+2\rho^2 r}{\sqrt{(1-\rho)(1-\rho r)(1+\delta-(1-\rho)r)\rho}}]
$$

It is obvious that $-2+\frac{1+\delta\times\rho-2\rho(1-\rho)r}{\sqrt{\rho(1-\rho)(1-\rho\times r)[1+\delta-(1-\rho)r]}}>0$, when $0<\rho<1/(2+\delta)$, $\frac{\partial\alpha_N^*}{\partial r}<0$. On the contrary, when $1>\rho>1/(2+\delta)$, $\frac{\partial\alpha_N^*}{\partial r}>0$. □

Proof of Theorem 3.

We have proved in the proof of Theorem 2 that:

$$S_N=[\alpha^2+\rho r(1+\delta-(1-\rho)r)-\alpha(1-(1-(2+\delta)\rho))+(1+\delta-(1-\rho)r)\rho r]/\alpha(\alpha-1)$$
$$\frac{\partial S_N}{\partial\alpha}=\frac{r}{\alpha^2(1-\alpha)^2}\times f_5(\alpha)$$
$$f_5(\alpha)=\rho\left(1+\delta-(1-\rho)\,r\right)-2\rho\left(1+\delta-(1-\rho)\,r\right)\alpha+\left((2+\delta)\,\rho-1\right)\alpha^2$$
$$f_5(\rho)=(1-\rho)\times\rho\times[(1+\delta-r)-(2+\delta-2r)\rho]$$

Since α_N^* is the only maximum point of function S_N within $(0,1)$, $\frac{\partial S_N(\alpha_N^*)}{\partial\alpha}=0$, therefore we can conclude whether α_N^* is larger than ρ with the positivity of $\frac{\partial S_N(\rho)}{\partial\alpha}$. It is shown that when $\rho=\frac{1+\delta-r}{2+\delta-2r}$, $f_5(\rho)=0$, therefore $\frac{\partial S_N(\rho)}{\partial\alpha}=0$ and $\alpha_N^*=\rho$. When $\rho<\frac{1+\delta-r}{2+\delta-2r}$, $f_5>0$, $\frac{\partial S_N(\rho)}{\partial\alpha}>0$, ρ is on the left side of α_N^*, so $\alpha_N^*>\rho$; in the same way, when $\rho>\frac{1+\delta-r}{2+\delta-2r}$, $\alpha_N^*<\rho$. □

Proof of Proposition 5.

(i) When $\rho<\rho^*=\frac{1+\delta-r}{2+\delta-2r}$ and the first cohort is motivated, that is, $\alpha_N^*>\rho$ and $\epsilon_1>0$:

$$\frac{\partial\epsilon_1}{\partial\rho}=\frac{\partial(\alpha_N^*-\rho)/\rho}{\partial\rho}=\frac{A_1-B_1*C_1}{2\rho[(2+\delta)\rho-1]^2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}}$$
$$A_1=-1-6\rho(-1+r)+r-2\rho^3r^2+2\rho^2(-2+2r+r^2)$$
$$+\delta^2\rho\left[3+\rho^2r-2\rho(1+r)\right]-\delta\left[1+3\rho(-3+r)+2\rho^3(-1+r)r+\rho^2(6+r-2r^2)\right]$$
$$B_1=\left[\delta^2\rho+(2-r)\rho+\delta\rho(3-r)\right]$$
$$C_1=2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}$$

Thus, we only need to proof $A_1-B_1\times C_1\le 0$, We can write $A_1-B_1\times C_1$ as $A_1-B_1\times D_1-B_1\times(C_1-D_1)$, where $D_1=1+\delta\times\rho-2\rho\times r+2\rho^2\times r$, according to our proof in the earlier proposition, obviously $C_1\le D_1, B_1>0$, so $B_1\times(C_1-D_1)\le 0$, and we can conclude $A_1-B_1\times D_1=-\left[-1+(2+\delta)\rho\right]^2\times[1+\delta+(-1+\rho)r]\le 0$ after simplification. So $A_1-B_1\times C_1\le 0$ is equivalent to $(A_1-B_1\times D_1)^2\ge B_1^2\times(C_1-D_1)^2$.

$$(A_1-B_1\times D_1)^2-B_1^2\times(C_1-D_1)^2=[-1+(2+\delta)\rho]^4\times[1+\delta+(-1+\rho)r]^2$$
$$-(1+\delta)^2\times\rho^2\times(2+\delta-r)^2\times$$
$$\left[1+\delta\times\rho-2\rho\times r+2\rho^2\times r-2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}\right]^2$$

Implementing the formula for the difference of squares:

Since

$$
\begin{aligned}
&[-1+(2+\delta)\rho]^2 \times [1+\delta+(-1+\rho)r] + (1+\delta)\times\rho\times(2+\delta-r) \\
&\times\left[1+\delta\times\rho-2\rho\times r+2\rho^2\times r-2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}\right] > 0
\end{aligned}
$$

Thus, we only need to prove:

$$
\begin{aligned}
M = &[-1+(2+\delta)\rho]^2 \times [1+\delta+(-1+\rho)r] - (1+\delta)\times\rho\times(2+\delta-r) \\
&\times\left[1+\delta\times\rho-2\rho\times r+2\rho^2\times r-2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}\right] \\
&= [-1+(2+\delta)\rho]^2 \times \\
&\left[[1+\delta+(-1+\rho)r] - \frac{(1+\delta)\times\rho\times(2+\delta-r)}{1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}}\right] \geq 0
\end{aligned}
$$

We divide our proof into two parts:

Part I When $0<\rho<\frac{1}{2+\delta}$, because $(1+\delta+(-1+\rho)r)-(1+\delta)\times\rho\times(2+\delta-r) = -(-1+(2+\delta)\rho)(1+\delta-r)$, then $(1+\delta+(-1+\rho)r) > (1+\delta)\rho(2+\delta-r)$ under this condition.

To prove $M>0$, we scale M as follow:

$$
\begin{aligned}
M \geq M_1 = &[-1+(2+\delta)\rho]^2 \times \\
&\left[(1+\delta)\times\rho\times(2+\delta-r) - \frac{(1+\delta)\times\rho\times(2+\delta-r)}{1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}}\right] \\
&= [-1+(2+\delta)\rho]^2 \times (1+\delta)\times\rho\times(2+\delta-r)\times \\
&\left[1 - \frac{1}{1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}}\right]
\end{aligned}
$$

$$
\begin{aligned}
M \geq 0 \Leftarrow M_1 > 0 \Leftarrow\ & 1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]} > 1 \\
\Leftarrow\ & \rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right] - (\delta\times\rho-2\rho\times r+2\rho^2\times r)^2 > 0 \\
\Leftarrow\ & 4\delta(-1+\rho)+\delta^2\times\rho+4(-1+\rho+r-\rho\times r) < 0 \\
\Leftarrow\ & \rho < \frac{4+4\delta-4r}{(2+\delta)^2-4r} \\
\Leftarrow\ & \frac{4+4\delta-4r}{(2+\delta)^2-4r} > \frac{1}{2+\delta} \\
\Leftarrow\ & r < 1 < \frac{(2+\delta)^2}{4(1+\delta)}
\end{aligned}
$$

Consequently $M \geq 0$ and $\frac{\partial(\alpha_N^*-\rho)/\rho}{\partial\rho} \geq 0$. Thus, we can conclude $\frac{\partial\epsilon_1}{\partial\rho} \geq 0$ when $0<\rho<\frac{1}{2+\delta}$.

Part II When $\frac{1}{2+\delta} < \rho < \frac{1+\delta-r}{2+\delta-2r}$, then we have $(1+\delta+(-1+\rho)r) > \rho \times (2+\delta-r)$ under this condition. To prove $M>0$, we scale M as follow:

$$\begin{aligned}
M > M_2 &= [-1+(2+\delta)\rho]^2 \times \\
&\left[\rho \times (2+\delta-r) - \frac{(1+\delta)\times\rho\times(2+\delta-r)}{1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}}\right] \\
&= [-1+(2+\delta)\rho]^2 \times \rho \times (2+\delta-r) \times \\
&\left[1 - \frac{(1+\delta)}{1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}}\right]
\end{aligned}$$

$$\begin{aligned}
M \geq 0 \Leftarrow M_2 > 0 &\Leftarrow 1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]} > 1+\delta \\
&\Leftarrow 4\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right] - (\rho-1)^2(\delta+2\rho\times r)^2 > 0 \\
&\Leftarrow \delta^2(\rho-1)+4\rho(1-r)+4\delta\times\rho\times(1-r) > 0 \\
&\Leftarrow \frac{\delta^2}{(4+4\delta+\delta^2-4r-4\delta\times r)} < \rho < 1 \\
&\Leftarrow \frac{\delta^2}{(4+4\delta+\delta^2-4r-4\delta\times r)} < \frac{1+\delta-r}{2+\delta-2r} \\
&\Leftarrow r < 1 < \frac{4+8\delta+3\delta^2}{4+4\delta}
\end{aligned}$$

Consequently $M \geq 0$ and $\frac{\partial(\alpha_N^*-\rho)/\rho}{\partial\rho} \geq 0$. Thus, we can conclude $\frac{\partial\epsilon_1}{\partial\rho} \geq 0$ when $\frac{1}{2+\delta} < \rho < \frac{1+\delta-r}{2+\delta-2r}$. So far we have proved that when C_1 is motivated, $\frac{\partial\epsilon_1}{\partial\rho} \geq 0$, and we next prove the case when C_2 is motivated.
(ii) When $\rho > \rho^* = \frac{1+\delta-r}{2+\delta-2r}$ and the second cohort is motivated, that is, $\alpha_N^* < \rho$ and $\epsilon_2 > 0$:

$$\begin{aligned}
\frac{\partial\epsilon_2}{\partial\rho} &= \frac{\partial(\rho-\alpha_N^*)/(1-\rho)}{\partial\rho} = -\frac{A_2 - B_2 * C_2}{2(1-\rho)[(2+\delta)\rho-1]^2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}} \\
A_2 &= 1-r+2\rho^3r^2+\delta^2\rho(1-2\rho+\rho^2 r)-4\rho^2(1-r+r^2) \\
&\quad -2\rho(-1+r-r^2)+\delta\left[1+3\rho^2(-2+r)+2\rho^3 r+3\rho(1-r)\right] \\
B_2 &= (1-\rho)(2-r+\delta) \\
C_2 &= 2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}
\end{aligned}$$

Thus, we only need to prove $A_2 - B_2 \times C_2 \leq 0$, We can write $A_2 - B_2 \times C_2$ as $A_2 - B_2 \times D_2 - B_2(C_2 - D_2)$, where $D_2 = 1+\delta\times\rho-2\rho\times r+2\rho^2\times r$, according to our proof in an earlier proposition, obviously $C_2 \leq D_2, B_2 > 0$, so $B_2(C_2-D_2) \leq 0$, and we can conclude $A_2 - B_2 \times D_2 = -[(2+\delta)\rho-1]^2(1-\rho\times r) \leq 0$ after simplification. So $A_2 - B_2 \times C_2 \leq 0$ is equivalent to $(A_2 - B_2\times D_2)^2 \geq B_2^2(C_2-D_2)^2$.

$$\begin{aligned}
(A_2 - B_2\times D_2)^2 - B_2^2(C_2-D_2)^2 &= [(2+\delta)\rho-1]^4(1-\rho r)^2 - (1-\rho)^2(2+\delta-r)^2 \times \\
&\left[1+\delta\times\rho-2\rho\times r+2\rho^2\times r-2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}\right]^2
\end{aligned}$$

Implementing the formula for the difference of square:

Since

$$[(2+\delta)\rho-1]^2(1-\rho r)+(1-\rho)(2+\delta-r) \times \left[1+\delta\times\rho-2\rho\times r+2\rho^2\times r-2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}\right]>0$$

Thus, we only need to prove

$$\begin{aligned} M_3 &= [(2+\delta)\rho-1]^2\times(1-\rho\times r)-(1-\rho)(2+\delta-r) \\ &\times\left[1+\delta\times\rho-2\rho\times r+2\rho^2\times r-2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}\right] \\ &= [(2+\delta)\rho-1]^2\times \\ &\left[(1-\rho\times r)-\frac{(1-\rho)(2+\delta-r)}{1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}}\right]\geq 0 \end{aligned}$$

When $\rho>\rho^*=\frac{1+\delta-r}{2+\delta-2r}$, we have $(1-\rho)(2+\delta-r)<1-\rho\times r$.

$$\begin{aligned} M_3>M_4 &= [(2+\delta)\rho-1]^2\times \\ &\left[(1-\rho)(2+\delta-r)-\frac{(1-\rho)(2+\delta-r)}{1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}}\right] \\ &= [(2+\delta)\rho-1]^2\times(1-\rho)\times(2+\delta-r)\times \\ &\left[1-\frac{1}{1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}}\right] \end{aligned}$$

$$\begin{aligned} M_3>0\Leftarrow M_4>0 &\Leftarrow 1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2\sqrt{\rho(1-\rho)(1-\rho r)\left[1+\delta-(1-\rho)r\right]}>1 \\ &\Leftarrow 1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2(1-\rho)\sqrt{\rho(2+\delta-r)\left[1+\delta-(1-\rho)r\right]}>1 \\ &\text{(Because } (1-\rho)(2+\delta-r)<1-\rho\times r \text{ when } \rho>\rho^*=\frac{1+\delta-r}{2+\delta-2r}) \\ &\Leftarrow 1+\delta\times\rho-2\rho\times r+2\rho^2\times r+2(1-\rho)\left[1+\delta-(1-\rho)r\right]>1 \\ &\text{(Because } \rho(2+\delta-r)>\left[1+\delta-(1-\rho)r\right] \text{ when } \rho>\rho^*=\frac{1+\delta-r}{2+\delta-2r}) \\ &\Leftarrow \rho<1<\frac{2\delta+2-2r}{\delta+2-2r} \end{aligned}$$

Consequently $M_3\geq 0$ and $\frac{\partial(\rho-\alpha_N^*)/(1-\rho)}{\partial\rho}\geq 0$. Thus, we can conclude that $\frac{\partial\epsilon_2}{\partial\rho}\geq 0$ when $\rho>\rho^*=\frac{1+\delta-r}{2+\delta-2r}$. $\square$