第四章：放射性与核衰变

李 阳

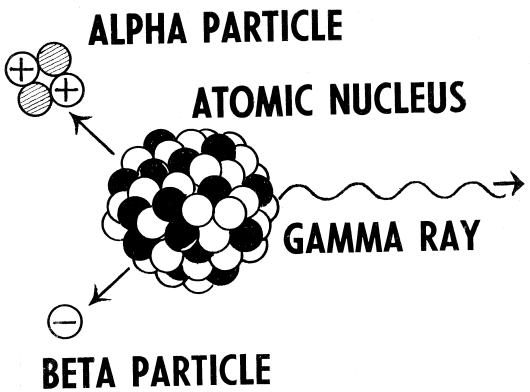
leeyoung1987@ustc.edu.cn

原子核物理导论 • 2026 年春

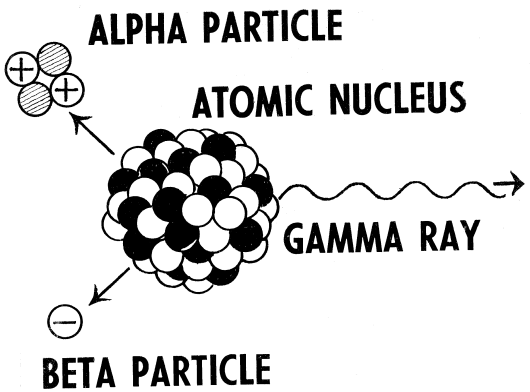
中国科学技术大学

Chapter 4: Radioactivity and Nuclear decay

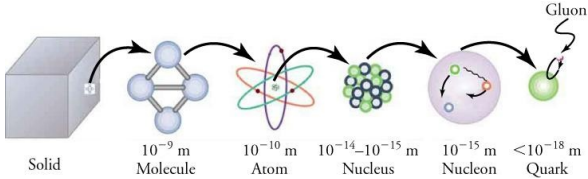
- Introduction
- α decay
- β decay
- γ decay



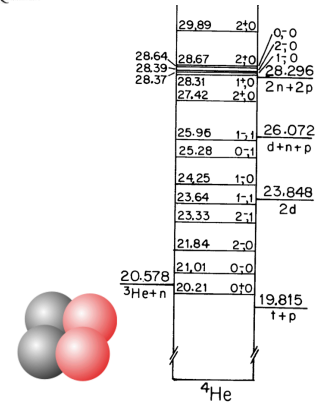
- **Introduction**
- α decay
- β decay
- γ decay



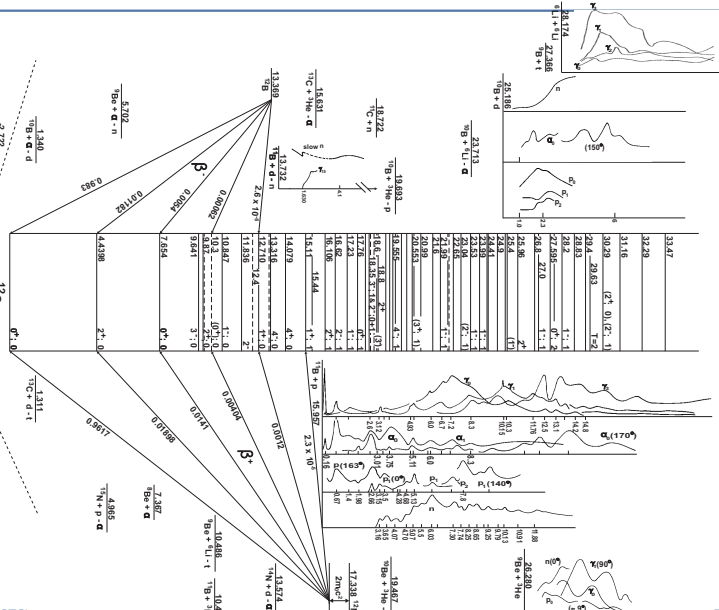
Atomic Nuclei



- Made of Z protons and N neutrons ($^{A}_{Z}X_{N}$) $\rightarrow$ bound states of the strong nuclear force (strong interaction)
- Global properties: mass $M \approx Am_u + B(Z, A)$, charge Ze, spin and parity J^{π}, isospin I (and its projection $I_3 = Z - N$)
- Moments: r.m.s. charge radius $\sqrt{\langle r^2 \rangle}$, magnetic moment μ_A, electric and magnetic multipole moments
- Interactions: Shell model, optical potentials, realistic NN interaction, 3-body forces, chiral effective field theory, strong interaction (quantum chromodynamics)



Example: Carbon-12



Radioactivity

- Most nuclei are not stable and spontaneously decays by emitting particles. The phenomenon is called radioactivity.
- In addition to naturally occurring radioactivity, radioactive nuclei can also be produced in laboratory.
- Discovery of natural radioactivity: Henri Becquerel (**贝克勒尔**), Pierre and Marie Curie (**居里**) (Nobel prize 1903)
- Discovery of artificial radioactivity : Irène Joliot-Curie and Frédéric Joliot (Nobel prize 1935)

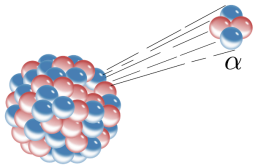
$$\alpha + {}^{26}_{13}\mathrm{Al} \rightarrow {}^{30}_{15}\mathrm{P} \rightarrow {}^{30}_{14}\mathrm{Si} + e^{+} + \nu_e$$

Henri Becquerel Pierre Curie Marie Curie

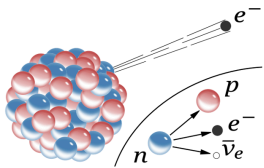
Frédéric Joliot Irène Joliot-Curie

Common types of radioactivity

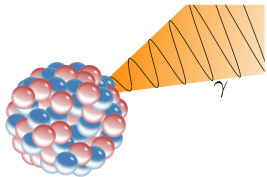
- α decay:
  - radioactivity emitting α particles, i.e. Helium-4 nuclei
  - proton-rich nuclei $Z \gtrsim 50$
  - strong interaction



- β decay:
  - radioactivity emitting electrons (β^- decay) or positrons, the anti-particle of the electron, (β^+ decay)
  - electron capture (EC): the orbital electrons captured by the atomic nucleus
  - weak interaction

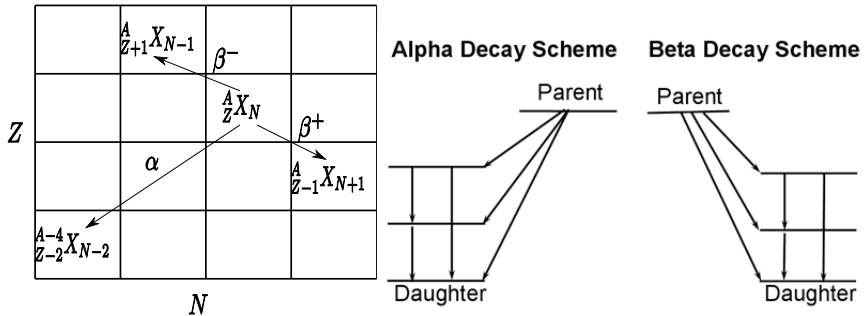


- γ decay:
  - radioactivity emitting high-energy photons
  - internal conversion (IC): the exited nucleus decays to a lower level by transferring the energy to the orbital electrons without photon emission
  - electromagnetic interaction



Types of radioactivity

Figure: Type of nuclear decays in the chart of nuclides. The scheme plots for alpha and beta decays are shown.



Other types of radioactivity

- Fission: spontaneous breakup of a heavy nucleus into two or more smaller nuclei. It is usually accompanied by the emission of neutrons.
- Neutron emission
- Proton emission
- Light cluster emission: ^{14}C, ^{20}O, ^{34}Si ...
- Double-beta decay

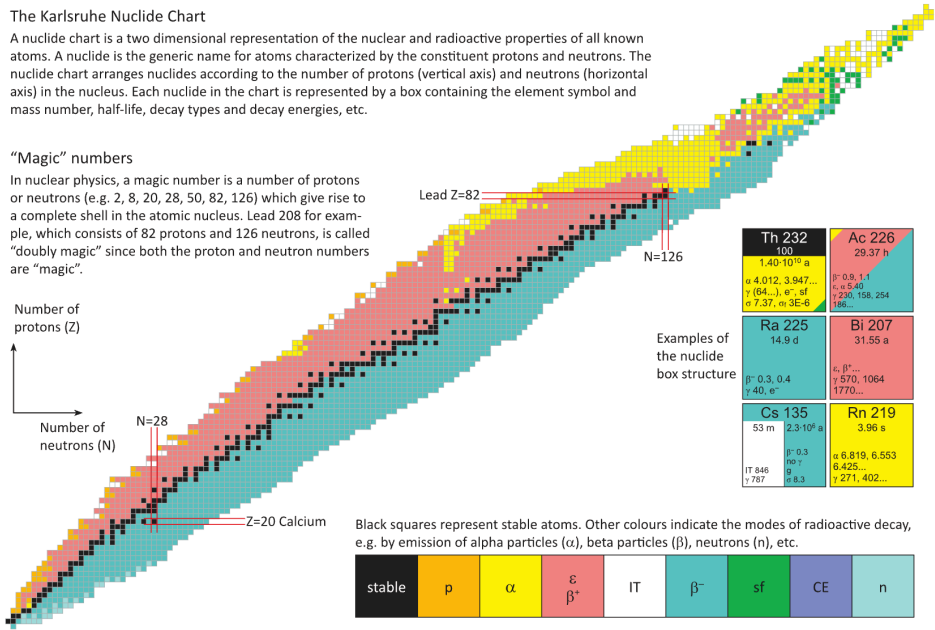
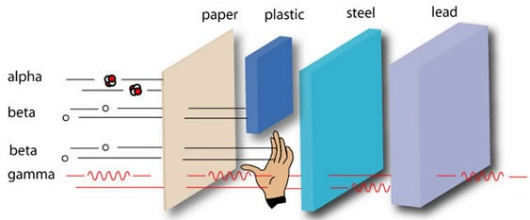


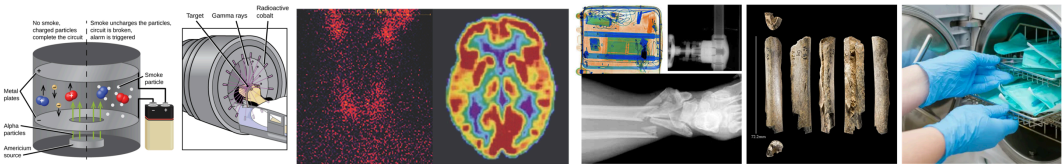
Fig. 1. Schematic diagram of the Karlsruhe Nuclide Chart showing the main features.

	Identity	Symbol	Charge	Rest Mass (amu)
α	helium nucleus	$^4_2\mathrm{He}$	+2	4.001506
β^-	electron	e or e^-	-1	0.000549
β^+	positron	e^+	+1	0.000549
γ	photon	γ	0	0
neutron	neutron	n or ^1_0n	0	1.008665
proton	proton/hydrogen nucleus	p or $^1_1\mathrm{H}$	+1	1.007276



Radioactive sources and applications

- α emitters: Am-241 (**镅**-241，smoke detectors), Ac-225 (**锕**-225，radiotherapy)
- β emitters: Y-90 (**钇**-90，radiotherapy), C-14 (**碳**-14，radiocarbon dating), F-18 (**氟**-18，positron emission tomography, PET)
- γ emitters: Co-60 (**钴**-60，medical sterilisation, industrial radiography), Cs-137/Ba-137 (**铯**-137/**钡**-137，radiotherapy), I-131 (**碘**-131，radioactive tracer)



Decay law

- Three years following the discovery of radioactivity in 1896, scientists found the decay rate of a pure radioactive substance decreases with time following an exponential law.
- It took several more years to realize that radioactivity represents changes in the individual atoms and not a change in the whole sample, and it was not chemical reaction.
- It took another two years to realize that the decay is statistical in nature, that it it impossible to predict when any given atom will disintegrate, and that this hypothesis leads directly to the exponential law. This lack of predictability of the behavior of single particles does not bother most scientists today, but this was difficult to accept before the development of quantum mechanics.

Decay law

- Radioactive decay is a random process at the level of single nucleus. In other words, radioactive nuclei have no memory. It does not age over time, and the probability of its breaking down per unit time stays constant over time. This constant is called the decay constant and is usually denoted by λ.
- Then the decay rate, the expected number of decayed nuclei per unit time, is,

$$\begin{aligned} \frac{dN}{dt} &= -\lambda N \\ \Rightarrow \quad N(t) &= N_0 e^{-\lambda t} \qquad (N_0 = N(0)) \end{aligned} \tag{1}$$

- Lifetime (mean decay time):

$$\bar{t} = \frac{\int dt\, t\, e^{-\lambda t}}{\int dt e^{-\lambda t}} = \frac{1}{\lambda} \tag{2}$$

- Half-life: the time that half of the original nuclei decay

$$t_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}, \quad \Rightarrow \quad N(t) = N_0 2^{-\frac{t}{t_{\frac{1}{2}}}} \tag{3}$$

nuclide	half-life (seconds)
H-5	8×10^{-23}
F-15	10^{-21}
Mg-19	5×10^{-12}
O-25	4×10^{-8}
Ra-205	2×10^{-1}
Na-26	1
C-11	10^{3}
Au-198	2×10^{5}
Ac-225	8×10^{5}
Co-60	2×10^{8}
H-3	4×10^{8}
Ra-226	5×10^{10}
U-238	10^{17}
Ca-48	7×10^{26}

Decay law

- Due to the random nature of radioactive decay, the experimentally observed decay number n over a period time T will not follow the decay law exactly,

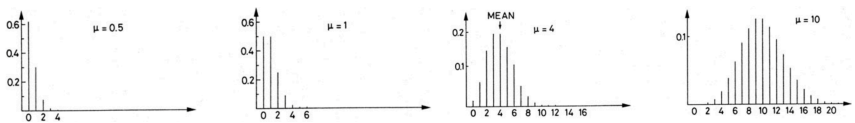
$$\mu = N_0\left[1 - \exp(-\lambda T)\right] \xrightarrow{T \ll t_{1/2}} N_0 \lambda T$$

- The observed decay number n follows the Poisson distribution:

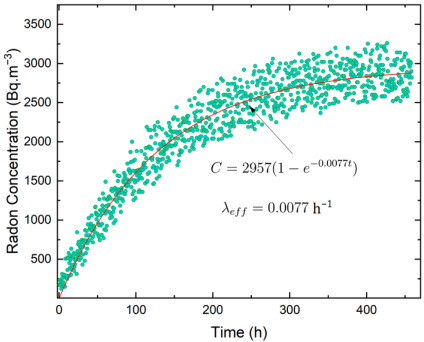
$$P(n) = \frac{\mu^n e^{-\mu}}{n!} \tag{4}$$

  The expectation value is $\langle n \rangle = \mu$ and the standard deviation is $\sigma = \sqrt{\mu}$

- For large μ, Poisson distribution approaches the normal distribution



Experimentally measured radon activity inside a chamber over a 15-day period with data collected every 30 min



Quantum theory of radioactive decay

- In quantum theory, the probability density $p(x)$ is the wave function squared. Therefore, for a system with radioactive decay, its wave function $\psi(x)$ acquires an exponential decay term,

$$\psi(t) = e^{-iE_0 t - \frac{1}{2}\lambda t}$$

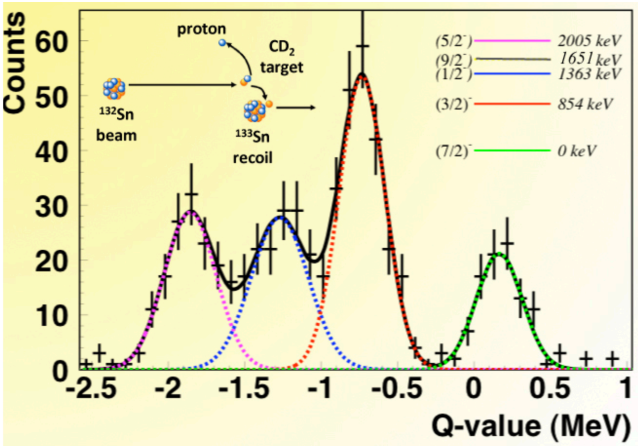
- The square of the Fourier transform of the wave function gives the probability to find the state in energy E:

$$P(E) = \frac{1}{(E - E_0)^2 + \frac{1}{4}\lambda^2}$$

- The energy density is broadened with a full-width $\Gamma = \lambda = 1/\bar{t}$, which is just the uncertainty relation for energy uncertainty Γ and time uncertainty $\bar{t}$:

$$\Gamma \cdot \bar{t} \sim \hbar$$

Quantum theory of radioactive decay



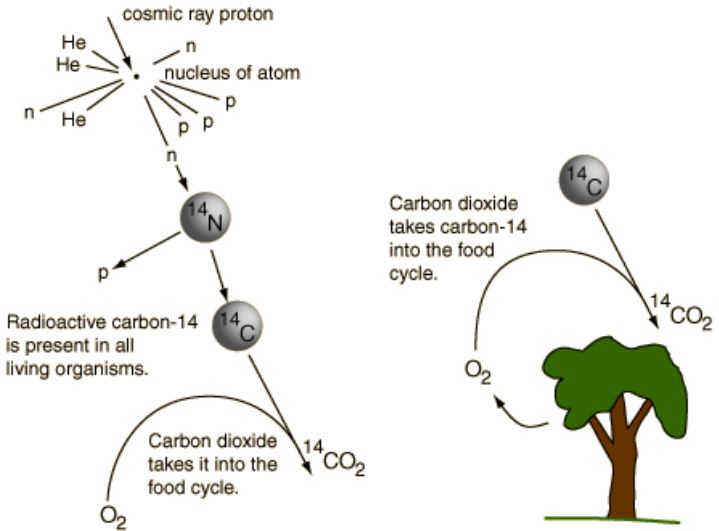
Radiocarbon dating

- Radiocarbon dating, pioneered by Willard Libby, is a method to determine the age of organic materials up to about 60,000 years using the radioactive isotope C-14. It plays a vital role in modern archaeology.
- Carbon in nature: C-12 (98.9%, stable), C-13 (1.06%, stable) & C-14 (10^{-12}, radioactive)
- C-14 is generated by cosmic rays bombarding N-14 in atmosphere, and decay back to N-14 with a half-life 5730 years

$$\mathrm{n} + {}^{14}_{7}\mathrm{N} \rightarrow \mathrm{p} + {}^{14}_{6}\mathrm{C},$$
$${}^{14}_{6}\mathrm{C} \rightarrow + {}^{14}_{7}\mathrm{N} + e^{-} + \bar{\nu}_{e}$$

The production rate of C-14 can be approximated as almost constant for thousands of years assuming that the flux of the cosmic ray does not vary much with time.

Radiocarbon dating



Radiocarbon dating

- All C-14 in atmosphere exists in the form of carbon dioxide $\mathbf{CO_2}$. The plants absorb C-14 by photosynthesis. The abundance of C-14 in living plants maintains the same level as in atmosphere. Animals consumes plants to update its C-14 abundance. Indeed, as carbon recycles, the C-14 abundance of entire ecosystem maintained an equilibrium. Once the plant or the animal dies, its C-14 abundance will not be updated and will decrease due to C-14 decay.
- The C-14:C-12 ratio can be used to date the period after the plant or animal died

$$r(t) = \frac{N_{14}(t)}{N_{12}(t)} = r(0)2^{-t/t_{\frac{1}{2}}}$$

Uranium-Lead dating

- Uranium-Lead dating provides a reliable method to date rocks about 1 million years to over 5 billion years ago
- It is based on zircon which incorporates uranium but strongly rejects lead when forming a crystal structure. As a result, all lead found in this mineral is generated from radioactive decay from uranium.
- It involves two decay chains: first from U-238 to Pb-206 (half-life 4.5 billion years) and second from U-235 to Pb-207 (half-life 700 million years).

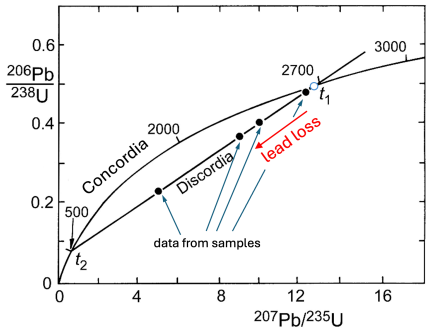
$$\frac{^{206}\mathrm{Pb}}{^{238}\mathrm{U}} = \exp(\lambda_{238} t) - 1, \tag{5}$$

$$\frac{^{207}\mathrm{Pb}}{^{235}\mathrm{U}} = \exp(\lambda_{235} t) - 1 \tag{6}$$

- Either ratio can be used to determine the time. One can also use the ratio of $^{206}\mathrm{Pb} : {}^{207}\mathrm{Pb}$ since the current ratio $^{238}\mathrm{U} : {}^{235}\mathrm{U} = 137.818$ is known precisely. However, there is the possibility of leakage of lead from the sample, which leads to uncertainties.

Concordia diagram

- One measures samples from the same rock, and plots the ratios as X-Y coordinates
- If there is no lead loss, the ratios fall to the theoretical curve called the concordant line. The second ratio serves as a cross check.
- If there is lead loss, the ratios from different samples may differ, and they form a line referred to as the discordance line.
- The upper intercept of the concordant line and discordant line is the time the rock initially formed



Activity

- Activity is defined as the number of atoms that decay and emit radiation in unit time

$$A = \frac{\Delta N}{\Delta t} = \lambda N = \frac{\ln 2}{t_{\frac{1}{2}}} N$$

- The SI unit for activity is the becquerel (Bq),

  1 Bq = 1 decay/s

  Examples:
  - Radioactivity released from Chernobyl: 10^{17} Bq = 10^{6} Ci
  - 1g of radium-226: 3.7×10^{10} Bq (1 Ci)
  - smoke detector (Am-241): 37 kBq (1 μCi)
  - 500g Banana (K-40): 65 Bq
  - Human (K-40, C-14): 8 kBq
- Note that Bq is a relative small unit. Radioactivity is commonly quantified in kBq (10^{3} Bq), rutherford (Rd) = MBq (10^{6} Bq), and GBq (10^{9} Bq) and curie (Ci): 1 Ci = 3.7037×10^{10} Bq = 37.037 GBq based on the activity of 1g of radium-226

Specific activity

Since it is easier to weigh a radioactive substance than to count its nuclei, it is useful to introduce specific activity: the activity per mass (Bq/g or Ci/g)

$$a = \frac{A}{M} = \frac{\ln 2}{t_{\frac{1}{2}}} \frac{N_A}{m_a}$$

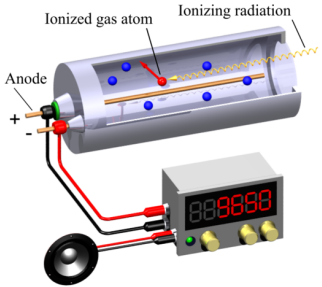
where, M_a (g/mol) is the molar mass of the radionuclide; $N_A = 6.02 \times 10^{23}\,\mathrm{mol}^{-1}$ is the Avogadro constant.

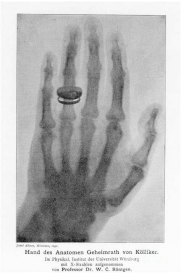
Isotope	activity per gram (Ci/g)
F-18	9.5×10^{7}
I-131	1.2×10^{5}
Y-90	5.4×10^{4}
Ac-225	5.8×10^{4}
Co-60	1132
H-3	9621
Cs-137	83
C-14	4.5
Am-241	3.43
Ra-226	1
K-40	6.4×10^{-6}
U-235	2.2×10^{-6}

https://www.nrc.gov/reading-rm/doc-collections/cfr/part071/part071-appa.html

Exposure

- Activity is not convenient to measure in practice. What is easier to quantify is the effects of the radiation, which is related to the energy released from the radiation.
- For X rays, one of the common quantification is the ability that the radiation ionizes the air, measured in terms of the ionized charge per unit mass of the air, called the exposure. The unit is C/kg or röntgen (R), 1 R = 2.58 $\times 10^{-4}$ C/kg
- Exposure is limited to X rays and gamma rays with energy less than 2.5 MeV. To quantify ionizing radiations of all form, the quantity dose was introduced





Radiation dose

- Absorbed dose is defined as the energy deposited in matter by ionizing radiation per unit mass

$$D = \frac{E}{M}$$

  The SI unit is gray (Gy), 1 Gy = 1 J/kg. Absorbed dose is widely used in radiation protection, radiology and radiation applications

- In radiation protection related areas, it is more useful to define equivalent dose (aka. dose equivalent) H, which takes into account the biological effects:

$$H = \sum_R D_R W_R$$

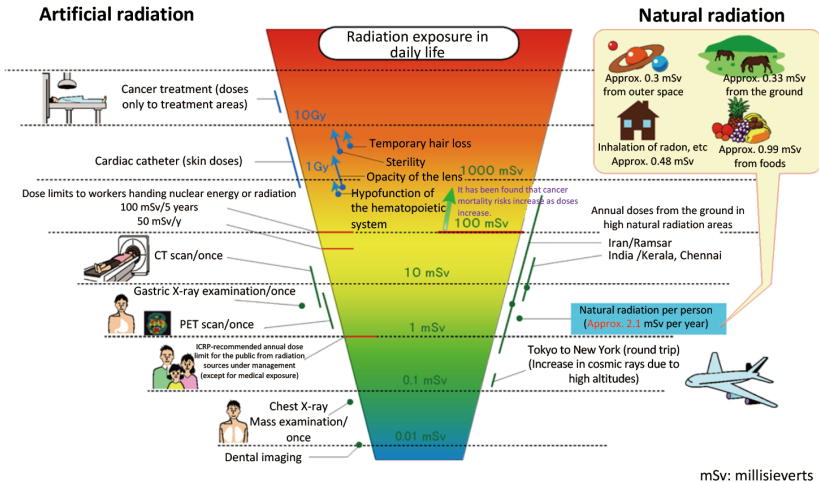
  where D_R is the absorbed dose of type R and W_R is its weighting factor. The SI unit of equivalent dose is sievert (Sv), 1 Sv = 1 J/kg

- Effective dose

Radiation weighting factors W_R according to ICRP report 103

radiation type	energy E (MeV)	W_R
X rays, gamma rays, β, μ		1
proton, $\pi^{\pm}$		2
neutrons	(< 1 MeV)	$2.5 + 18.2 \exp\left[\frac{1}{6}\ln^2 E\right]$
neutrons	(1 - 50 MeV)	$5.0 + 17.0 \exp\left[\frac{1}{6}\ln^2 2E\right]$
neutrons	(> 50 MeV)	$2.5 + 3.25 \exp\left[\frac{1}{6}\ln^2 0.04E\right]$
α, heavy nuclei		20

Natural radiation per person is 2 mSv per year. ICRP-recommended annual dose limit is 1 mSv for the public, which is also our national standard. For radiation workers, the limit is 20 mSv.



Homework 6

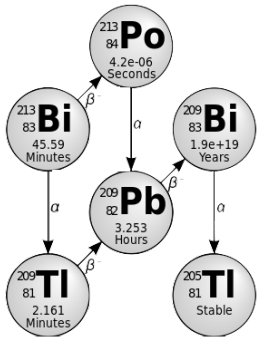
写一篇短文介绍放射性的一种应用，字数不少于一千字（中英文皆可）。

Decay chain

Typical radioactive nucleus does not decay directly to a stable nucleus. Rather it decays into other radioisotopes. Thus there is usually a series of decays until the atom has become a stable isotope. This series of decays is known as a decay chain

Example: decay chain of ^{213}Bi (铋-213)

$$^{213}\mathrm{Bi} \rightarrow {}^{213}\mathrm{Po} + e^- + 1.426\,\mathrm{MeV}$$
$$^{213}\mathrm{Bi} \rightarrow {}^{209}\mathrm{Tl} + {}^{4}\mathrm{He} + 5.982\,\mathrm{MeV}$$
$$^{213}\mathrm{Po} \rightarrow {}^{209}\mathrm{Pb} + {}^{4}\mathrm{He} + 8.537\,\mathrm{MeV}$$
$$^{209}\mathrm{Tl} \rightarrow {}^{209}\mathrm{Pb} + e^- + 3.980\,\mathrm{MeV}$$
$$^{209}\mathrm{Pb} \rightarrow {}^{209}\mathrm{Bi} + e^- + 0.644\,\mathrm{MeV}$$
$$^{205}\mathrm{Bi} \rightarrow {}^{205}\mathrm{Tl} + {}^{4}\mathrm{He} + 3.14\,\mathrm{MeV}$$



Natural radioactivity

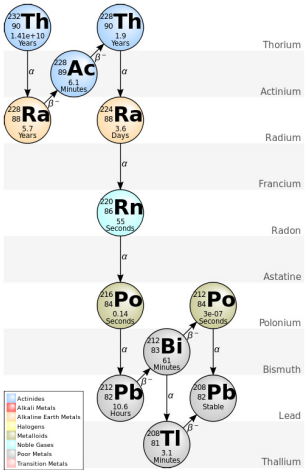
- On Earth, naturally occurring radionuclides fall into three categories: primordial radionuclides, secondary radionuclides, and cosmogenic radionuclides
- Primordial radionuclides are those existed before Earth was formed
  - They were formed, along with stable nuclides, in nucleosynthesis, and were present in the interstellar medium from which the solar system was formed
  - There are about 286 primordial nuclides, 251 of which are stable and 35 are radioactive
  - Only those have a half-life comparable to the age of the earth (5 billion years) have been preserved, such as K-40 (钾-40, 1.27×10^9 yr.), Th-233 (钍-233, 1.39×10^{10} yr.), U-238 (铀-238，4.51×10^9 yr.) and U-235 (铀-235，7.1×10^8 yr.), Rb-87 (铷-87，4.88×10^{10} yr.), Sm-147 (钐-147，06×10^{11} yr.), ...
- Secondary radionuclides are those arise in the decay chains of the primordial radionuclides
- cosmogenic radionuclides, such as C-14, are those formed by cosmic rays bombarding the atmosphere

Natural radioactivity

- Since only α decay can change the atomic mass number A, secondary radionuclides within the same chain must get the same remainder when their mass numbers modulo four
- Therefore, there are four decay chains based on the remainder, i.e. $A = 4n, 4n+1, 4n+2, 4n+3$
- Three main decay chains of primordial radionuclide are observed in nature: starting from Th-232 (**钍**-232, $4n$, 14 billion years), U-238 (**铀**-238, $4n+2$, 4.5 billion years) and U-235 (**铀**-235, $4n+3$ 700 million years), respectively
- There are no natural decay chains of the $4n+1$ series because this series Np-237 (**镎**-237) has a relatively short half-life 2.1 million years in comparison to the age of the earth. But this series can be generated artificially.

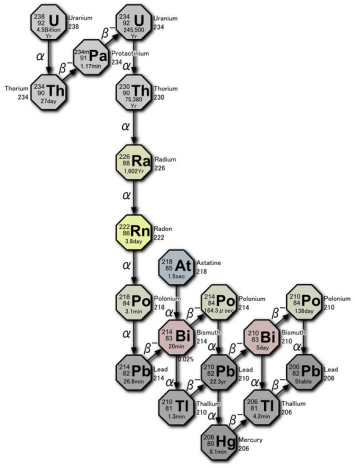
Thorium series 钍系

$4n$ chain, half-life 14 billion years



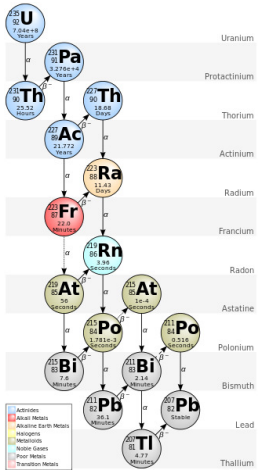
Uranium series 铀系

$4n+2$ series, half-life 4.5 billion years, also known as radium series



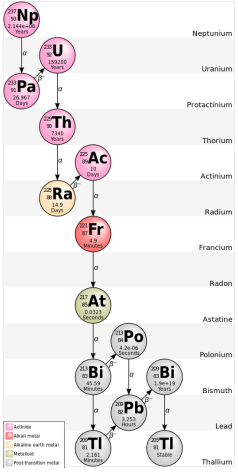
Actinium series 锕系

$4n + 3$ chain, half-life 700 million years



Neptunium series 镎系

$4n+1$ chain, half-life 2.1 million years, not observed naturally



Decay from multiple channels

Many nuclei can undergo different radioactive processes. Each process is called a decay channel.

- For simplicity, let us consider a radioactive nucleus undergoing two decay channels and the decay constants are λ_a and λ_b, respectively. The total decay rate is,

$$\frac{dN}{dt} = -N(\lambda_a + \lambda_b) \qquad \Rightarrow \qquad N = N_0 e^{-(\lambda_a+\lambda_b)t}$$

- For decay with an arbitrary number channels, the total decay constant is sum of the decay constant from each channel

$$\lambda = \sum_a \lambda_a$$

Decay from multiple channels

The relative intensities of the competing decays are called branching ratio.

- The number of decay product from channel a is,

$$N_a(t) = \frac{\lambda_a}{\lambda} N_0 \left[1 - e^{-\lambda t}\right] = \frac{\lambda_a}{\lambda} N(t) \left[e^{\lambda t} - 1\right]$$

- Note that the ratio of any two products is a constant:

$$N_a(t) : N_b(t) = \lambda_a : \lambda_b$$

- The branching ratio of channel a is:

$$\mathcal{B}_a = \frac{N_a}{\sum_a N_a} = \frac{\lambda_a}{\lambda}$$

Decay in presence of nuclear production

In presence of nuclear production, the number of radioactive nuclei follows the law:

$$\frac{dN}{dt} = R - \lambda N$$

where, R is the production rate.
To solve this equation, we perform a change of variable: $r \equiv R - \lambda N$,

$$\begin{aligned} \frac{dr}{dt} &= -\lambda \frac{dN}{dt} = -\lambda r \\ \Rightarrow \quad N(t) &= \frac{1}{\lambda} R(1 - e^{-\lambda t}) + N(0) e^{-\lambda t} \\ &= N(\infty)(1 - e^{-\lambda t}) + N(0) e^{-\lambda t} \end{aligned}$$

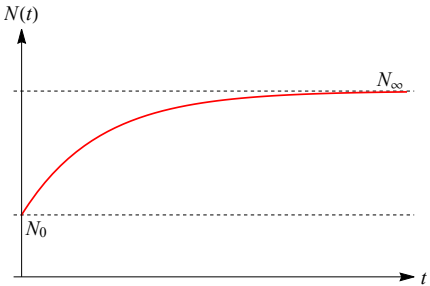
After a very long time $t \to \infty$, the number of radioactive nuclei approaches a constant $N(\infty) = R/\lambda$, called secular equilibrium.

Decay in presence of nuclear production

Activity:

$$A = \lambda N \approx \begin{cases} \lambda(N_0 + Rt), & t \ll t_{1/2} \\ R, & t \gg t_{1/2} \end{cases}$$

We see the activity is linear in time for short time and reach equilibrium (secular) for long time.



Chain decay

Consider a chain decay $P \xrightarrow{\lambda_P} D \xrightarrow{\lambda_D} G$. The decay rates are

$$\frac{dN_P}{dt} = -\lambda_P N_P, \quad \Rightarrow \quad N_P(t) = N_p(0)e^{-\lambda_P t}$$
$$\frac{dN_D}{dt} = \lambda_P N_P - \lambda_D N_D$$

We assume the number of the daughter radionuclei takes the form,

$$N_D(t) = c_1 e^{-\lambda_P t} + c_2 e^{-\lambda_D t} + c_3$$

The coefficients can be determined by matching the coefficients of the exponentials as well as the initial condition,

$$N_D(t) = N_P(0)\frac{\lambda_P}{\lambda_D - \lambda_P}\left[e^{-\lambda_P t} - e^{-\lambda_D t}\right] + N_D(0)e^{-\lambda_D t}.$$

Secular equilibrium

For $\lambda_P \ll \lambda_D$, i.e. the half-life of the daughter is much shorter than the half-life of the parent, the long time behavior of the daughter nuclei N_D are,

$$\begin{aligned} N_D(t) =& e^{-\lambda_P t}\left\{N_P(0)\frac{\lambda_P}{\lambda_D-\lambda_P}(1-e^{-(\lambda_D-\lambda_P)t}) + N_D(0)e^{(\lambda_D-\lambda_P)}\right\} \\ \approx& N_P(0)\frac{\lambda_P}{\lambda_D}e^{-\lambda_P t} \\ =& \frac{\lambda_P}{\lambda_D}N_P(t) \end{aligned}$$

Therefore the activity is the same for the parent and daughter

$$\lambda_P N_P \approx \lambda_D N_D. \tag{7}$$

Since N_P decays very slowly, the number of the daughter radionuclei reaches an equilibrium: its production rate equal to its decay rate.

Decay chain

Consider a decay chain $A \xrightarrow{\lambda_1} B \xrightarrow{\lambda_2} C \xrightarrow{\lambda_3} D$(stable nuclei), the decay rates are

$$\begin{aligned} \frac{dN_1}{dt} &= -\lambda_1 N_1 \\ \frac{dN_2}{dt} &= \lambda_1 N_1 - \lambda_2 N_2 \\ \frac{dN_3}{dt} &= \lambda_2 N_2 - \lambda_3 N_3 \end{aligned} \tag{8}$$

A general solution has the form,

$$\begin{aligned} N_1 &= a_{11} e^{-\lambda_1 t} \\ N_2 &= a_{21} e^{-\lambda_1 t} + a_{22} e^{-\lambda_2 t} \\ N_3 &= a_{31} e^{-\lambda_1 t} + a_{32} e^{-\lambda_2 t} + a_{33} e^{-\lambda_3 t} \end{aligned} \tag{9}$$

Decay chain

We assume that there are only A species at the initial time. The initial conditions are

$$\begin{aligned} a_{11} &= N_0 \\ a_{21} + a_{22} &= 0 \\ a_{31} + a_{32} + a_{33} &= 0 \end{aligned} \tag{10}$$

Inserting the solution (9) into Eq. (8) we can solve a_{ij}, the second and third lines becomes

$$\begin{aligned} -a_{21}\lambda_1 e^{-\lambda_1 t} &= \lambda_1 a_{11} e^{-\lambda_1 t} - a_{21}\lambda_2 e^{-\lambda_1 t} \\ -a_{31}\lambda_1 e^{-\lambda_1 t} - a_{32}\lambda_2 e^{-\lambda_2 t} &= a_{21}\lambda_2 e^{-\lambda_1 t} + a_{22}\lambda_2 e^{-\lambda_2 t} \\ & \quad -a_{31}\lambda_3 e^{-\lambda_1 t} - a_{32}\lambda_3 e^{-\lambda_2 t} \end{aligned} \tag{11}$$

Decay chain

Then we get

$$
\begin{aligned}
a_{21} &= \frac{\lambda_1}{\lambda_2-\lambda_1}a_{11} \\
a_{22} &= -a_{21} \\
a_{31} &= \frac{\lambda_2}{\lambda_3-\lambda_1}a_{21} = \frac{\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}a_{11} \\
a_{32} &= \frac{\lambda_2}{\lambda_3-\lambda_2}a_{22} = \frac{\lambda_1\lambda_2}{(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)}a_{11} \\
a_{33} &= -a_{31}-a_{32} = \frac{\lambda_1\lambda_2}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)}a_{11}
\end{aligned}
\tag{12}
$$

Decay chain

Then Eq. (8) becomes

$$
\begin{aligned}
N_1 &= N_0 e^{-\lambda_1 t} \\
N_2 &= N_0 \frac{\lambda_1}{\lambda_2 - \lambda_1}(e^{-\lambda_1 t} - e^{-\lambda_2 t}) \\
N_3 &= N_0 \frac{\lambda_1 \lambda_2}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)} e^{-\lambda_1 t} + N_0 \frac{\lambda_1 \lambda_2}{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_2)} e^{-\lambda_2 t} \\
&\quad + N_0 \frac{\lambda_1 \lambda_2}{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)} e^{-\lambda_3 t}
\end{aligned}
\tag{13}
$$

Bateman equation

We can generalize to arbitrarily long decay chains:

$$\frac{dN_1}{dt} = -\lambda_1 N_1, \tag{14}$$

$$\frac{dN_{i+1}}{dt} = \lambda_i N_i - \lambda_{i+1} N_{i+1} \qquad (i = 1, 2, \cdots, n) \tag{15}$$

The solution is given by Bateman equation,

$$N_k = N_0 \lambda_1 \lambda_2 \cdots \lambda_{k-1} \sum_{i=1}^{k} \prod_{j(i \neq j)}^{k} \frac{1}{(\lambda_j - \lambda_i)} e^{-\lambda_i t} \tag{16}$$

Bateman equation

To obtain the solution, we assume N_{k+1} has the following form,

$$N_{k+1}(t) = \sum_{i=1}^{k+1} a_{k+1,i} e^{-\lambda_i t} \tag{17}$$

Inserting the above into Eq. (15) we obtain

$$\begin{aligned} -\sum_{i=1}^{n} a_{n+1,i}\lambda_i e^{-\lambda_i t} &= \sum_{i=1}^{n} (\lambda_n a_{n,i} - \lambda_{n+1} a_{n+1,i}) e^{-\lambda_i t} \\ \rightarrow \quad -a_{n+1,i}\lambda_i &= \lambda_n a_{n,i} - \lambda_{n+1} a_{n+1,i} \\ \rightarrow \quad a_{n+1,i} &= \frac{\lambda_n}{\lambda_{n+1} - \lambda_i} a_{n,i},\ i = 1, \cdots, n \end{aligned} \tag{18}$$

Bateman equation

From the initial condition, $N_{n+1}(0) = 0$, we can obtain $a_{n+1,n+1}$,

$$a_{n+1,n+1} = -\sum_{i=1}^{n} a_{n+1,i} = N_0 \lambda_1 \lambda_2 \cdots \lambda_{n+1} \prod_{j \neq i}^{n+1} \frac{1}{\lambda_j - \lambda_{n+1}} \tag{19}$$

So we finally get (Bateman equation)

$$N_{n+1} = N_0 \lambda_1 \lambda_2 \cdots \lambda_{n+1} \sum_{i=1}^{n+1} \prod_{j \neq i}^{n+1} \frac{1}{\lambda_j - \lambda_i} e^{-\lambda_i t} \tag{20}$$

Assuming $\sum_{i=1}^{n} a_{n,i} = 0$, one can prove Eq. (19) (proof by induction). The trick for the proof is to replace $a_{n,1}$ in the last line of Eq. (18) with $-\sum_{i=2}^{n} a_{n,i}$.

Chain decay

The above analysis assumes there is one production channel for each radionuclide. In general, there may be multiple channels of production and multiple branches of decay. The decay rate of the i-th nuclide can be written as,

$$\frac{d}{dt}N_i = \sum_{j(j<i)} \lambda_{ij} N_j - \lambda_i N_i. \tag{21}$$

To obtain the general solution, we can write this differential equation in a matrix form as,

$$\frac{d}{dt}N = -\Lambda . N \tag{22}$$

where, $N = (N_1, N_2, \cdots, N_n)^T$ and matrix Λ is,

Chain decay

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & \\ -\lambda_{12} & \lambda_2 & & & & \\ -\lambda_{13} & -\lambda_{23} & \lambda_3 & & & \\ \vdots & \vdots & \vdots & & & \\ -\lambda_{1i} & \cdots & -\lambda_{ji} & \cdots & \lambda_i & \\ \vdots & \vdots & \vdots & & & \ddots \end{pmatrix} \tag{23}$$

Then, the general solution is,

$$N(t) = e^{-\Lambda t} N(0) \tag{24}$$

Chain decay

The exponential of a matrix can be defined from the matrix series,

$$e^{-\Lambda t} = I - \Lambda t + \frac{1}{2!}(-\Lambda t)^2 + \frac{1}{3!}(-\Lambda t)^3 + \cdots \tag{25}$$

The infinite series can be terminated at n-th order with a reshuffle of coefficients due to Cayley-Hamilton theorem, where n is the dimension of the matrix Λ.

Chain decay

To evaluate the matrix exponential, apart from the definition (25) there are several methods,

- Matrix diagonalization. If Λ is diagonalizable,

$$\Lambda = S^{-1} D S$$

where, $D = \mathbf{diag}\{d_1, d_2, \cdots, d_n\}$ is a diagonal matrix. Its exponential is also diagonal: $e^{-Dt} = \mathbf{diag}\{e^{-d_1 t}, e^{-d_2 t}, \cdots, e^{-d_n t}\}$. Then,

$$e^{-\Lambda t} = S^{-1} e^{-Dt} S.$$

Chain decay

- Sylvester's formula. Again, if Λ is diagonalizable, the Sylvester's formula states,

$$e^{-\Lambda t} = \sum_i e^{-d_i t} L_i$$

where L_i is the Lagrange interpolation matrix (called Frobenius covariants of Λ),

$$L_i = \prod_{j(j \neq i)} \frac{1}{d_i - d_j} (\Lambda - d_j I)$$

Chain decay

- Putzer Algorithm.

$$e^{-\Lambda t} = \sum_i r_i(t) P_i$$

where,

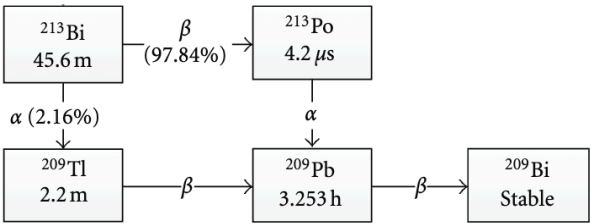
$$P_i = \prod_{k=2}^{i} (d_i I - \Lambda), \quad P_1 = I$$

and

$$\begin{aligned} \dot{r}_1 &= -d_1 r_1, \quad r_1(0) = 1 \\ \dot{r}_{i+1} &= -d_{i+1} r_{i+1} + r_i, \quad r_{i+1}(0) = 0, \quad (i = 1, 2, \cdots n-1) \end{aligned}$$

Note the algorithm does not require that the matrix Λ be diagonalizable and bypasses complexities of the Jordan canonical forms normally utilized.

Homework 7



The decay chain of Bismuth-213 is shown above. Assume that we have $1\,\mathrm{g}$ Bi-213 at the initial time $t = 0$. How much Bi-209 are there after 2 hours?