第四章：放射性与核衰变

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原子核物理导论・2026 年春

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- Introduction
- α decay
- β **decay**
- γ decay

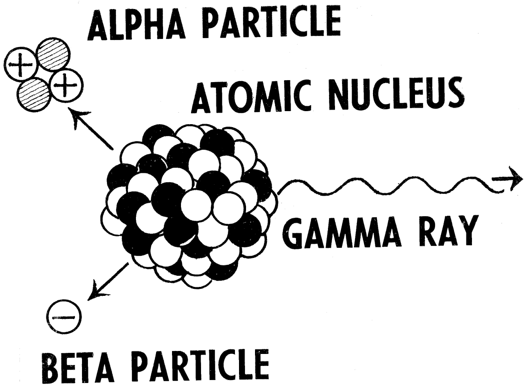
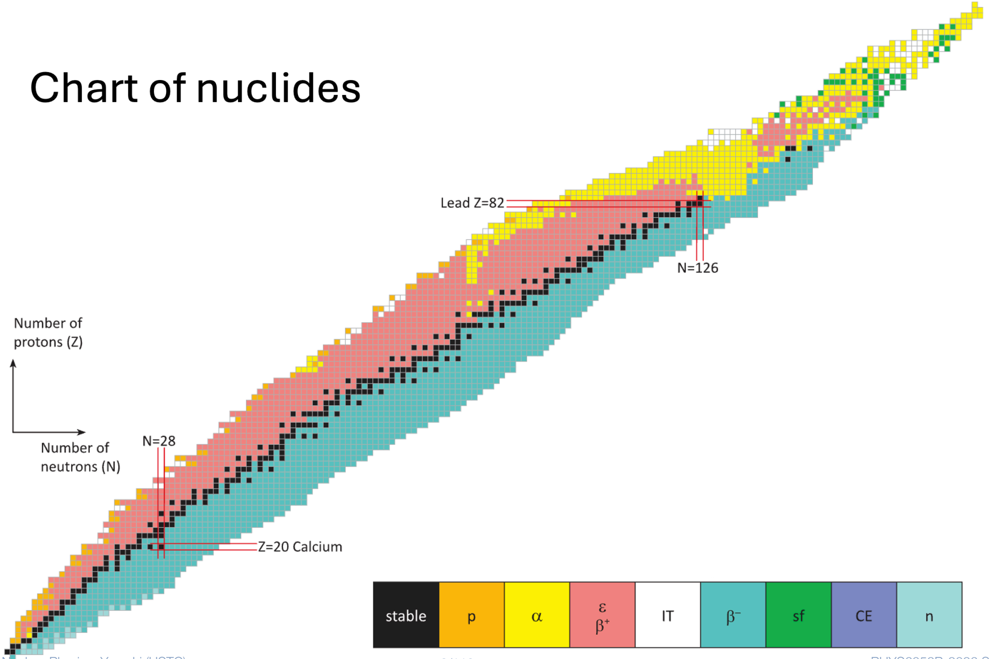


Chart of nuclides



History of β decay

- In 1899, Rutherford separated radioactive emissions into two types: α and β (now β^-), based on their penetration ability
- In 1900, Becquerel measured the m/e of the β particles and found that it is the same as the electron, and therefore identified β-rays as electron beams
- In 1911, Lise Meitner and Otto Hahn measured the spectrum of the β emission and found it is continuous, in contrast to α and γ decay. It was later realized this is in apparent contradiction to energy conservation
- In 1930, Pauli proposed a new extremely light neutral particle (later called neutrino) to resolve the energy lost puzzle in β decay
- In 1933, Charles Ellis and Nevill Mott found that the β spectrum has an upper bound in energy, which ruled out Bohr's theory that energy conservation is statistical

History of β decay

- In 1933, Fermi published his landmark theory of beta decay
- In 1934, Joliot-Curies discovered the β^+ decay (Nobel Prize 1935)
- In 1934, Wick, Yukawa and others proposed electron capture theory, which was observed in 1937 by Alvarez
- In 1942, Wang Ganchang (**王淦昌**) proposed to use beta capture to detect neutrinos. In 1956, neutrino was detected directly in experiments by Cowan and Reines (Nobel Prize 1995)
- In 1956, T.D. Lee and C.N. Yang proposed parity violation in weak interaction. Chien-Shiung Wu and others measured the beta decay of $^{60}\mathrm{Co}$ (the Wu experiment) and confirmed Lee & Yang's theory (Nobel Prize in 1957)

Chien-Shiung Wu 吴健雄

- In 1912, Chien-Shiung Wu was born in Taicang, Jiangsu. Her father was an engineer, revolutionary and educator, and her mother was a teacher
- In 1934, Wu graduated from National Central University (later split into Nanjing University and Southeast University). She worked as a research assistant at Zhejiang University and the Institute of Physics of the Academia Sinica after graduation

Chien-Shiung Wu 吴健雄

- In 1936 - 1940, she did her Ph.D. at Berkeley under the supervision of Ernest Lawrence and Emilio Segrè
- In 1942, Wu married Luke Chia-Liu Yuan (**袁家骝**), physicist and grandson of Yuan Shikai. Wu became the first female faculty member of Princeton university.
- In 1944, Wu joined the Manhattan Project, and played a critical role
- In 1945, Wu joined Columbia university and remain at Columbia for the rest of her career
- In 1949, Wu first verified quantum entanglement to resolve the EPR paradox
- In 1945-1949, Wu experimentally verified Fermi's theory of beta decay
- In 1957, Wu conducted the experiment to verify parity violation, which led to Lee and Yang's Nobel Prize in 1957. Many including Jack Steinberger (Nobel Prize 1988) called the omission of Wu in Nobel Prize the biggest mistake of the Nobel committee.

Chien-Shiung Wu 吴健雄

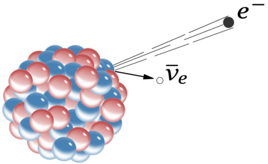
- In 1958, Wu experimentally verified the conservation of vector current proposed by Feynman and Gell-Mann
- In 1964, Wu was awarded the Comstock Prize in Physics
- In 1966, Wu published her monograph *Beta Decay*
- In 1975, Wu received the National Medal of Science and served as the president of the American Physical Society
- In 1978, Wu won the inaugural Wolf Prize in physics
- In 1981, Wu retired from Columbia University
- In 1997, Wu passed away in New York, survived by her husband and children

``Beta decay was...like a dear old friend. There would always be a special place in my heart reserved especially for it."

——*C.-S. Wu*

Introduction to β decay

- α decay is a two-body decay, i.e. there are two final products, the α particle and the daughter nucleus. From energy and momentum conservations, we conclude that the energy of the α particle is discrete
- Unlike α decay, the energy spectrum of the β particle from β decay is continuous. Only two products, the β particle and the daughter nucleus are detected in the experiments. If β decay is a truly two-body decay, then some energy must be lost in the process. This poses a long-standing puzzle for physicists
- Pauli proposed that there is a neutral particle that escape the detector. This particle is later called the neutrino, meaning little neutral one in Italian. With neutrino, β decay is now recognized as a three-body decay and energy & momentum are conserved,



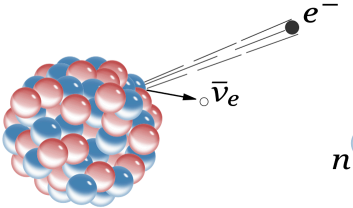
$$^A_ZX \quad \to \quad ^{A}_{Z+1}Y + e^- + \overline{\nu}_e \tag{1}$$

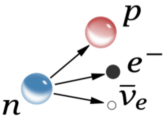
Introduction to β decay

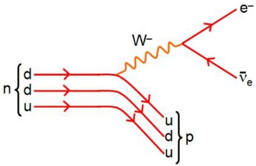
- The most elementary β decay is the decay of the neutron into a proton, an electron and an electron anti-neutrino,

$$n \to p + e^- + \overline{\nu}_e$$

- The modern theory of β decay is based on the Standard Model, which unifies the weak and electromagnetic interactions. β decay is caused by the weak interaction between the quark and the W boson which then decays into an electron and electron anti-neutrino
- In Standard Model, the neutrinos are massless. Recent experiments confirm that neutrinos have finite masses. However, their masses are very small, no more than 1 eV







Neutrino

Neutrinos are one of the most fascinating and elusive particles known to physics

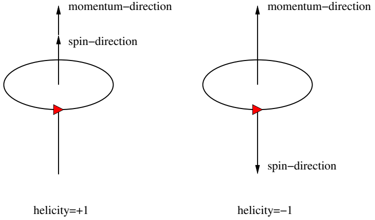
- They are spin-1/2, charge neutral and nearly massless ($m_\nu \lesssim 1\,\mathrm{eV}$) point-like particles
- They only participate the weak interaction and gravitational interaction. So their interaction with normal matter is extremely weak, therefore making them incredibly difficult to detect

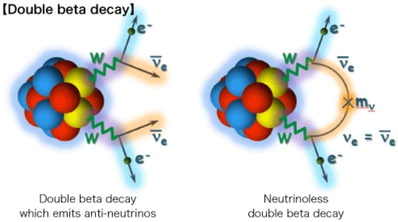
$$\sigma_{\nu N} \sim 10^{-27}\,\mathrm{barn}, \quad \sigma_{NN} \sim 0.1\,\mathrm{barn}$$

- There are three types (called flavor) of neutrinos, electron neutrino ν_e, muon neutrino ν_μ and tau neutrino ν_τ. Each flavor of neutrino corresponds to a flavor of charged lepton, i.e. the electron e, the muon μ and the tau τ. The number of each lepton flavor is conserved
- Each flavor of neutrino has an anti-particle with an opposite flavor
- Neutrino was detected in experiments by Cowan and Reines in 1956 (Nobel Prize 1995)

Neutrino

- Since parity is violated in weak interaction, neutrinos are handed. Specifically, all neutrinos are left-handed, i.e. their chirality is negative; whereas all anti-neutrinos are right-handed.
- It remains a mystery whether there are right-handed neutrinos and left-handed anti-neutrinos. It is possible there are but they are heavy and do not interact with matter via weak force (sterile neutrinos)
- Another mystery surrounding neutrino is whether neutrinos and their anti-neutrino are in fact the same (like the photon), known as Majorana neutrino
- An important test whether neutrinos are Majorana is that whether there exists neutrinoless double-beta decay



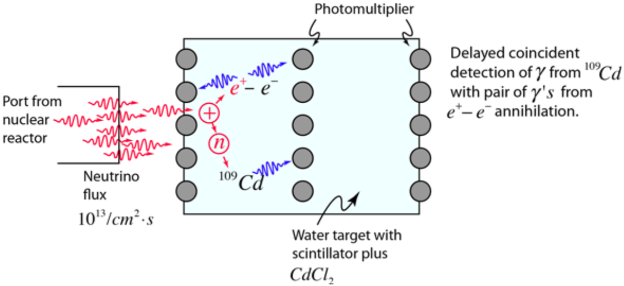


Neutrino

- Neutrinos oscillate between different flavors in flight. For example, an electron neutrino produced in a beta decay reaction may interact in a distant detector as a muon or tau neutrino, as defined by the flavor of the charged lepton produced in the detector. This oscillation occurs because the three mass state components of the produced flavor travel at slightly different speeds.
- In research that spanned from 1967 –1985, Raymond Davis consistently found only one-third of the neutrinos that standard theories predicted. His results threw the field of astrophysics into an uproar, and, for nearly three decades, physicists tried to resolve the so-called "solar neutrino puzzle." Experiments in the 1990s using different detectors around the world eventually confirmed the solar neutrino discrepancy.
- Raymond Davis Jr. shared the Nobel Prize in Physics in 2002 (88 years old then) for the detection of cosmic neutrinos, and resolving the solar neutrino problem in the Homestake Experiment.

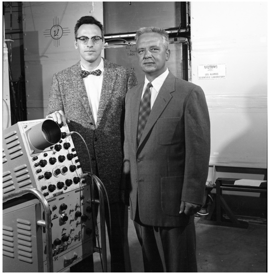
Cowan-Reines neutrino experiment

$$
\begin{aligned}
\overline{\nu} + p &\rightarrow n + e^{+} \\
e^{+} + e^{-} &\rightarrow \gamma + \gamma, \\
n + {}^{108}_{48}\,\mathrm{Cd} &\rightarrow {}^{108m}\mathrm{Cd} \rightarrow {}^{109}\mathrm{Cd} + \gamma
\end{aligned}
$$



Cowan-Reines neutrino experiment

They made use of the fact that nuclear reactors were expected to produce neutrino fluxes on the order of $10^{12} - 10^{13}$ neutrinos per second per cm^2, far higher than any attainable flux from radioactive sources. They used 1400 liter water as the proton source. The positrons will annihilate with electrons and emit a pair of 0.511 MeV photons, which will be detected by scintillation material (as gamma detector) in the water. At the same time, the neutrons are captured by cadmium (a neutron absorber), which will produce a de-excitation gamma within 5 μs.



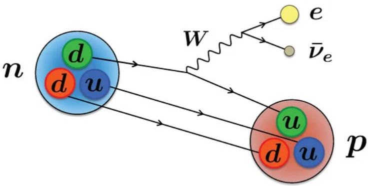
Classification of β decay

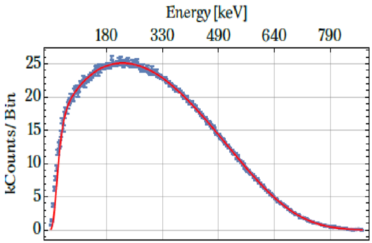
- β^- decay (notation β^-):
$$ {}^A_Z X \to {}^{A}_{Z+1}Y + e^- + \bar{\nu}_e $$
Example: free neutron decay $n \to p + e^- + \bar{\nu}_e$
- β^+ decay (notation β^+):
$$ {}^A_Z X \to {}^{A}_{Z-1}Y + e^+ + \nu_e $$
Example: ${}^{30}\mathrm{P} \to {}^{30}\mathrm{Si} + e^+ + \nu_e$
- Electron capture (notation ε):
$$ {}^A_Z X + e^- \to {}^{A}_{Z-1}Y + \nu_e $$
Example: ${}^{81}\mathrm{Kr} + e^- \to {}^{81}\mathrm{Br} + \nu_e$

Free neutron decay

$$n \to p + e^- + \bar{\nu}_e$$

- A free neutron is not stable. It's mass $M_n = 939.6\,\mathrm{MeV}$ is slightly larger than that of the proton $938.3\,\mathrm{MeV}$ by about $1.3\,\mathrm{MeV}$.
- The electron spectrum is continuous up to $0.782 \pm 0.013\,\mathrm{MeV}$
- The lifetime of the free neutron is about $\tau_n = 878\,\mathrm{s} \approx 15\,\mathrm{min}$

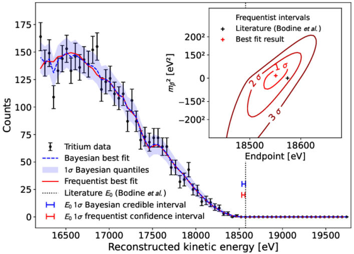


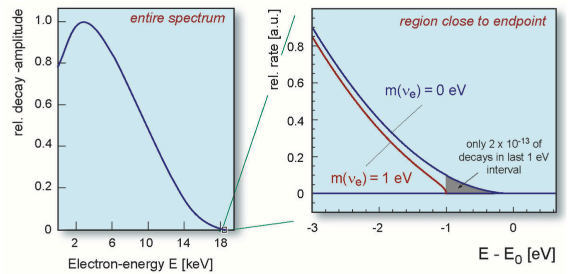


β decay of tritium

$$^3_1\mathrm{H} \to {}^3_2\mathrm{He} + e^- + \overline{\nu}_e + 18.6\,\mathrm{keV}$$

The energy released from this process is fairly low, making it a good tool to determine the neutrino mass. The upper limit of the electron neutrino mass set from this method is 2 eV



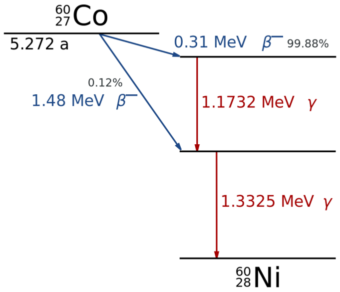


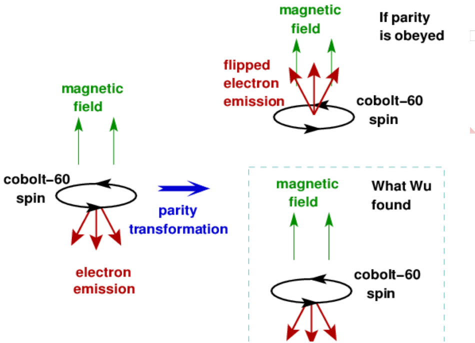
Co-60 β decay and parity violation

- Co-60 decays to two excited states of Ni-60 through beta emission with a half life of 5.27 years. The excited states of Ni-60 emit two gamma rays with energies at 1.1732 MeV and 1.33 MeV. The decay energies are 0.31 MeV and 1.48 MeV respectively.

$$^{60}\mathrm{Co} \to {}^{60}\mathrm{Ni}^* + e^- + \bar{\nu}_e \to {}^{60}\mathrm{Ni} + e^- + \bar{\nu}_e + \gamma$$

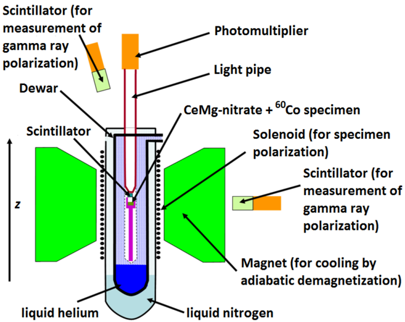
- Co-60 β decay is used by C.-S. Wu in her famous experiment to demonstrate parity violation

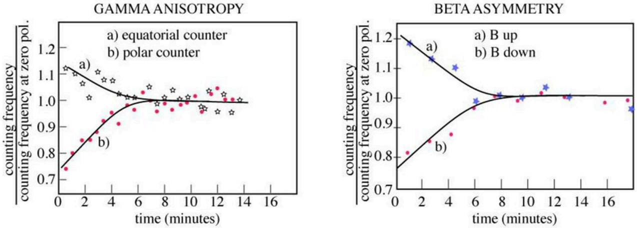




Wu experiment

- Turn on B to polarize the Co-60 atoms and then turn off B to measure the γ and β distribution
- γ distribution is used as additional calibration





Experiment performed during Christmas break 1956 at the NBS (National Bureau of Standards) in Washington, D.C. C.S. Wu cancelled her vacation.

Wu, C. S. et al. (1957); *Phys. Rev.* **105** 1413 and *ibid.* **106** (1957) 1361

Kinematics of the β^- decay

$$ {}^A_Z X \to {}^{A}_{Z+1} Y + e^- + \overline{\nu}_e $$

- Energy and momentum conservation in the c.m. frame implies,

$$ \vec{p}_e + \vec{p}_\nu + \vec{p}_Y = 0, \qquad T_e + T_\nu + T_Y = Q $$

where, $Q = m_X - m_Y - m_e - m_\nu \approx [M(Z,A) - M(Z+1,A)]m_u - m_e$.

- For the kinetic energy of the leptons, we adopt

$$ T_l = E_l - m_l = \sqrt{\vec{p}_l^2 + m_l^2} - m_l \quad \Rightarrow \quad \vec{p}_l^2 = T_l^2 + 2m_l T_l $$

- For the recoil energy daughter nucleus, i.e. its kinetic energy, we adopt

$$ T_Y \approx \frac{\vec{p}_Y^2}{2m_Y} $$

Kinematics of the β^- decay

The electron kinetic energy and the daughter nucleus recoil energy satisfy the constraint,

$$(Q - T_e - T_Y)^2 = T_e^2 + 2m_eT_e + 2m_YT_Y - 2\sqrt{2m_YT_Y(T_e^2 + 2m_eT_e)}\cos\Theta_{eY}$$

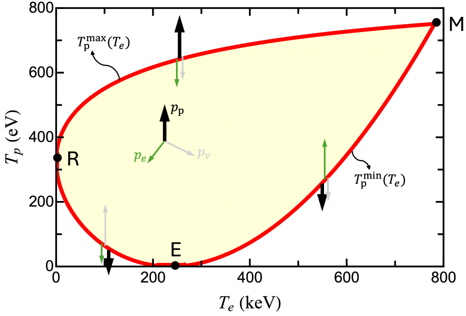


Figure: The Dalitz plot of the free neutron decay

Kinematics of the β decay

The maximum momentum and maximum kinetic energy of the electron are

$$|\vec{p}_e|_{\max} = \frac{1}{m_X}\sqrt{Q(2m_e + Q)(m_Y + \tfrac{1}{2}Q)(m_e + m_Y + \tfrac{1}{2}Q)}, \quad T_{e,\max} = \frac{Q(m_Y + \frac{1}{2}Q)}{m_X}$$

where, $Q = m_X - m_Y - m_e$. The maximum kinetic energy is achieved when the neutrino and the recoil daughter nucleus is traveling in the same direction

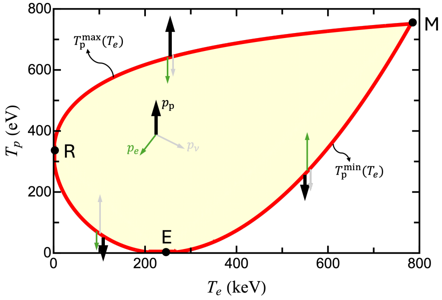
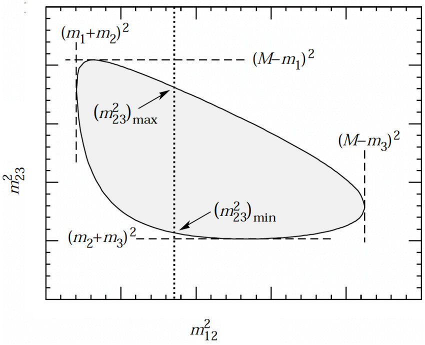


Figure: The Dalitz plot of the free neutron decay

General relativistic three-body decay

For a general relativistic three-body decay, it is useful to adopt Lorentz invariants: $m_{ij}^2 = (p_i + p_j)^2$

- $m_{12}^2 = (P - p_3)^2 = M^2 + m_3^2 - 2ME_3$
- $m_{12}^2 + m_{23}^2 + m_{13}^2 = M^2 + m_1^2 + m_2^2 + m_3^2$
- $|\vec{p}_3| = [(M^2 - (m_{12} + m_3)^2)(M^2 - (m_{12} - m_3)^2)]^{1/2}/2M$



Q-value

The Q-value for β^- is given in terms of the nuclear masses as,

$$\begin{aligned} Q(\beta^-) &= m_N(A,Z) - m_N(A,Z+1) - m_e - m_\nu \\ &\approx m(A,Z) - m(A,Z+1) \\ &= B(A,Z+1) - B(A,Z) + \delta_{nH} \end{aligned}$$

where, $m_N(A,Z)$ is the mass of the nucleus and $m(A,Z) = m_N(A,Z) + Zm_e - \sum_i W_i$ is the mass of the neutral atom, which already includes the mass of the Z electrons and the electron binding energies. $B(A,Z) = m(A,Z) - Zm_H - Nm_n$ is the binding energy.
$\delta_{nH} = m_n - m_H = 0.782\,\mathrm{MeV}$ is the mass difference between the neutron and the Hydrogen atom. We have assumed that the neutrino mass is zero and we ignored the electron binding energy in the .

Q-value

The Q-value for β^+ is,

$$Q(\beta^+) = M(A,Z) - M(A,Z-1) - 2m_e = B(A,Z-1) - B(A,Z) - 2m_e - \delta_{nH}$$

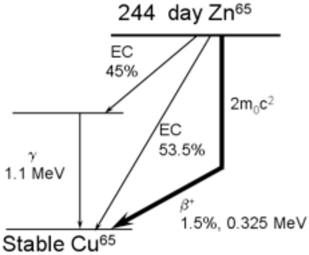
For electron capture, the binding energy of the electrons must be taken into account. This is because the atom will be left in an excited state after capturing the electron, and the binding energy of the captured innermost electron is significant. The Q-value for electron capture:

$$Q(\varepsilon) = M(A,Z) - M(A,Z-1) - W_i = B(A,Z-1) - B(A,Z) - \delta_{nH} - W_i$$

Because the binding energy of the electron is much less than the mass of the electron, nuclei that can undergo β^+ decay can always also undergo electron capture, but the reverse is not true, i.e. some nuclides can undergo electron capture even through they are energetically forbidden for β^+ decay.

β^+ decay and EC of Zn-65

$$
\begin{aligned}
{}^{65}_{30}\mathrm{Zn} &\rightarrow {}^{65}_{29}\mathrm{Cu} + e^+ + \nu_e \\
{}^{65}_{30}\mathrm{Zn} + e_i^- &\rightarrow {}^{65}_{29}\mathrm{Cu} + \nu_e
\end{aligned}
$$



There are two EC channels, whose decay energies are 0.236 MeV (45%) and 1.34 MeV (53.5%). The decay energy of the β^+ decay is 0.325 MeV.

β decay of Cu-64

Cu-64 can undergo all three types of β decay: β^- decay, β^+ decay and electron capture

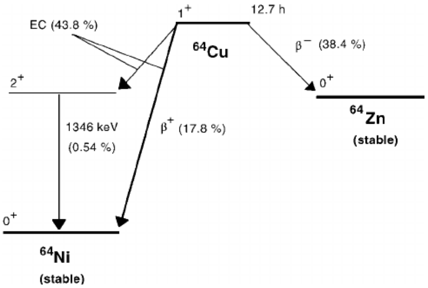


Figure: The Q-value of the β-decay for all known nuclides

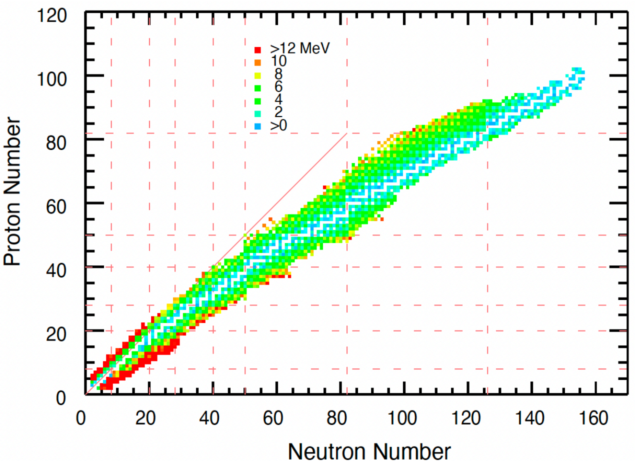
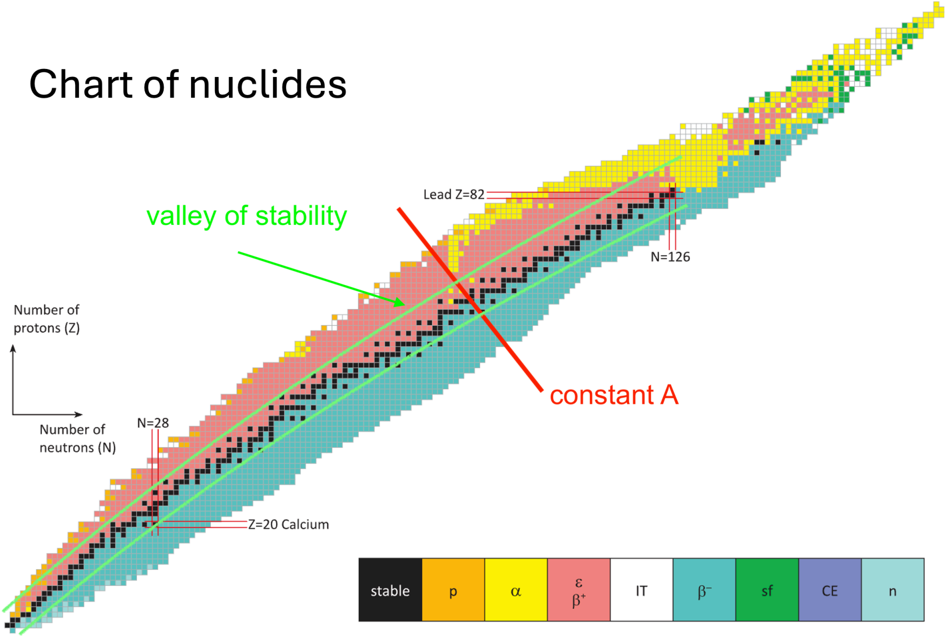


Chart of nuclides



Isobaric mass parabola

$$m(A,Z) = Zm_p + (A-Z)m_n - a_V A + a_S A^{2/3} + a_C Z^2 A^{-1/3} + a_{sym}(A-2Z)^2 A^{-1} - s a_P A^{-1/2}$$

- For fixed A, the nuclear mass m is a quadratic function of Z, i.e. a parabola
- The parabola is concave, since

$$\frac{\partial^2 m}{\partial Z^2} = 2a_C A^{-1/3} + 8a_{sym} A^{-1} = 1.42 A^{-1/3} + 186.4 A^{-1} > 0$$

- The location of the minimum:

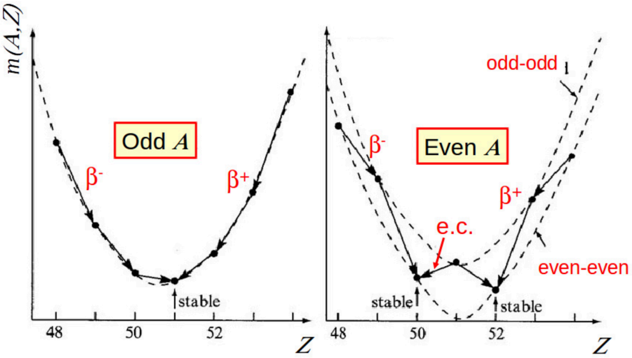
$$\left.\frac{\partial m}{\partial Z}\right|_A = 0 \quad \Rightarrow \quad m_p - m_n + 2a_C Z A^{-1/3} - 4a_{sym} + 8a_{sym} Z A^{-1} = 0$$

$$\Rightarrow \quad Z_0 = A\frac{m_n - m_p + 4a_{sym}}{2a_C A^{2/3} + 8a_{sym}} = A\frac{94.5}{186.4 + 1.42A^{2/3}}$$

β decay stability

Recall the isobaric mass parabolas from the liquid drop model:

Figure: The isobaric mass parabolas



Due to the presence of the pairing term, odd-A isobars have different pattern from the even-A isobars

Double beta decay

There are two kinds of double β decays, one with two neutrinos produced ($\beta\beta2\nu$) and another without neutrinos $\beta\beta0\nu$.

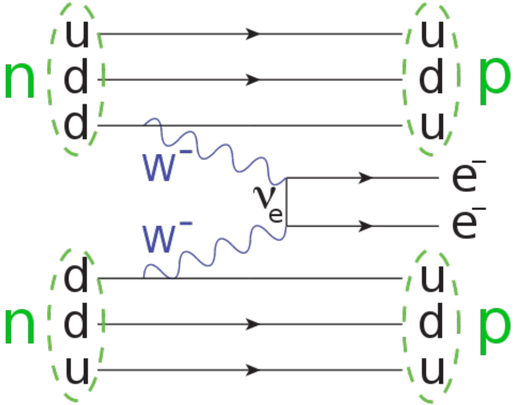
$$\begin{aligned} {}^{A}_{Z}X &\rightarrow {}^{A}_{Z+2}Y + 2e^- + 2\overline{\nu}, \quad (\beta\beta2\nu) \\ {}^{A}_{Z}X &\rightarrow {}^{A}_{Z+2}Y + 2e^-, \qquad (\beta\beta0\nu) \end{aligned} \tag{2}$$

The neutrinos in $\beta\beta2\nu$ are Dirac neutrinos for which anti-neutrinos and neutrinos are different fermions, those in $\beta\beta0\nu$ are Majorana ones for which anti-neutrinos and neutrinos are identical. The neutrinoless double β deay ($\beta\beta0\nu$) is the combination of two processes,

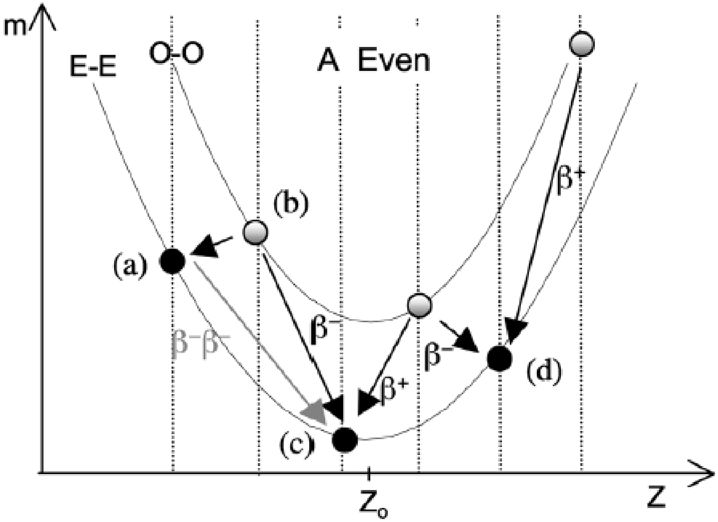
$$\begin{aligned} n &\rightarrow p + e^- + \overline{\nu} \\ \nu + n &\rightarrow p + e^- \end{aligned} \tag{3}$$

Double beta decay

Figure: Double beta decay. From: http://en.wikipedia.org/wiki/Double_beta_decay.



Double beta decay



Double beta decay

No experiments so far have succeeded in proving that neutrinos are of Dirac or Majorana types. In β decay, the final nucleus must have a larger binding energy than the initial nucleus. For some nuclei the β decay cannot happen since the nucleus with one more proton and one less neutron has smaller binding energy, the nucleus with two more protons and two less neutrons has larger binding energy so that the double β decay can happen. Germanium-76 is an example, which can in principle decay to selenium-76 via the double β decay. Here are isotopes which have been observed to have double β decay: ${}^{48}_{20}$Ca, ${}^{76}_{32}$Ge, ${}^{82}_{34}$Se, ${}^{96}_{40}$Zr, ${}^{100}_{42}$Mo, ${}^{116}_{48}$Cd, ${}^{128}_{52}$Te, ${}^{130}_{52}$Te, ${}^{150}_{60}$Nd.

β spectrum

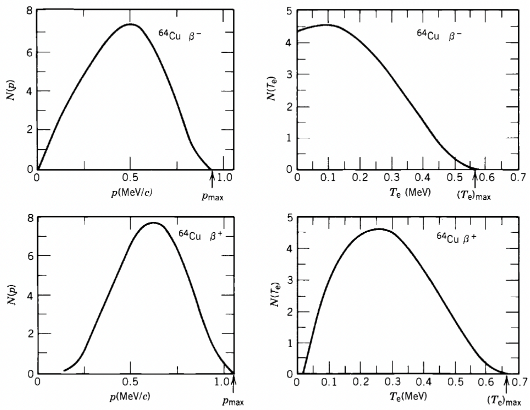


Figure 9.3 Momentum and kinetic energy spectra of electrons and positrons emitted in the decay of ^{64}Cu. Compare with Figure 9.2; the differences arise from the Coulomb interactions with the daughter nucleus. From R. D. Evans, *The Atomic Nucleus* (New York: McGraw-Hill, 1955).

Fermi's theory of β decay: historical background

- In 1921, quantum mechanics was born and applied to problems of atoms
- In the realm of nuclear physics, the proton was identified by Rutherford in 1920
- In 1932, Chadwick discovered the neutron. Immediately, Heisenberg proposed that the atomic nucleus consists of Z protons and $A - Z$ neutrons, establishing the correct
- In 1930, Pauli proposed neutrino as the invisible particle in β decay
- In 1934, Fermi proposed a theory to describe the β decay based on Pauli's neutrino hypothesis and Heisenberg's model of nucleus
- In 1935, Gamow and Teller generalized Fermi's theory to tensor and axial vector interactions to account for transitions that changes the nuclear spin

Fermi's theory of β decay: historical background

- In 1949, Lee, Rosenbluth and Yang, and independently Wheeler and Tiomno discovered that Fermi's theory can be describe meson decays, e.g., μ decay, μ capture, and π decay, which leads to the idea of universal Fermi type interaction, i.e. the weak interaction
- In 1956, Lee and Yang proposed the idea parity violation
- Based on Lee and Yang's work, in 1957, Sudarshan and Marshak, and Feynman and Gell-Mann, proposed the vector minus axial vector (V-A) theory as the universal Fermi theory for weak interaction.
- In the 60s, Glashow, Salam, Weinberg and others proposed a unified theory for electromagnetic and weak interactions based on the non-abelian gauge theory proposed by Yang and Mills in 1954. The electroweak theory predicted the existence of a neutral vector meson Z boson, which was discovered in 1973 at CERN

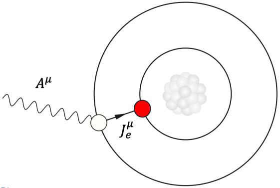
Fermi interaction: origin

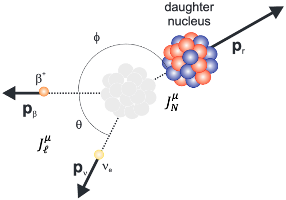
- Fermi's theory of β decay is inspired by Dirac's theory of atomic radiation, where the proton and the neutron play the role of the ground state and the excited state of the atoms. The light leptons play the role of radiation
- Dirac's theory features the interaction Hamiltonian,

$$H_{\text{int}} = \int d^3x \, e\overline{\psi}_e \gamma_\mu \psi_e A^\mu$$

Fermi replaced the electron current with nuclear current $\overline{\psi}_e \gamma_\mu \psi_e \to \bar{p}\gamma_\mu n$ and replaced the radiation field with a lepton current $A^\mu \to \bar{e}\gamma^\mu \nu$. Hence, Fermi's interaction Hamiltonian reads,

$$H_{\text{int}} = \int d^3x \, G(\bar{p}\gamma_\mu n)(\bar{e}\gamma^\mu \nu) + \text{H.c.}$$





Fermi interaction: V-A

- According to Gamow and Teller, to account for β decay with nuclear spin flip, the above interaction can be generalized to different types, i.e. replacing γ^μ with other Dirac spinors,

$$H_{\text{int}} = \int d^3x \sum_i G_i (\bar{p}\Gamma^i n)(\bar{e}\Gamma^i \nu) + \text{H.c.},$$

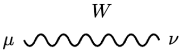
$$\begin{aligned}
\Gamma^i = 1, &\quad \text{scalar (S)}, \quad P = +1, \quad \Delta J = 0; \\
\gamma_5, &\quad \text{pseudo-scalar (P)}, \quad P = -1, \quad \text{no contribution in the NR limit} \\
\gamma^\mu, &\quad \text{vector (V)}, \quad P = -1, \quad \Delta J = 0; \\
\gamma^\mu\gamma_5, &\quad \text{axial vector (A)}, \quad P = +1, \quad \Delta J = 1; \\
\sigma^{\mu\nu}, &\quad \text{tensor (T)}, \quad P = +1, \quad \Delta J = 1.
\end{aligned}$$

- Parity violation and V-A theory suggests that the correct form of interaction is,

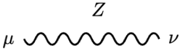
$$H_{\text{int}} = \int d^3x \frac{G}{\sqrt{2}} [\bar{p}\gamma^\mu (g_V - g_A\gamma_5) n] [\bar{e}\gamma_\mu (1-\gamma_5)\nu] + \text{H.c.}$$

Fermi interaction: SM

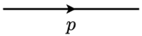
$$-i\left[\frac{g_{\mu\nu}}{k^2+i\epsilon}-(1-\xi_A)\frac{k_\mu k_\nu}{(k^2)^2}\right]$$



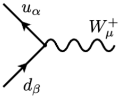
$$-i\,\frac{1}{k^2-m_W^2+i\epsilon}\left[g_{\mu\nu}-(1-\xi_W)\frac{k_\mu k_\nu}{k^2-\xi_W m_W^2}\right]$$



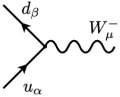
$$-i\,\frac{1}{k^2-m_Z^2+i\epsilon}\left[g_{\mu\nu}-(1-\xi_Z)\frac{k_\mu k_\nu}{k^2-\xi_Z m_Z^2}\right]$$



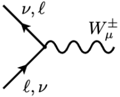
$$\frac{i(\not{p}+m_f)}{p^2-m_f^2+i\epsilon}$$



$$-i\,\eta\,\frac{g}{\sqrt{2}}\gamma_\mu P_L\,V_{\alpha\beta}$$



$$-i\,\eta\,\frac{g}{\sqrt{2}}\gamma_\mu P_L\,V_{\alpha\beta}^*$$



$$-i\,\eta\,\frac{g}{\sqrt{2}}\gamma_\mu P_L$$

Figure: Relevent Feynman rules of the Standard Model (SM)

Fermi interaction: SM

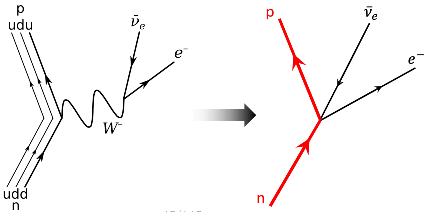
- From the modern perspective, Fermi's theory is a low-energy effective theory of the electroweak interaction:

$$\mathcal{M}_{fi} = \frac{g^2}{8} V_{ud} \bar{U}_u \gamma^\mu (1-\gamma_5) U_d \frac{g_{\mu\nu} + q_\mu q_\nu / M_W^2}{q^2 + M_W^2} \bar{U}_e \gamma^\nu (1-\gamma_5) V_\nu + O(g^4)$$

- The intermediate W is heavy in comparison to the characteristic energies involved in the nuclear beta decay. Therefore, the propagator can be approximated by $1/M_W^2$, i.e.,

$$\mathcal{M}_{fi} = \frac{G}{\sqrt{2}} V_{ud} [\bar{U}_u \gamma^\mu (1-\gamma_5) U_d][\bar{U}_e \gamma_\mu (1-\gamma_5) V_\nu] + O(g^4, q^2/M_W^2)$$

where, $G/\sqrt{2} = g^2/(8M_W^2)$ is Fermi's constant, $V_{ud} = 0.9741$ is the CKM matrix element



Fermi interaction: nuclear matrix elements (NMEs)

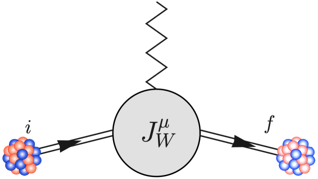
- For nuclear β decay, i.e. from quarks to nuclei,

$$\mathcal{M}_{fi} = \frac{GV_{ud}}{\sqrt{2}} \langle f|J_W^\mu|i\rangle \bar{U}_e \gamma_\mu (1-\gamma_5) V_\nu + O(g^4)$$

  where, $|i\rangle$ and $|f\rangle$ are the state vectors of the initial and final state nuclei, respectively. $\langle f|J_W^\mu|i\rangle$ is known as the nuclear matrix element.

- Weak current consists of a vector current V^μ and an axial-vector current A^μ,

$$\begin{aligned} J_W^\mu &= \bar{u}\gamma^\mu(1-\gamma_5)d + \text{H.c.} \\ &= \bar{u}\gamma^\mu d - \bar{u}\gamma^\mu\gamma_5 d + \text{H.c.} \equiv V^\mu - A^\mu \end{aligned}$$



Fermi interaction: hadronic matrix elements (HMEs)

- The simplest case is the free nucleon (proton and nucleon). The most general form of its HMEs subject to symmetries (Lorentz symmetries, Hermiticity, CPT etc) is,

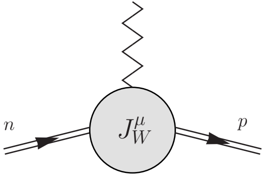
$$\langle p(k', s')|J_W^\mu|n(k,s)\rangle = j_V^\mu - j_A^\mu$$

$$j_V^\mu = \bar{U}_{s'}(k')[g_V(q^2)\gamma^\mu + \frac{g_M(q^2)}{2M}\sigma^{\mu\nu}q_\nu + \frac{g_S(q^2)}{2M}q^\mu]U_s(k)$$

$$j_A^\mu = \bar{U}_{s'}(k')[g_A(q^2)\gamma^\mu\gamma_5 + \frac{g_T(q^2)}{2M}\sigma^{\mu\nu}\gamma_5 q_\nu + \frac{g_P(q^2)}{2M}\gamma_5 q^\mu\Big]U_s(k)$$

where, $M = (M_p + M_n)/2$ is the average nucleon mass; $q = p_n - p_p$. $g_i(q^2)$ are called the form factors (FFs), which can be computed in theory, e.g. Lattice QCD, or measured in experiments.

- For β-decay, we are primarily interested in the low-energy constants (LECs) $g_i(q^2 \to 0)$



g_V	g_M	g_S	g_A	g_T	g_P
1.0	3.70	1.02	1.278	0.987	349

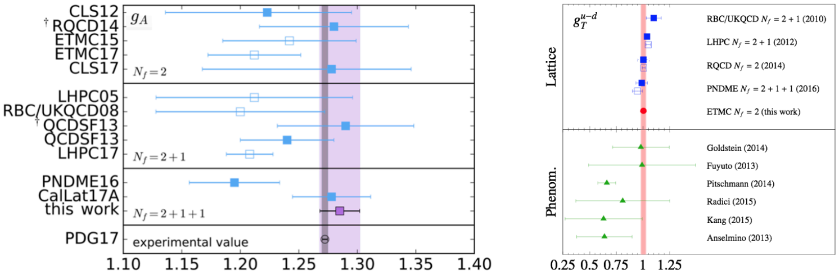


Figure: Lattice determination of nucleon axial charges

Dipole approximation:

$$g_A(Q^2) \approx \frac{g_A(0)}{\left(1 + Q^2/M_A^2\right)^2}$$

where, the nucleon axial mass $M_A = 1.068(17)\,\mathrm{GeV}$

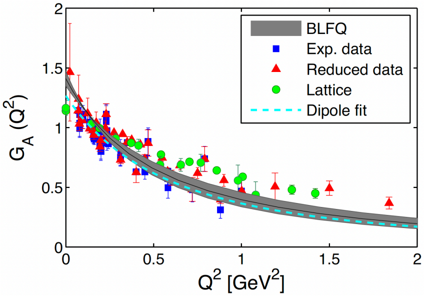


Figure: Theoretical evaluation of axial form factor $G_A(Q^2)$

Isospin symmetry

- In SM, the u, d quarks have similar masses and they form an isospin doublet (similar to spin up and spin down in normal spin doublet). Nucleons, i.e. the neutron and the proton, can also be treated as an isospin doublet

$$q = \begin{pmatrix} u \\ d \end{pmatrix}, \qquad N = \begin{pmatrix} p \\ n \end{pmatrix}$$

- Similar to spin, we can introduce the 2-by-2 Pauli matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. But to avoid confusion, we use a different notation for isospin

$$\vec{\sigma} \ \rightarrow \ \vec{\tau}, \qquad \vec{s} = \frac{\vec{\sigma}}{2} \ \rightarrow \ \vec{I} \equiv \vec{t} \equiv \frac{\vec{\tau}}{2}$$

- Using isospin, a nucleon N within a nucleus can be denoted by its isospin projection, similar to spin:

$$|N\rangle = |n, l, m, j = \tfrac{1}{2}, m_j, t = \tfrac{1}{2}, t_z\rangle$$

where, n, l, m are spatial quantum numbers.

- Similar to the addition of spins, the nucleus is also denoted by the isospin quantum numbers:

$$|\alpha\rangle = |J, m_J, T, T_z = (Z - N)/2\rangle$$

Conserved vector current (CVC)

- Using the isospin, the electromagnetic current and the vector current can be written as

$$J^\mu_{EM} = \frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d = \bar{q}\gamma^\mu \frac{\tau^3}{2} q + \frac{1}{6}\bar{q}\gamma^\mu q, \qquad V^\mu = \bar{u}\gamma^\mu d = \bar{q}\gamma^\mu \frac{\tau^-}{2} q$$

  where, $\tau^\pm = \tau^1 \pm i\tau^2$ is the isospin raising operator, i.e.

$$\frac{\tau^+}{2}|d\rangle = |u\rangle, \quad \frac{\tau^+}{2}|u\rangle = 0, \quad \frac{\tau^-}{2}|d\rangle = 0, \quad \frac{\tau^-}{2}|u\rangle = |d\rangle$$

- The strong force does not strongly modify the isospin symmetry, as indicated by the small nucleon mass difference $M_n - M_p = 1.3\,\mathrm{MeV}$. Therefore, in analogy to J^μ_{EM}, we can conjecture CVC $\partial_\mu V^\mu = 0$ on the nucleon level, which implies,
  - $g_V(0) = 1$, similar to charge conservation
  - $g_M(0) = \mu_p - 1 - \mu_n = 3.70$, where μ_N is the nucleon magnetic moment (in unit of $e/2M$)
  - $g_S = 0$

Partially conserved axial-vector current (PCAC)

- The axial current A^μ is not conserved unless in the exact chiral limit $m_q \equiv (m_u + m_d)/2 \to 0$

$$\partial_\mu A^\mu = 2m_q \bar{q} i\gamma_5 q + \text{anomaly} \equiv \phi_\pi + \text{anomaly}$$

  That is, the non-conserving part is due to the pion field.
- Goldberger-Treiman relation:

$$g_A M = f_\pi g_{\pi NN}$$

  where, f_π is the pion decay constant, $g_{\pi NN}$ is the π-N coupling constant
- PCAC also implies,

$$g_P = g_A(M_n + M_p)/(m_d + m_u) = 349(9)$$

Fermi interaction: from SM to V-A

- If we ignore the nuclear recoil $|\vec{q}| \ll M$, only $g_V(0)$ and $g_A(0)$ contribute. The leading order amplitude,

$$\mathcal{M}_{fi} = \frac{GV_{ud}}{\sqrt{2}}[\bar{U}_n\gamma^\mu(g_V - g_A\gamma_5)U_p]\bar{U}_e\gamma_\mu(1-\gamma_5)V_\nu + O(g^4)$$

- It is easy to show that the above tree-level amplitude is equivalent to the V-A Hamiltonian

$$\begin{aligned}
H_\beta &= \frac{GV_{ud}}{\sqrt{2}}\int d^3x\,[\bar{n}(x)\gamma^\mu(g_V - g_A\gamma_5)p(x)]\,[\bar{e}(x)\gamma_\mu(1-\gamma_5)\nu(x)] + \mathrm{H.c.} \\
&\equiv \frac{GV_{ud}}{\sqrt{2}}\int d^3x\,[\bar{N}(x)\gamma^\mu(g_V - g_A\gamma_5)\frac{\tau^-}{2}N(x)]\,[\bar{e}(x)\gamma_\mu(1-\gamma_5)\nu(x)] + \mathrm{H.c.} \\
&\equiv \frac{G_\beta}{\sqrt{2}}\int d^3x\,H^\mu(x)L^\mu(x) + \mathrm{H.c.}
\end{aligned}$$

where, H^μ and L^μ are the hadronic current and leptonic current, respectively, and $G_\beta = GV_{ud}$.

Fermi interaction: from quarks to nuclei

- Consider the β-decay transition matrix element between the initial atomic state $|I\rangle = |ii'\rangle$ and the final atomic state $|F\rangle = |ff'\rangle$ is

$$\langle F|H_\beta|I\rangle = \frac{G_\beta}{\sqrt{2}}\ell^\mu \int d^3x \, \langle f|e^{-i\vec{q}\cdot\vec{x}} H_\mu(x)|i\rangle$$

where, $\vec{q}$ is the momentum transfer to the leptons. $\ell^\mu = \langle f'|L^\mu|i'\rangle$ is a leptonic matrix element that depends on the initial and final state lepton wave functions $|f'\rangle$ and $|i'\rangle$. The remaining term $\langle f|H_\mu(x)|i\rangle$ is the nuclear matrix element of the weak interaction between initial and final nuclear many-body wave functions $|i\rangle$ and $|f\rangle$.

Fermi interaction: nuclear matrix elements (NMEs)

- Moving from the nucleon β decay ($n \to p + e^- + \bar{\nu}$) to nuclear β decay ($\alpha \to \beta + e^- + \bar{\nu}$), one in principle can again perform the Lorentz decomposition and determine the FFs from the experiments and/or theory

$$\begin{aligned}
\ell^\mu < \beta|V_\mu|\alpha > &= \left(a(q^2)\frac{P\cdot\ell}{2M_A} + e(q^2)\frac{q\cdot\ell}{2M_A} \right)\delta_{JJ'}\delta_{MM'} + i\frac{\tilde{b}(q^2)}{2M_A} C^{M'k;M}_{J'1;J}(\vec{q}\times\vec{\ell})_k \\
&+ C^{M'k;M}_{J'2;J}\left[\frac{f(q^2)}{2M_A} C^{nn';k}_{11;2}\ell_n q_{n'} + \frac{g(q^2)}{(2M_A)^3} P\cdot\ell\sqrt{\frac{4\pi}{5}} Y^k_2(\hat{q})\vec{q}^2 + \ldots \right] \\
\ell^\mu < \beta|A_\mu|\alpha > &= C^{M'k;M}_{J'1;J}\epsilon_{ijk}\epsilon_{ij\lambda\eta}\frac{1}{4M_A}\left[c(q^2)\ell^\lambda P^\eta - d(q^2)\ell^\lambda q^\eta + \frac{1}{(2M_A)^2} h(q^2) q^\lambda P^\eta q\cdot\ell \right] \\
&+ C^{M'k;M}_{J'2;J} C^{nn';k}_{12;2}\ell_n\sqrt{\frac{4\pi}{5}} Y^{n'}_2(\hat{q})\frac{\vec{q}^2}{(2M_A)^2} j_2(q^2) \\
&+ C^{M'k;M}_{J'3;J} C^{nn';k}_{12;3}\ell_n\sqrt{\frac{4\pi}{5}} Y^{n'}_2(\hat{q})\frac{\vec{q}^2}{(2M_A)^2} j_3(q^2) + \ldots, \qquad (2.18)
\end{aligned}$$

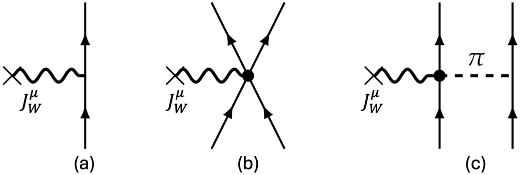
  where, ℓ^μ denotes the leptonic current and $M_A = (M_1 + M_2)/2$
- However, the LECs will be process-dependent

Impulse approximation

- Fortunately, the nuclear binding $\sim$ MeV is relatively small in comparison to the nucleon masses M_N and we can view the J_W^μ as a one-body operator, i.e. acting on one nucleon each time. This approximation is known as the impulse approximation. Consequently, the interaction Hamiltonian is one-body:

$$H = \sum_{i=1}^{A} h_i$$

- In doing so, we ignore the coupling of J_W^μ to meson fields within between nucleons, which in theory would induce two-body and more couplings, e.g. (b) & (c)



Non-relativistic reduction of currents

Table: Scheme of the reduction of nucleonic part of a beta-decay matrix element in non-relativistic approximation.

current	C	P	T	covariant form	non-relativistic approximation
S	+1	+1	+1	$\bar{U}U$	$\xi^\dagger\xi$
V	-1	-1	+1	$\bar{U}\gamma^\mu U$	$\xi^\dagger\xi$
A	+1	+1	+1	$\bar{U}\gamma^\mu\gamma_5 U$	$\xi^\dagger\vec{\sigma}\xi$
T	+1	+1	+1	$\bar{U}\sigma^{\mu\nu}U$	$\xi^\dagger\vec{\sigma}\xi$
P	+1	-1	-1	$\bar{U}\gamma_5 U$	0

Non-relativistic reduction

The recoil is small $|\vec{q}| \ll MA$ and the leading terms of the one-body Hamiltonian are,

$$h_{\text{int}} = \int d^3x \, \frac{G_\beta}{\sqrt{2}} \sum_{a=\pm} [\bar{N}\gamma^\mu (g_V - g_A\gamma_5)\frac{\tau^a}{2} N]\ell_\mu$$
$$\approx \frac{G_\beta}{\sqrt{2}} \int d^3x \sum_{a=\pm} N^\dagger \left\{ g_V \ell^0 - g_A \vec{\sigma}\cdot\vec{\ell} \right\} \frac{\tau^a}{2} N$$

There are two contributions: Fermi type and Gamow-Teller type.

- The Fermi operator does nothing on spin or spatial part of the nuclear wave function. It only converts a neutron to a proton or vice versa.
- The Gamow-Teller operator not only flip isospin, but also flip the spin

$$O_F = \frac{\tau^\pm}{2}, \qquad O_{GT} = \vec{\sigma}\frac{\tau^\pm}{2}$$

- Another feature of the interaction is that it is a contact interaction:

Fermi's Golden rule

- In order to compute half-life of β decay, we need to relate the nuclear matrix element to the decay width
- Fermi treated the weak interaction as a perturbation on the strong nuclear interaction and invoked Dirac's second order perturbation theory to compute the width.

$$H_0 \to H_0 + H_{\text{int}}, \qquad H_{\text{nucl}} \to H_{\text{nucl}} + H_\beta$$

- The formula he adopted is (Fermi's golden rule),

$$\dot{P}_{i\to f} = \frac{2\pi}{\hbar} \left| \langle f | H_{\text{int}} | i \rangle \right|^2 \rho(E)$$

where, $P_{i\to f}$ is the transition probability from state i to state f around the same energy $E_i = E_f = E$. $\rho(E) = dn(E)/dE$ is the energy density of states.

- This formula is first derived by Dirac. Fermi called this formula the golden rule #2. Golden rule #1 according to Fermi is the second order energy shift.

Derivation of Fermi's golden rule

- Let's denote the original part of the Hamiltonian as H_0. In the case of β decay, H_0 would be the strong nuclear interaction. Let's suppose we know the eigenstate of H_0,

$$H_0|n\rangle = E_n|n\rangle$$

  In our case, these eigenstates are the physical levels of the nuclei.

- A perturbative interaction, H_{int} is added to H_0 and caused the quantum transition. The full Hamiltonian now reads,

$$H = H_0 + H_{\text{int}}$$

- The quantum transition is controlled by Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

  with initial state $|\psi(0)\rangle = |i\rangle$ chosen to be one of the eigenstate of H_0. In the case of β decay, we start with a parent nucleus at certain quantum state $|i\rangle$. If H_{int} is not small, in general the above equation is difficult to solve. However, if H_{int} is much smaller than H_0, we can use perturbation theory to obtain the approximate solution

Quantum theory of transition

At a later time t, the wave function is in the form,

$$|\psi(t)\rangle = \sum_n C_{ni}(t) \exp\left(-\frac{i}{\hbar}E_n t\right)|n\rangle$$

So we have $C_{ni}(0) = \delta_{ni}$. We can calculate the amplitude $C_{nk}(t)$ by substituting the above wave function into the Schrödinger equation,

$$i\hbar \sum_{n_1} \frac{d}{dt} C_{n_1 i}(t) \exp\left(-\frac{i}{\hbar}E_{n_1} t\right)|n_1\rangle = \sum_{n_1} C_{n_1 i}(t) \exp\left(-\frac{i}{\hbar}E_{n_1} t\right) H_{\text{int}}|n_1\rangle \tag{4}$$

Quantum theory of transition

We can determine $\dot{C}_{ni}(t)$ by projecting the above state on $|n\rangle$,

$$\begin{aligned} \dot{C}_{ni}(t) &= \frac{1}{i\hbar}\sum_{n_1} C_{n_1 i}(t)\exp\left[\frac{i}{\hbar}(E_n - E_{n_1})t\right]\langle n|\, H_{\text{int}}\, |n_1\rangle \\ &\approx \frac{1}{i\hbar}\exp\left[\frac{i}{\hbar}(E_n - E_i)t\right]\langle n|\, H_{\text{int}}\, |i\rangle \end{aligned} \tag{5}$$

where we have used $C_{ni}(t) \approx C_{ni}(0) = \delta_{ni}$. Then we can solve $C_{ni}(t)$ as

$$C_{ni}(t) = \delta_{ni} + \frac{1}{i\hbar}\int_0^t dt\, \exp\left[\frac{i}{\hbar}(E_n - E_i)t\right]\langle n|\, H_{\text{int}}\, |i\rangle \tag{6}$$

Quantum theory of transition

If $n \neq i$, we get the transition probability for $|i\rangle \to |n\rangle$,

$$P_{ni} = \frac{1}{\hbar^2}\left|\int_0^t dt\, \exp\left[\frac{i}{\hbar}(E_n - E_i)t\right] \langle n| H_{\rm int} |i\rangle\right|^2 \tag{7}$$

If $H_{\rm int}$ is independent of time, the probability for $t \to \infty$ becomes,

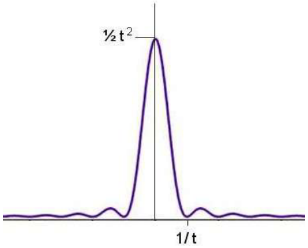
$$\begin{aligned} P_{ni} &= \frac{1}{\hbar^2} |\langle n| H_{\rm int} |i\rangle|^2 \lim_{t\to\infty} \left|\int_0^t dt'\, \exp\left[\frac{i}{\hbar}(E_n - E_i)t'\right]\right|^2 \\ &= \frac{1}{\hbar^2} |\langle n| H_{\rm int} |i\rangle|^2 \lim_{t\to\infty} \frac{\sin^2[(E_n - E_i)t/(2\hbar)]}{(E_n - E_i)^2/(4\hbar^2)} \\ &\to \frac{1}{\hbar} |\langle n| H_{\rm int} |i\rangle|^2\, 2\pi t \delta(E_n - E_i). \end{aligned} \tag{8}$$

Fermi golden rule

where we have used

$$\lim_{t\to\infty} \frac{\sin^2(tx)}{tx^2} = \pi\delta(x) \qquad (9)$$

with $x \equiv (E_n - E_i)/(2\hbar)$.



Fermi's golden rule

Taking the density of states for the final state into account, we obtain

$$d\lambda \equiv \lim_{t\to\infty} \frac{dP_{i\to n}}{t} = 2\pi \left|\langle n|\, H_{\text{int}}\, |i\rangle\right|^2 \delta(E_n - E_i) dn$$
$$= 2\pi \left|\langle n|\, H_{\text{int}}\, |i\rangle\right|^2 \delta(E_n - E_i)\rho(E_n) dE_n \qquad (10)$$

where $\frac{dn}{dE} = \rho(E)$ is the energy density of states. The integrated transition rate is now

$$\dot{P}_{i\to n} \quad = \quad 2\pi\, \rho(E_n)\big|_{E_n=E_i} \left|\langle n|\, H_{\text{int}}\, |i\rangle\right|^2 \qquad (11)$$

This is called the Fermi's golden rule.

Initial and final state in β decay

Now we deal with the β decay with the Fermi golden rule. Initial state and final state of the transition

$$\begin{aligned} |\mathrm{i}\rangle &= \left|{}^{A}_{Z}X\right\rangle = u_{\mathrm{i}}(\mathrm{x}) \\ |\mathrm{f}\rangle &= \left|{}^{A}_{Z+1}Y, \mathrm{e}^{-}, \overline{\nu}_{\mathrm{e}}\right\rangle = u_{\mathrm{f}}(\mathrm{x})\phi_{\mathrm{e}}(\mathrm{x}_{\mathrm{e}})\phi_{\nu}(\mathrm{x}_{\nu}) \end{aligned} \tag{12}$$

Here $u_{f,i}$ are eigenstates of mother and daughter nuclei, and ϕ_e and ϕ_ν are the wave function of the electron and neutrino. Accoding to the Fermi golden rule the differential decay rate reads

$$d\lambda = 2\pi \left|\langle \mathrm{f}| H_{\mathrm{int}} |\mathrm{i}\rangle\right|^2 \delta(E_{\mathrm{f}} + E_{\mathrm{e}} + E_{\nu} - E_{\mathrm{i}}) dn(\mathrm{e}^{-}, \overline{\nu}_{\mathrm{e}}) \tag{13}$$

where $dn(e^{-}, \bar{\nu}_e)$ is the state density element of final states for electrons and (anti-)neutrinos.

Transition Matrix element

The matrix element of the Hamiltonian between the initial and final state is

$$\begin{aligned}\mathcal{M}_{\rm fi} &= \langle {\rm f}| H_{\rm int} |{\rm i}\rangle \\ &= \int d^3{\rm x} d^3{\rm x}_{\rm e} d^3{\rm x}_\nu u_{\rm f}^*({\rm x})\phi_{\rm e}^*({\rm x}_{\rm e})\phi_\nu^*({\rm x}_\nu) H_{\rm int} u_{\rm i}({\rm x}) \\ &= G\int d^3{\rm x} d^3{\rm x}_{\rm e} d^3{\rm x}_\nu u_{\rm f}^*({\rm x})\phi_{\rm e}^*({\rm x}_{\rm e})\phi_\nu^*({\rm x}_\nu) u_{\rm i}({\rm x}) \\ &\quad \times\delta^{(3)}({\rm x}_{\rm e}-{\rm x})\delta^{(3)}({\rm x}_\nu-{\rm x}) \\ &= G\int d^3{\rm x} u_{\rm f}^*({\rm x})\phi_{\rm e}^*({\rm x})\phi_\nu^*({\rm x}) u_{\rm i}({\rm x}) \end{aligned} \tag{14}$$

where we assumed the interaction part of the Hamiltonian is in a point contact form

$$H_{\rm int} = G\delta^{(3)}({\rm x}_{\rm e}-{\rm x})\delta^{(3)}({\rm x}_\nu-{\rm x}) \tag{15}$$

Here, $G = G_{F,GT}$, i.e. we focus on the spatial part of the interaction

Decay rate

The differential decay rate for emitting an electron within the momentum range $[k_e, k_e + dk_e]$ is

$$\begin{aligned} d\lambda &= G^2 \left| \int d^3\mathrm{x} u_\mathrm{f}^*(\mathrm{x}) \phi_\mathrm{e}^*(\mathrm{x}) \phi_\nu^*(\mathrm{x}) u_\mathrm{i}(\mathrm{x}) \right|^2 \\ &\quad \times 2\pi\delta(E_\mathrm{f} + E_\mathrm{e} + E_\nu - E_\mathrm{i}) dn(\mathrm{e}^-, \overline{\nu}_\mathrm{e}) \end{aligned} \tag{16}$$

Assuming that the electron and the neutrino are free particles, their wave functions can be approximated by the plane waves

$$\begin{aligned} \phi_e &= \frac{1}{\sqrt{V}} e^{i\mathrm{k}_e \cdot \mathrm{x}} \\ \phi_\nu &= \frac{1}{\sqrt{V}} e^{i\mathrm{k}_\nu \cdot \mathrm{x}} \end{aligned} \tag{17}$$

Decay rate

The density element of state for electrons and neutrinos

$$
\begin{aligned}
dn(\mathrm{e}^-,\overline{\nu}_\mathrm{e}) &= \frac{Vd^3\mathrm{k}_\mathrm{e}}{(2\pi)^3}\frac{Vd^3\mathrm{k}_\nu}{(2\pi)^3} \\
&= k_\mathrm{e}^2 k_\nu^2 dk_\mathrm{e} dk_\nu \frac{Vd\Omega_\mathrm{e}}{(2\pi)^3}\frac{Vd\Omega_\nu}{(2\pi)^3}
\end{aligned} \tag{18}
$$

The differential decay rate

$$
\begin{aligned}
d\lambda &= G^2\frac{1}{V^2}\left|\int d^3\mathrm{x}\, u_\mathrm{f}^*(\mathrm{x})u_\mathrm{i}(\mathrm{x})e^{-i(\mathrm{k}_\mathrm{e}+\mathrm{k}_\nu)\cdot\mathrm{x}}\right|^2 \\
&\quad \times 2\pi\delta(E_\mathrm{f}+E_\mathrm{e}+E_\nu-E_\mathrm{i})dn(\mathrm{e}^-,\overline{\nu}_\mathrm{e}) \\
&= G^2\frac{1}{V^2}\frac{Vd^3\mathrm{k}_\mathrm{e}}{(2\pi)^3}\frac{Vd^3\mathrm{k}_\nu}{(2\pi)^3}2\pi\delta(E_\mathrm{f}+E_\mathrm{e}+E_\nu-E_\mathrm{i})|M_\mathrm{fi}|^2
\end{aligned}
$$

Decay rate: electron spectrum

where we have used

$$|M_{\text{fi}}|^2 \equiv \left| \int d^3\text{x} u_f^*(\text{x}) u_i(\text{x}) e^{-i(\text{k}_e + \text{k}_\nu)\cdot \text{x}} \right|^2 \quad (19)$$

If we neglect the neutrino mass $m_\nu \approx 0$ and integrate over $d\Omega_\text{e}$ and $d^3\text{k}_\nu$, we obtain the differential rate for electrons in the momentum range $[k_e, k_e + dk_e]$

$$\begin{aligned} d\lambda &= 2\pi G^2 k_\text{e}^2 dk_\text{e} \int \frac{d\Omega_\text{e}}{(2\pi)^3} \int \frac{d^3\text{k}_\nu}{(2\pi)^3} \delta(E_\text{f} + E_\text{e} + E_\nu - E_\text{i}) |M_{\text{fi}}|^2 \\ &= \frac{G^2}{2\pi^3} dk_\text{e} k_\text{e}^2 (E_\text{i} - E_\text{f} - E_\text{e})^2 |\overline{M}_{\text{fi}}|^2 \end{aligned} \quad (20)$$

Decay rate: electron spectrum

where $|\overline{M}_{\rm fi}|^2$ is the average matrix element over neutrino momentum and electron momentum direction. The momentum spectrum for electrons is

$$\frac{d\lambda}{dk_e} = \frac{G^2}{2\pi^3}k_e^2(\Delta E - E_e)^2|\overline{M}_{\rm fi}|^2 \tag{21}$$

where $\Delta E = E_{\rm i} - E_{\rm f}$ is the energy between the mother and daughter nucleus in the beta decay.

Coulomb modification factor

Here we have neglected the influence of the Coulomb field on electrons from the nucleus. Normally the plane wave function of an electron can be distorted in the Coulomb field from protons inside the nucleus. The distortion to the electron wave function can be described by a Coulomb modification factor $F(Z, E)$,

$$F(Z, E_e) = \frac{x}{1 - e^{-x}} \tag{22}$$

where $x = \pm\frac{2\pi Z e^2}{v_e}$ with $v_\mathrm{e} = E_\mathrm{e}^{-1}\sqrt{E_\mathrm{e}^2 - m_\mathrm{e}^2}$ for the $\beta^\mp$ decay.

Coulomb modification factor

Including the Coulomb modification factor, the final form of the probability distribution is now

$$\frac{d\lambda}{dk_e} = \frac{G^2}{2\pi^3} k_e^2 (\Delta E - E_e)^2 F(Z, E_e) |\overline{M}_{\rm fi}|^2 \tag{23}$$

At small kinetic energy $E_e \to m_e$ or $k_e \to 0$ or $v_e \to 0$, for β^- decay, the factor behaves like $F(Z, E_{\rm e}) \approx 2\pi Z e^2 E_{\rm e}/k_{\rm e}$, we have

$$k_e^2 (\Delta E - E_e)^2 F(Z, E_e) \sim 2\pi Z e^2 k_e Q^2 m_e \tag{24}$$

For β^+ decay, we have

$$k_e^2 (\Delta E - E_e)^2 F(Z, E_e) \sim 2\pi Z e^2 Q^2 m_e k_e \exp\left(-\frac{2\pi Z e^2 m_e}{k_e}\right) \tag{25}$$

Kurie plot

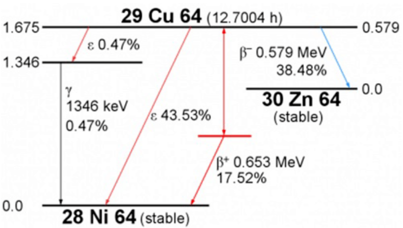
We can re-write the above spectrum as

$$\sqrt{\frac{d\lambda}{dk_e}/[k_e^2 F(Z, E_e)]} \sim C(\Delta E - E_e)$$
$$= C(Q - T_e) \qquad (26)$$

where C is a constant, $Q = \Delta E - m_e$ and $E_e = T_e - m_e$. We can see that $\sqrt{(d\lambda/dk_e)/[k_e^2 F(Z, E_e)]}$ is in linear relation to E_e or T_e, which is known as the Kurie plot. From the Kurie plot we can determine Q-value from the intercept with the axis of E_e or T_e.

Homework

Try to derive the kinetic energy spectra from the momentum spectra for electrons. Also try to obtain the maximum values of electron's and positron's momenta in MeV from the plot below. Note that the unit in the plot is Gauss·cm.



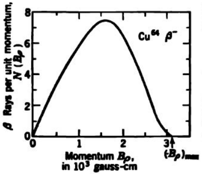


Fig. 1.3 Momentum spectrum of the negatron β rays from Cu⁶⁴, an allowed transition. [*From Reitz* (R14).]

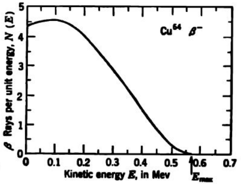


Fig. 1.4 Energy spectrum of the negatron β rays from Cu⁶⁴. (Calculated from Fig. 1.3.)

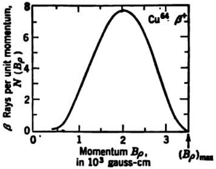


Fig. 1.5 Momentum spectrum of the positron β rays from Cu⁶⁴, an allowed transition. [*From Reitz* (R14).]

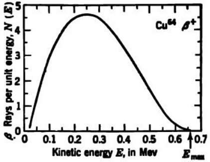
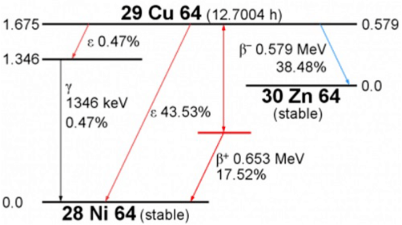


Fig. 1.6 Energy spectrum of the positron β rays from Cu⁶⁴. (Calculated from Fig. 1.5.)

β spectra

Figure: Momentum and kinetic energy spectra of electrons and positrons in $\beta^{\pm}$ decays of ${}^{64}_{29}\mathrm{Cu}$. Taken from Fig. 5.9 of K. Heyde, *Basic ideas and concepts in nuclear physics*, second edition, IOP Publishing Ltd 1994, 1999.



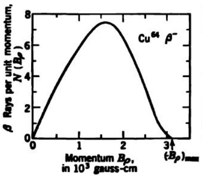


Fig. 1.3 Momentum spectrum of the negatron β rays from Cu64, an allowed transition. [*From Reitz* (R14).]

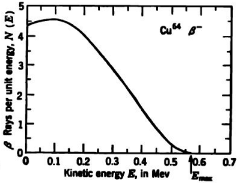


Fig. 1.4 Energy spectrum of the negatron β rays from Cu64. (Calculated from Fig. 1.3.)

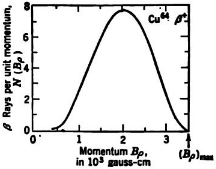


Fig. 1.5 Momentum spectrum of the positron β rays from Cu64, an allowed transition. [*From Reitz* (R14).]

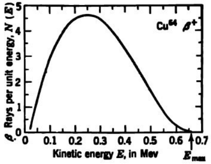
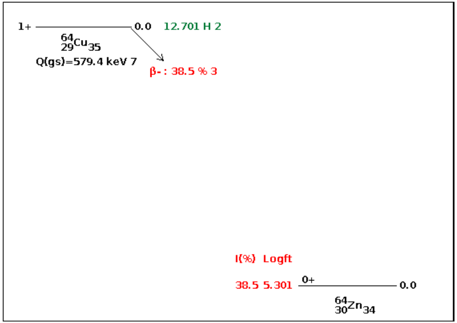


Fig. 1.6 Energy spectrum of the positron β rays from Cu64. (Calculated from Fig. 1.5.)

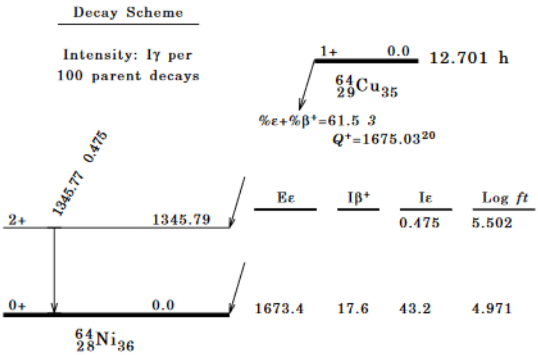
Scheme plot of β decay

Figure: Scheme plot for β^- decays of ${}^{64}_{29}\mathrm{Cu}$.



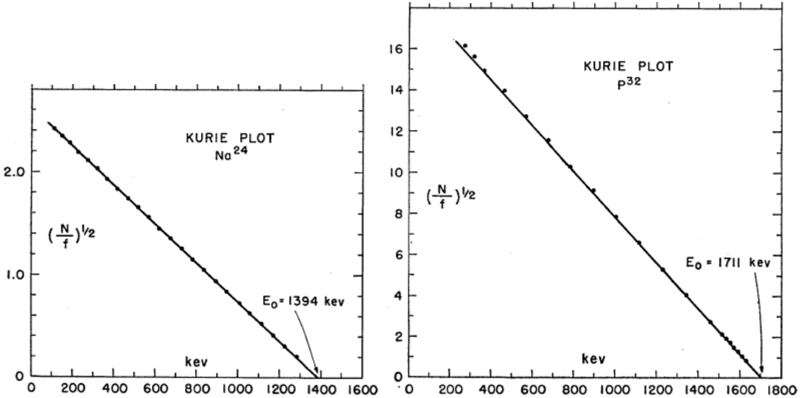
Scheme plot of β decay

Figure: Scheme plot for β^+ decays of ${}^{64}_{29}\mathrm{Cu}$.



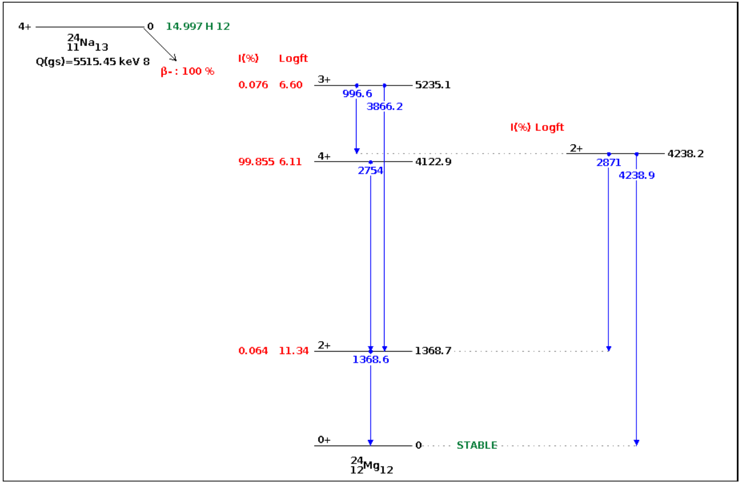
Kurie plot

Figure: Kurie plot for beta decay of Na-24 (to Mg-24) and P-32 (to S-32).



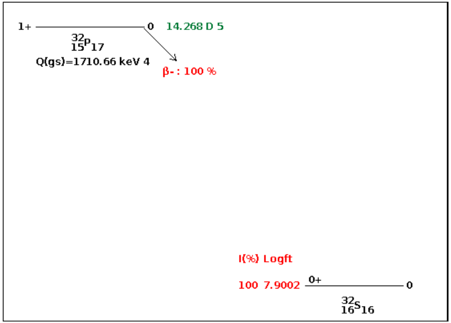
Scheme plot of β decay

Figure: Scheme plot for β^- decays of Na-24.



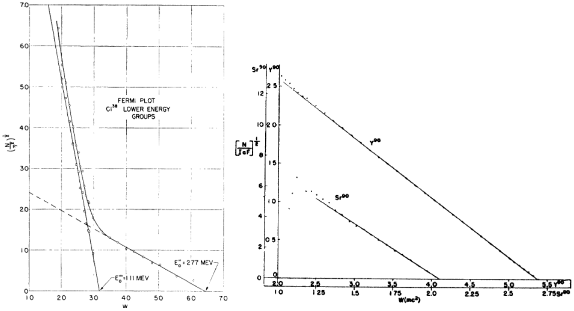
Scheme plot of β decay

Figure: Scheme plot for β^- decays of P-32.



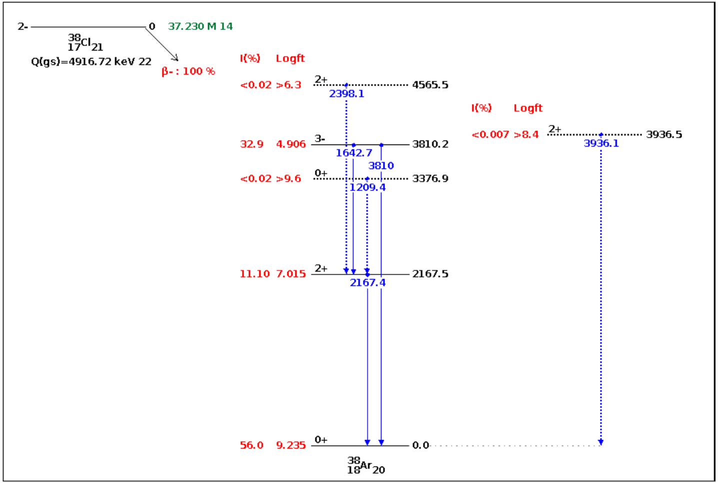
Kurie plot

Figure: Kurie plot for beta decay of Cl-38 (to Ar-38) with $Q = 1.11, 2.77$ MeV, Sr-90 (to Y-90) and Y-90 (to Zr-90) with $Q = 0.537, 2.23$ MeV.



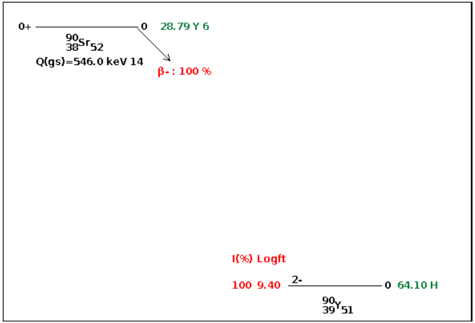
Scheme plot of β decay

Figure: Scheme plot for β^- decays of Cl-38.



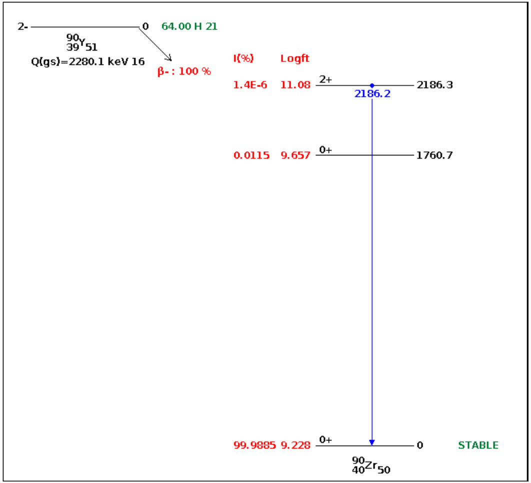
Scheme plot of β decay

Figure: Scheme plot for β^- decays of Sr-90.



Scheme plot of β decay

Figure: Scheme plot for β^- decays of Y-90.



Decay rate: neutrino mass

If the neutrino has a mass we have

$$d^3\mathrm{k}_\nu = d\Omega_\nu dE_\nu \sqrt{E_\nu^2 - m_\nu^2} E_\nu$$

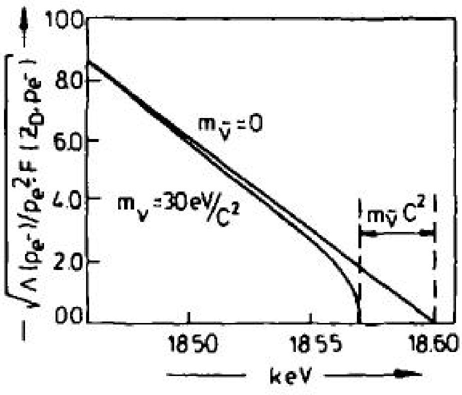
Then Eq. (23) we obtain

$$\frac{d\lambda}{dk_e} = \frac{G^2}{2\pi^3} k_\mathrm{e}^2 (\Delta E - E_\mathrm{e})^2 F(Z, E_\mathrm{e}) \sqrt{1 - \frac{m_\nu^2}{(\Delta E - E_\mathrm{e})^2}} |\overline{M}_\mathrm{fi}|^2 \tag{27}$$

where we have used $k_\nu^2 dk_\nu = k_\nu E_\nu dE_\nu$. The zero points read $E_e^{(0)} = \Delta E$ and $E_e^{(0)} = \Delta E \pm m_\nu$, where the lowest zero point $E_e^{(0)} = \Delta E - m_\nu$ is physical. So we can determine the neutrino mass from the intercept with energy axis once we know ΔE.

Decay rate: neutrino mass

Figure: Illustration of the Fermi-Kurie plot for ${}^3\mathrm{H} \to {}^3\,\mathrm{He} + e^- + \bar{\nu}_e$. Suppose m_ν =30 eV, there is difference between massive and massless neutrinos. Taken from Fig. 5.11 of K. Heyde, *Basic ideas and concepts in nuclear physics*, second edition, IOP Publishing Ltd 1994, 1999.



Decay rate

We can obtain the integrated rate by using dimensionless variable $w \equiv E_e/m_e$ and $w_0 = \Delta E/m_e$,

$$\begin{aligned} \lambda &= \frac{G^2}{2\pi^3}\int_{m_\mathrm{e}}^{\Delta E} dE_\mathrm{e} F(Z,E_\mathrm{e}) E_\mathrm{e} k_\mathrm{e} (\Delta E - E_\mathrm{e})^2 |\overline{M}_\mathrm{fi}|^2 \\ &\approx \frac{G^2 m_\mathrm{e}^5}{2\pi^3} f(Z,w_0)|\overline{M}_\mathrm{fi}|^2 \end{aligned} \tag{28}$$

where $f(Z,w_0)$ is defined as

$$f(Z,w_0) = \int_1^{w_0} dw F(Z,w)\sqrt{w^2-1}(w_0-w)^2 w \tag{29}$$

Relative half life

If w_0 is very large, we have $f(Z, w_0) \sim w_0^5/5$. The half life is given by

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{2\pi^3 \ln 2}{G^2 m_e^5 f(Z, w_0) |\overline{M}_{fi}|^2} \tag{30}$$

and $f(Z, w_0)T_{1/2}$ is called relative half life which is almost a constant for certain type of beta decays

$$\begin{aligned} f(Z, w_0)T_{1/2} &= \frac{2\pi^3 \ln 2}{G^2 m_e^5 |\overline{M}_{fi}|^2} \\ &= \frac{197 \times 10^{12} 2\pi^3 \ln 2}{1.1664^2 \times 10^{-10} \times 0.511^5 |\overline{M}_{fi}|^2} \text{ fm}/c \\ &\approx \frac{5955}{|\overline{M}_{fi}|^2} \text{ s.} \end{aligned} \tag{31}$$

Fermi constant

where

$$G = 1.1664 \times 10^{-5}\,\mathrm{GeV}^{-2}$$
$$Gm_p^2 \approx 1.026 \times 10^{-5}$$

If we know $T_{1/2}$ and $|M_{fi}|^2$, we can determine the coupling G. So the dimensionless coupling constant is $(Gm_p^2)^2 \sim 10^{-10}$ with nucleon mass m_p, which is much smaller than the counlings in strong and electromagnetic interactions whose coupling constants are of the order 1 (strong) and 10^{-2} (electromagnetic). So in the β decay, the so-called weak interaction is at work.

Decay rate of free neutron

We can estimate the free neutron half life from the beta decay by Eqs. (29,30). We can neglect $F(Z, w)$. We can further neglect the recoil effect and approximate

$$\begin{aligned} G^2|M_{\text{fi}}|^2 &\to G_F^2|M_F|^2 + G_{GT}^2|M_{GT}|^2 \\ |M_{\text{fi}}|^2 &\sim 1 + 3g_A^2/g_V^2 \sim 5.8 \end{aligned} \tag{32}$$

with $g_A/g_V \approx -1.27$ and $\Delta E \approx 1.29$ MeV, and

$$f(Z, w_0) = \int_1^{w_0} dw\sqrt{w^2-1}(w_0-w)^2 w \approx 1.63, \tag{33}$$

Decay rate of free neutron

We obtain the neutron's half life

$$T_{1/2}(n) \sim \frac{197 \times 10^{12} \times 2\pi^3 \ln 2}{1.15^2 \times 10^{-10} \times 0.51^5 \times 1.63 \times 5.8} (\mathrm{fm}/c) \approx 654\, s.$$

The data give $T_{1/2}(n) \approx 661\, s$. We see the perfect agreement between the data and the theoretical value.

Selection rules of β decay

Let us discuss about the transition amplitude $|M_{fi}|^2$. Note that the phase is very small $(\mathrm{k}_e + \mathrm{k}_\nu) \cdot \mathrm{x} \sim 0.1 - 0.01 \ll 1$, so we can make an expansion of the phase factor $e^{i(\mathrm{k}_e+\mathrm{k}_\nu)\cdot\mathrm{x}}$ in terms of powers of $(\mathrm{k}_e + \mathrm{k}_\nu) \cdot \mathrm{x}$. We can also make an expansion of it in terms of spherical harmonic functions,

$$
\begin{aligned}
e^{i(\mathrm{k}_e+\mathrm{k}_\nu)\cdot\mathrm{x}} &= \sum_{L=0} (2L+1)(-i)^L j_L(|\mathrm{k}_e + \mathrm{k}_\nu| x) P_L(\cos\theta) \\
&\approx \sum_{L=0} \frac{(2L+1)(-i)^L}{(2L+1)!!} |\mathrm{k}_e + \mathrm{k}_\nu|^L x^L P_L(\cos\theta) \qquad (34)
\end{aligned}
$$

Selection rules of β decay

Therefore the amplitude can be written as

$$\begin{aligned} M_{fi} &= \sum_{L=0} \frac{(2L+1)(-i)^L}{(2L+1)!!} \int d^3\mathrm{x} u_f^*(\mathrm{x}) u_i(\mathrm{x}) |\mathrm{k}_e + \mathrm{k}_\nu|^L x^L P_L(\cos\theta) \\ &= \sum_{L=0} M_{fi}^L \end{aligned} \tag{35}$$

The matrix elements corresponding to two lowest partial waves are

$$\begin{aligned} M_{fi}^{L=0} &= \int d^3\mathrm{x} u_f^*(\mathrm{x}) u_i(\mathrm{x}) \\ M_{fi}^{L=1} &= -i \int d^3\mathrm{x} u_f^*(\mathrm{x}) u_i(\mathrm{x}) |\mathrm{k}_e + \mathrm{k}_\nu| x P_L(\cos\theta) \end{aligned} \tag{36}$$

Selection rules of β decay

The partial wave contribution $M_{\rm fi}^L$ drops very fast as L increases. The suppression of higher order contributions can be seen by

$$\frac{M_{fi}^{L+1}}{M_{fi}^{L}} \sim \frac{kr}{2L+1} \sim 10^{-2} \tag{37}$$

for $r \sim 5$ fm and $k \sim 1$ MeV. If $M_{fi}^{L=0} \neq 0$, it is called the allowed transition. If $M_{fi}^{L=1} \neq 0$, it is called the first order forbidden transition and etc.. The partial wave contribution M_{fi}^L drops very fast as L increases. So the lowest order, $L = 0, 1$, contributions play the dominant role.

Selection rules of β decay

We do not introduce the spin part into Eq. (35). Noticing that nucleons and leptons are fermions, we can recover the spin part in Eq. (35),

$$|M_{fi}|^2 = \sum_{L,S_{e\nu}} |M_{fi}^{L,S_{e\nu}}|^2 \tag{38}$$

where $\mathrm{S}_{e\nu} = \mathrm{S}_e + \mathrm{S}_\nu$ is the lepton spin. The angular momentum conservation reads

$$\mathrm{J}_i = \mathrm{J}_f + \mathrm{S}_{e\nu} + \mathrm{L} \tag{39}$$

where $\mathrm{J}_i(\mathrm{J}_f)$ is the nuclear spin of the mother (daughter) nucleus or the intitial (final) nuclear state. Parity selection rule for mother and daughter parity:

$$P_\mathrm{i} P_\mathrm{f} = (-1)^L \tag{40}$$

Selection rules of β decay

For the mixed type decays, there are both Fermi type and Gamow-Teller type contributions. We can replace the matrix element squared as

$$G^2|M_{\rm fi}|^2 \to G_F^2|M_F|^2 + G_{GT}^2|M_{GT}|^2 \tag{41}$$

The classification for the beta decay is in Table 2. For various types of transitions, the decay strengths are as follows:

$$\begin{aligned} \log_{10}(T_{1/2}f) &= 2.9-3.7 \quad (super-allowed) \\ \log_{10}(T_{1/2}f) &= 4.4-6 \quad (allowed) \\ \log_{10}(T_{1/2}f) &= 6-10 \quad (1st\ forbidden) \\ \log_{10}(T_{1/2}f) &> 15 \quad (2nd\ forbidden) \end{aligned} \tag{42}$$

Selection rules of β decay

The spin of the leptonic system $S_{e\nu}$ can be 0 (singlet) or 1 (triplet), which correspond to Fermi and Gamow-Teller transition respectively. For $S_{e\nu} = 0$ and the allowed transition ($L = 0$), we get $J_i - J_f = 0$ and $P_\mathrm{i} = P_\mathrm{f}$, this is called Fermi selection rule. For $S_l = 1$ and the allowed transition ($L = 0$), we get $J_i - J_f = 0, \pm 1$ and $P_\mathrm{i} = P_\mathrm{f}$, this is called Gamow-Teller seletion rule.

Selection rules of β decay

Table: Classification for the beta decay.

	$S_l = 0$: Fermi	$S_l = 1$: Gamow-Teller
$L = 0$: allowed	$J_i - J_f = 0$ $0^+ \to 0^+$: super-allowed	$J_i - J_f = 0, \pm 1$ $0^+ \to 1^+$: unique Gamow-Teller
$L = 1$: 1st forbidden	$J_i - J_f = 0, \pm 1$	$J_i - J_f = 0, \pm 1, \pm 2$

Examples of β decay

Here are some examples for the allowed β-decays.
(1) super-allowed Fermi-type transitions $0^+ \to 0^+$

$$\begin{aligned} {}^{14}_{8}\mathrm{O} &\to {}^{14}_{7}\mathrm{N}^* \\ {}^{10}_{6}\mathrm{C} &\to {}^{10}_{5}\mathrm{B}^* \\ {}^{18}_{10}\mathrm{Ne} &\to {}^{18}_{9}\mathrm{F}^* \\ {}^{34}_{17}\mathrm{Cl} &\to {}^{34}_{16}\mathrm{S} \end{aligned} \tag{43}$$

The scheme plots for these decays are shown in Fig. 20.

Examples of β decay

(2) Gamow-Teller transitions

$$\begin{aligned}
{}^{10}_{6}\mathrm{C} &\rightarrow {}^{10}_{5}\,\mathrm{B}\,(0^+ \rightarrow 1^+)\\
{}^{6}_{2}\mathrm{He} &\rightarrow {}^{6}_{3}\,\mathrm{Li}\,(0^+ \rightarrow 1^+)\\
{}^{13}_{5}\mathrm{B} &\rightarrow {}^{13}_{6}\,\mathrm{C}\,(3/2^- \rightarrow 1/2^-)\\
{}^{28}_{12}\mathrm{Mg} &\rightarrow {}^{28}_{13}\,\mathrm{Al}^*\,(0^+ \rightarrow 1^+)\\
{}^{60}_{24}\mathrm{Cr} &\rightarrow {}^{60}_{25}\,\mathrm{Mn}\,(0^+ \rightarrow 1^+)
\end{aligned} \tag{44}$$

The scheme plots for these decays are shown in Fig. 21.

Examples of β decay

There are mixed type of decays, here are some examples. (3) Mirror decays. The relative ratio of Fermi type to GT is fixed.

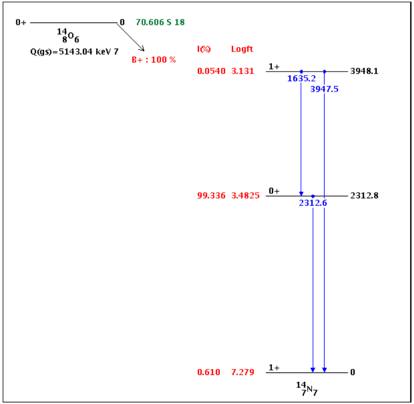
$$
\begin{aligned}
\mathrm{n} &\to \mathrm{p}\,(1/2^+ \to 1/2^+) \\
{}^{3}_{1}\mathrm{H} &\to {}^{3}_{2}\,\mathrm{He}\,(1/2^+ \to 1/2^+) \\
{}^{13}_{7}\mathrm{N} &\to {}^{13}_{6}\,\mathrm{C}\,(1/2^- \to 1/2^-) \\
{}^{21}_{11}\mathrm{Na} &\to {}^{21}_{10}\,\mathrm{Ne}\,(3/2^+ \to 3/2^+) \\
{}^{41}_{21}\mathrm{Sc} &\to {}^{41}_{20}\,\mathrm{Ca}\,(4^+ \to 4^+)
\end{aligned}
\tag{45}
$$

Examples of β decay

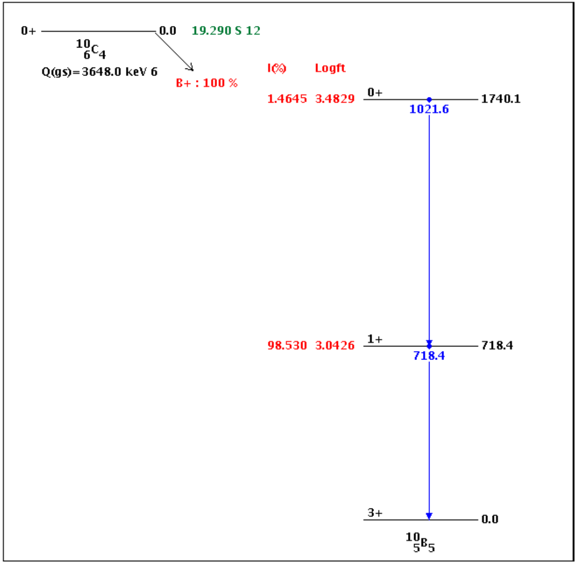
Non-mirror decays, dominated by GT

$$
\begin{aligned}
{}^{24}_{11}\mathrm{Na} &\to {}^{24}_{12}\,\mathrm{Mg}\,(3/2^+ \to 3/2^+) \\
{}^{41}_{11}\mathrm{Ar} &\to {}^{41}_{12}\,\mathrm{K}\,(7/2^- \to 7/2^-) \\
{}^{46}_{21}\mathrm{Sc} &\to {}^{46}_{22}\,\mathrm{Ti}\,(4^+ \to 4^+) \\
{}^{52}_{25}\mathrm{Mn} &\to {}^{52}_{24}\,\mathrm{Cr}\,(2^+ \to 2^+) \\
{}^{65}_{28}\mathrm{Ni} &\to {}^{65}_{29}\,\mathrm{Cu}\,(5/2^- \to 5/2^-)
\end{aligned}
\tag{46}
$$

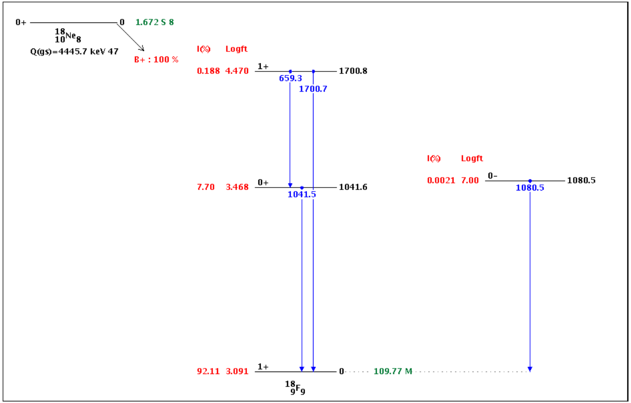
Figure: The scheme plots for the allowed β-decays of the Fermi type. The half-lives are shown as $\log_{10}(fT_{1/2})$. From IAEA database NuDat 2.6 at ``http://www.nndc.bnl.gov/nudat2/''.



3.5 β decay: weak interaction at work (51)



3.5 β decay: weak interaction at work (52)



3.5 β decay: weak interaction at work (53)

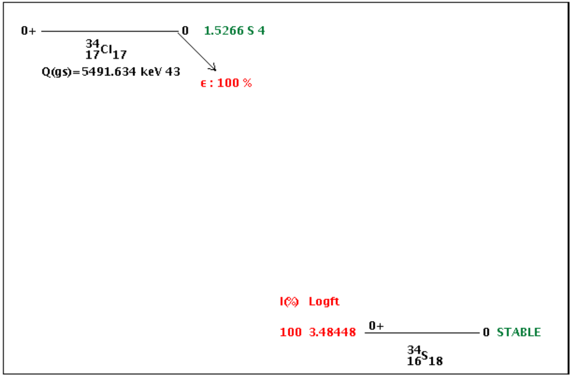
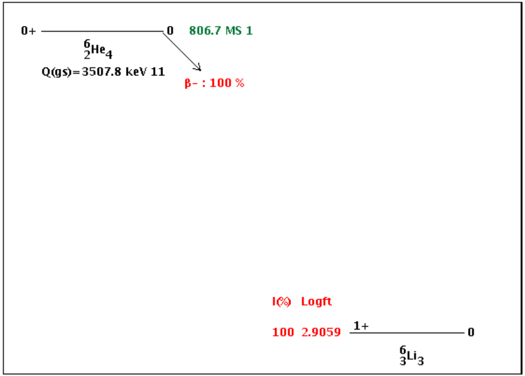
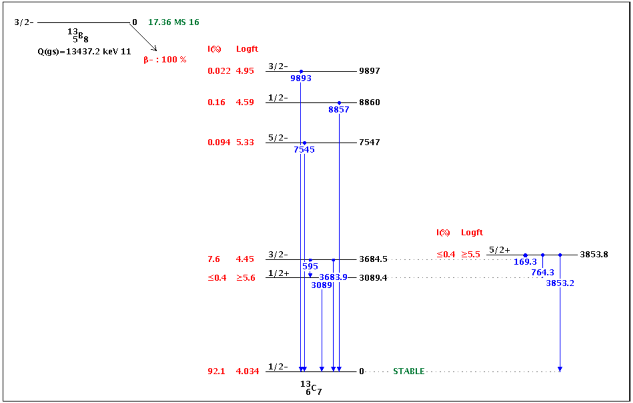


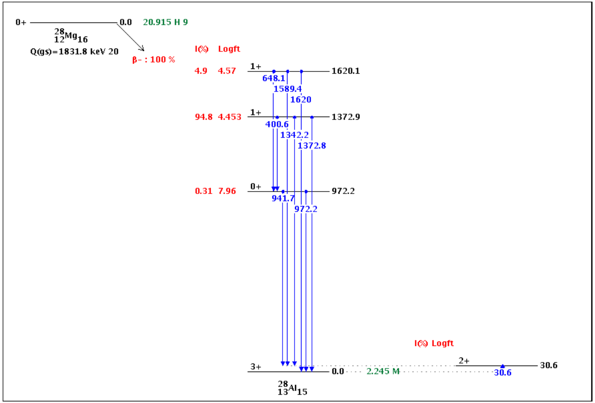
Figure: The scheme plots for the allowed β-decays of the Gamow-Teller type. The half-lives are shown as $\log_{10}(fT_{1/2})$. From IAEA database NuDat 2.6 at ``http://www.nndc.bnl.gov/nudat2/''.



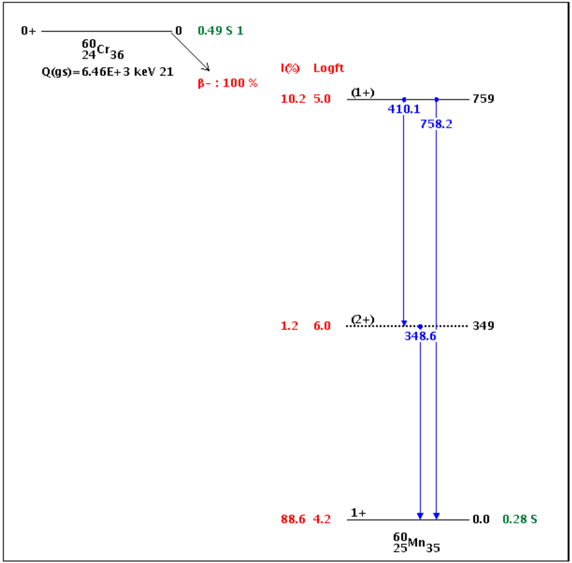
3.5 β decay: weak interaction at work (55)



3.5 β decay: weak interaction at work (56)



3.5 β decay: weak interaction at work (56)



EC

The electron capture process leads to an electron vacancy in the orbit. The filling of this vacancy by a free electron can release the binding energy by emission of X-rays or Auger electrons. The decay constant can be obtained through the Fermi golden rule,

$$d\lambda_{\rm EC} = 2\pi|\mathcal{M}_{\rm fi}|^2\delta(E_\nu - E_0)\frac{dn}{dE_\nu}dE_\nu, \tag{47}$$

where E_0 is the Q-value of the process and

$$\begin{aligned} \mathcal{M}_{\rm fi} &= G\int d^3{\rm x}\, u_{\rm f}^*({\rm x})\phi_\nu^*({\rm x})\phi_{\rm e}({\rm x})u_{\rm i}({\rm x}), \\ \frac{dn}{dE_\nu} &= V\frac{1}{2\pi^2}E_\nu^2. \end{aligned} \tag{48}$$

Here $\phi_{\rm e}({\rm x})$ is the electron wave function in the atomic shell. Capture is most likely for 1s electrons in the K-shell, since the s-wave function is maximal at the origin,

$$\phi_{\rm e}({\rm x}) \quad = \quad \frac{1}{\sqrt{\pi}}(Ze^2m_{\rm e})^{3/2}\exp(-Ze^2m_{\rm e}|{\rm x}|). \tag{49}$$

The neutrino wave function is still a plane wave, $\phi_\nu = \frac{1}{\sqrt{V}}e^{i{\rm k}_\nu\cdot{\rm x}}$. Then we have

$$\begin{aligned} |\mathcal{M}_{\rm fi}|^2 &= G^2\frac{1}{V}\frac{1}{\pi}(Ze^2m_{\rm e})^3|M_{\rm fi}|^2, \\ M_{\rm fi} &= \int d^3{\rm x}u_{\rm f}^*({\rm x})u_{\rm i}({\rm x})\exp(-i{\rm k}_\nu\cdot{\rm x}-Ze^2m_{\rm e}|{\rm x}|), \end{aligned} \tag{50}$$

where $M_{\rm fi}$ is dimensionless.

EC

So after integration M_{fi} is replaced by its average value, the decay rate or constant is then

$$\lambda_{\text{EC}} = \frac{G^2}{\pi^2}(Ze^2 m_{\text{e}})^3 |\bar{M}_{\text{fi}}|^2 E_\nu^2.$$

We have taken into account that there are two 1s electrons in the K-shell. We see that the decay constant is proportional to Z^3.

Parity violation in β decay

In 1950s there was a $\tau - \theta$ puzzle that seemingly identical strange mesons θ^+ and τ^+ can decay into two and three pions respectively,

$$\begin{aligned} \theta^+ &\rightarrow \pi^+ + \pi^0 \\ \tau^+ &\rightarrow \pi^+ + \pi^+ + \pi^- \end{aligned} \tag{51}$$

Considering a pion has negative parity and the decay occurs in s-wave, it seems that θ^+ and τ^+ are different particles with opposite parity. But actually they are all kaons and parity is violated in weak decays.

Parity violation in β decay

Kaon is a superposition of two parity eigenstates

$$\begin{aligned} |K^+\rangle &= c_1 |K^+_{\text{even}}\rangle + c_2 |K^+_{\text{odd}}\rangle, \\ \hat{P}|K^+\rangle &= c_1 |K^+_{\text{even}}\rangle - c_2 |K^+_{\text{odd}}\rangle \neq |K^+\rangle \end{aligned} \tag{52}$$

where c_1 and c_2 satisfy $c_1^2 + c_2^2 = 1$. This means $[\hat{P}, \hat{H}_{\text{weak}}] \neq 0$, partity is not conserved in the weak interaction.

Parity violation in β decay

If the state is a partiy eigenstate, we have $\hat{P}\psi(\mathrm{r}) = \pm\psi(\mathrm{r})$, no pseudo-scalar operator $\hat{O}_{\mathrm{ps}}$ exsists in the system. A pseudo-scalar operator $\hat{O}_{\mathrm{ps}}$ transforms under spatial reflection as $\hat{P}\hat{O}_{\mathrm{ps}}\hat{P} = -\hat{O}_{\mathrm{ps}}$. Here is the proof

$$\begin{aligned}
\langle\hat{O}_{\mathrm{ps}}\rangle &= \int d^3\mathrm{r}\psi^*(\mathrm{r})\hat{O}_{\mathrm{ps}}\psi(\mathrm{r}) \\
&= -\int d^3\mathrm{r}\psi^*(\mathrm{r})\hat{P}\hat{O}_{\mathrm{ps}}\hat{P}\psi(\mathrm{r}) \\
&= -\int d^3\mathrm{r}[\hat{P}\psi(\mathrm{r})]^\dagger\hat{O}_{\mathrm{ps}}\hat{P}\psi(\mathrm{r}) \\
&= -\int d^3\mathrm{r}\psi^*(\mathrm{r})\hat{O}_{\mathrm{ps}}\psi(\mathrm{r}) = 0
\end{aligned} \tag{53}$$

So we conclude, if the state is not parity eigenstate, pseudo-scalar operators such as $\mathrm{S}\cdot\mathrm{p}$ must exsist in the system, they have non-zero expectation values e.g. $\langle\mathrm{S}\cdot\mathrm{p}\rangle \neq 0$.

Parity violation in β decay

In 1957 C.S. Wu and her collaborators measured the electron distribution in the β decay of polarized cobalt ,

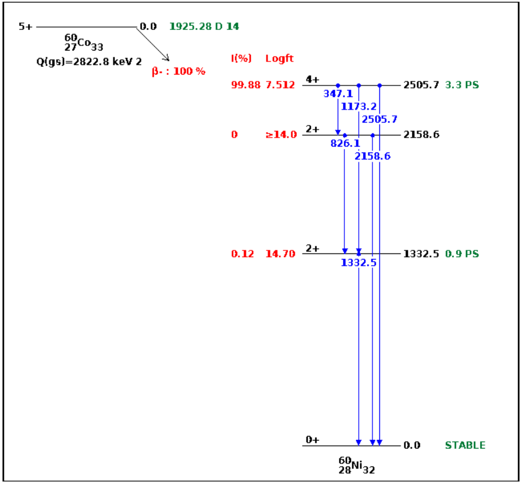
$$^{60}_{27}\mathrm{Co} \quad \rightarrow \quad ^{60}_{28}\mathrm{Ni} + e^- + \overline{\nu}_e \tag{54}$$

They found that electrons are predominantly emitted opposite to the nuclear spin. Under the parity transformation $\mathrm{r} \rightarrow -\mathrm{r}$, the magnetic field and the spin are invariant while the mometum changes the direction, i.e.

$$\begin{aligned} \mathrm{B} &\rightarrow \mathrm{B} \\ \mathrm{s} &\rightarrow \mathrm{s} \\ \mathrm{p} &\rightarrow -\mathrm{p} \end{aligned} \tag{55}$$

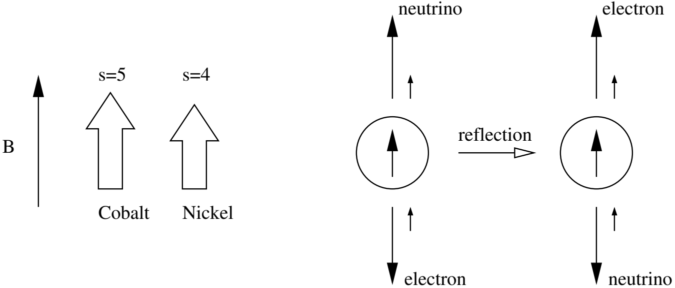
Parity violation in β decay

Figure: The parity violation in the β decay.



Parity violation in β decay

Figure: The parity violation in the β decay.



Homework

Read and translate two famous articles: (1) Wu et. al., Phys. Rev. 105, 1413 (1957). (2) Lee and Yang, Phys. Rev. 104, 254 (1956).