第四章：放射性与核衰变

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原子核物理导论・2026 年春

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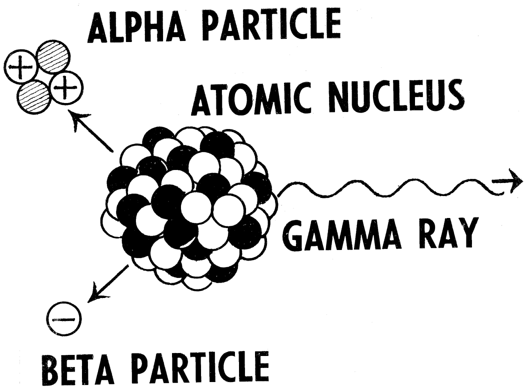
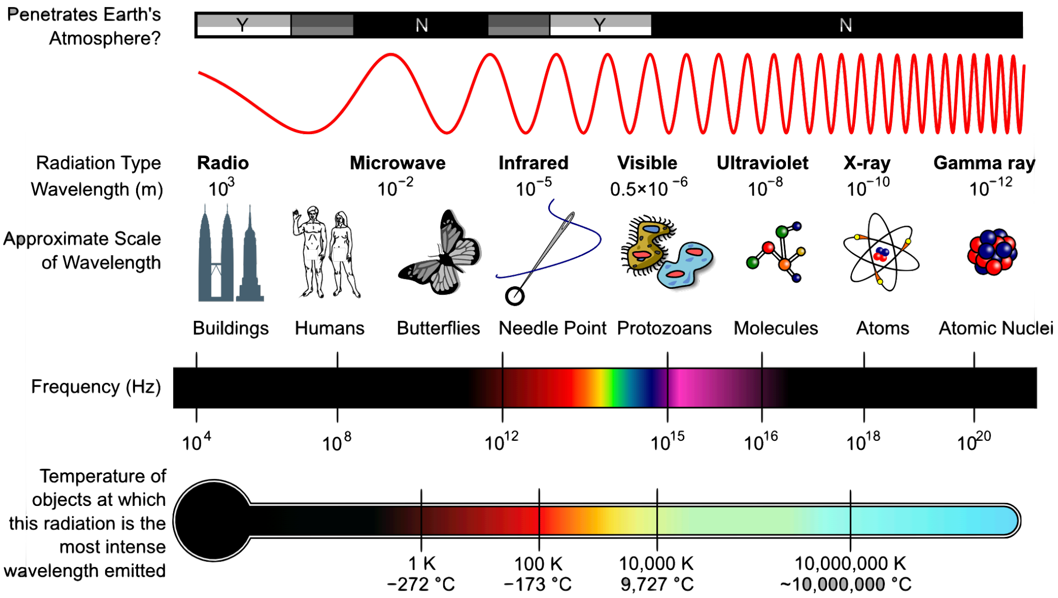
- Introduction
- α decay
- β decay
- γ **decay**




Penetrates Earth's Atmosphere?
Y
N
Y
N
Radiation Type
Wavelength (m)
Radio
10^3
Microwave
10^{-2}
Infrared
10^{-5}
Visible
0.5×10^{-6}
Ultraviolet
10^{-8}
X-ray
10^{-10}
Gamma ray
10^{-12}
Approximate Scale of Wavelength
Buildings
Humans
Butterflies
Needle Point
Protozoans
Molecules
Atoms
Atomic Nuclei
Frequency (Hz)
10^4
10^8
10^{12}
10^{15}
10^{16}
10^{18}
10^{20}
Temperature of objects at which this radiation is the most intense wavelength emitted
1 K
−272 °C
100 K
−173 °C
10,000 K
9,727 °C
10,000,000 K
~10,000,000 °C

γ decay: history

- 1895: Wilhelm Röntgen discovered X-rays
- 1899: Paul Villard discovered a third component of radiation from radium
- 1903: Rutherford named this radiation gamma rays based on their relatively strong penetration of matter. Rutherford also discovered that gamma rays were not deflected by a magnetic field, another property making them unlike alpha and beta rays
- 1905: Einstein proposed the photon, the quantum of the light, which was proved by Compton and Wu in 1922

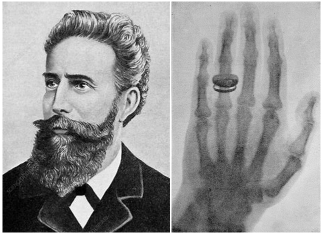


Figure: Wilhelm C. Röntgen and the world first X-ray photograph

γ decay: history

- 1912: Max von Laue et. al. observed the diffraction of X-rays by crystals and showed that X-rays are short wavelength electromagnetic radiations
- 1914: Rutherford observed the reflection of gamma rays from crystal surfaces, proving that they were electromagnetic radiation
- 1920s: Dirac developed the quantum theory of atomic radiation
- 1930s: Heisenberg established the model of atomic nucleus after the discovery of the neutron by Chadwick
- 1951: Weisskopf and Moszkowski derived the multipole radiation of gamma decay based on single-particle approximation

γ decay

- γ decay is a de-excitation from an excited bound-state nucleus to a lower energy nuclear state, preceded by some decay or reaction. During the process, Z & A of the nucleus remain unchanged
$$^{A}Z^{*} \rightarrow {}^{A}Z + \gamma$$
- The most often emitted radiations are a single photon (γ). Conversion electron (electron from the atom), $e^{+}e^{-}$ pair and two-photon ($\gamma\gamma$) decays are also possible
- γ ray energies can span anywhere from several keV to several MeV
- γ decay lifetimes are typically extremely short (10^{-12} s), with the exception of isomeric states
- A nuclear isomer is a metastable state ($T_{1/2} \gtrsim 10^{-9}$ s) of an atomic nucleus, in which one or more nucleons (protons or neutrons) occupy excited state (higher energy) levels. An extreme case is $^{180m}\mathrm{Ta}$ (钽), which has a lifetime at least 10^{15} years!

γ decay kinematics

$$\vec{p}_\gamma + \vec{p}_d = 0,$$

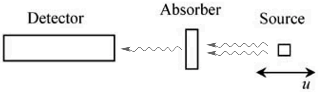
$$E_\gamma + \frac{p_d^2}{2m_A} = Q_\gamma,$$

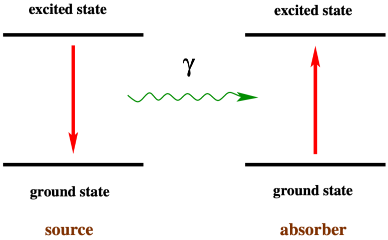
$$E_\gamma = p_\gamma c$$

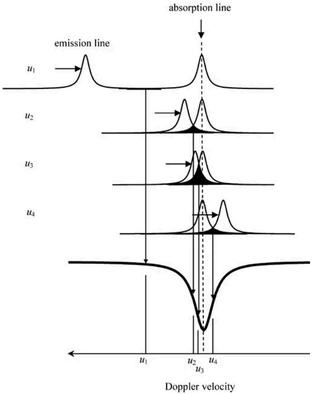
- Gamma photon energy: 10 keV - 1 MeV
- Recoil energy: $T_{\rm rec} = E_\gamma^2/2M_A \sim$ eV -- small in comparison to the gamma energy
- Recoil-free gamma decay: Mössbauer effect
- gamma wavelength: $\lambda_\gamma = 2\pi\hbar/E_\gamma \sim 2\pi \times 197\,\text{MeV}\cdot\text{fm}/E_\gamma \sim 120\,\text{fm}$; nuclear radii:
  $r_A = 1.2A^{1/3}\,\text{fm} \sim 5\,\text{fm}$
  $\lambda_\gamma \gg r_A \quad \Rightarrow \quad$ no diffraction

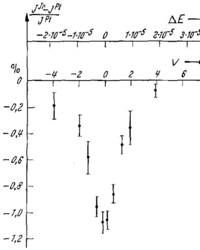
Mössbauer effect

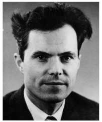
Mössbauer discovered the recoil-free emission and absorption of gamma rays by nuclei in 1958 in his Ph.D. work (Nobel prize 1961). Mössbauer effect allows one to make high precision measurement as shown immediately after his publication by Pound and Rebka in 1960 to measure the gravitational redshift of photons.







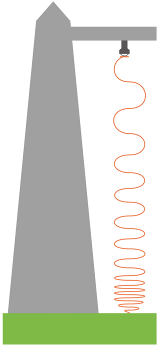


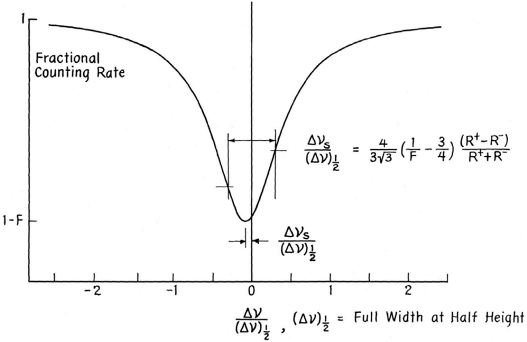


Mössbauer effect

In 1960, Pound and Rebka used Mössbauer effect to experimentally verify the gravitational redshift.

$$\frac{\Delta\nu}{\nu} = \frac{gH}{c^2} \sim 10^{-15}$$





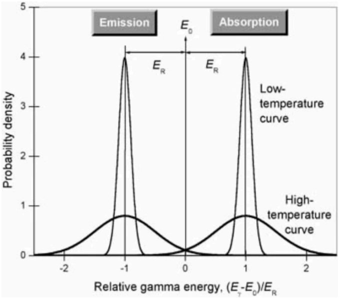
Mössbauer effect

- If the nucleus recoil is ignored, the energy of the emitted gamma photon E_γ is the same as E_0 the energy difference between the two energy levels. Therefore, the gamma photon can be absorbed by the same type nuclide
- However, in gamma decay, there is a small recoil energy $E_R/E_\gamma = E_\gamma/2M_A \sim 10^{-8}$, which is still large enough in comparison to the width of the nuclear energy level $\Gamma/E_\gamma \sim 10^{-8}$.
- Furthermore, in gases, the gamma photons are Doppler shifted, which broadens the gamma photon energy spectrum: $\Delta E_\gamma = \sqrt{2TE_R}$.

$$^{57}\mathrm{Fe} \quad \frac{3}{2}^+ \rightarrow \frac{1}{2}^+ :$$

$$E_0 = 14.4\,\mathrm{keV}, \quad \Gamma = 4.67\times 10^{-9}\,\mathrm{eV},$$

$$E_R = 1.92\times 10^{-3}\,\mathrm{eV}.$$



Mössbauer effect

- Prior to the discovery of the Mössbauer effect, many attempts had been made to improve the chances for γ ray resonance fluorescence. The idea was to make the overlap between the emission and absorption lines larger by setting the emitter/absorber nuclei in motion (in order to compensate for the energy loss due to recoil) either mechanically (Moon 1950) or thermally (Malmfors 1953).
- The key problem, i.e., how to avoid recoil and Doppler broadening at the same time, was solved by Rudolf Mössbauer (Mössbauer 1958) when he embedded the γ ray emitting and absorbing nuclei in the lattice of a solid. In this way, he succeeded to show that a recoil free line exists at E_0, which is called the Mössbauer line.
- Since the energy of lattice vibrations is quantized approximately at the Einstein energy $E_E = \pi\hbar c_s/a$, the relatively low recoil energy cannot be transferred to the solid to increase its internal vibrational energy.

Mössbauer effect

Let us consider, e.g., the 14 keV transition of Fe-57 and the 800 keV transition of Fe-58. What we find is that the recoil energy of Fe-57 ($E_R = 0.002\,\mathrm{eV}$) is small compared with the Einstein energy ($E_E = 0.04\,\mathrm{eV}$) and, consequently, recoil-free emission can occur. For Fe-58, on the other hand, the recoil energy ($E_R = 6\,\mathrm{eV}$)

H																	He
Li	Be	Unsuitable										B	C	N	O	F	Ne
Na	Mg	Mössbauer-active probe										Al	Si	P	S	Cl	Ar
K	Ca	Sc	Ti	V	Cr	Mn	Fe	Co	Ni	Cu	Zn	Ga	Ge	As	Se	Br	Kr
Rb	Sr	Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd	Ag	Cd	In	Sn	Sb	Te	I	Xe
Cs	Ba	*	Hf	Ta	W	Re	Os	Ir	Pt	Au	Hg	Tl	Pb	Bi	Po	At	Rn
Fr	Ra	**	104~														
*Lanthanide		La	Ce	Pr	Nd	Pm	Sm	Eu	Gd	Tb	Dy	Ho	Er	Tm	Yb	Lu	
**Actinide		Ac	Th	Pa	U	Np	Pu	Am	Cm		Cf	Es	Fm	Md	No	Lr	

Mössbauer effect

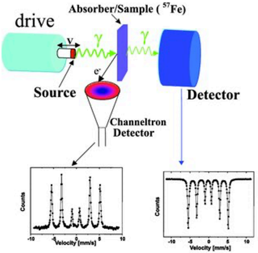
- The width of the absorption curve

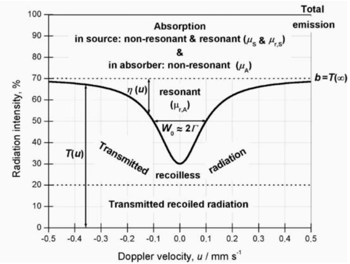
$$W = \frac{2\Gamma}{E_0}c$$

- The area of the curve,

$$A = \frac{\pi}{2}\Gamma f_s \tau_A$$

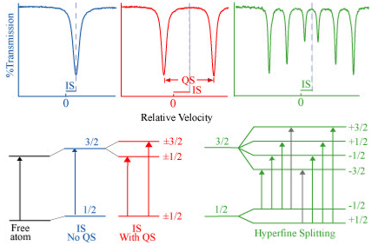
where, τ_A is the effective thickness of the absorber and f_S is the recoilless fraction of the radiation.

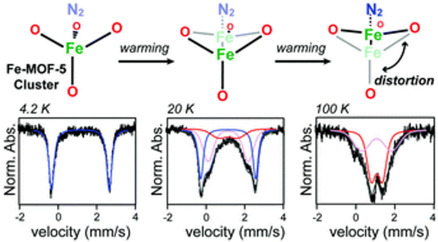




Application of Mössbauer effect

- The interaction between the electrons and the nucleus causes a very small perturbation of the nuclear energy levels in comparison with the energy of the nuclear transition. Such interactions can be sensitively monitored by Mössbauer spectroscopy. The measurement of hyperfine interactions is the key of the utilization of Mössbauer spectroscopy in a wide range of applications.





Mössbauer effect

For the resonant absorption to occur, on should have

$$\begin{aligned} E_{\gamma,\mathrm{em}} &= E_0 - E_R \\ E_{\gamma,\mathrm{ab}} &= E_0 + E_R \\ E_{\gamma,\mathrm{ab}} - E_{\gamma,\mathrm{em}} &= 2E_R < 2\Gamma \end{aligned} \tag{1}$$

It was a great breakthrough to realize that one could get resonance absorption of gamma rays by putting the source nuclei in a crystal in low temperature. To see how many iron nuclei would have to recoil together to keep the gamma within the natural linewidth:

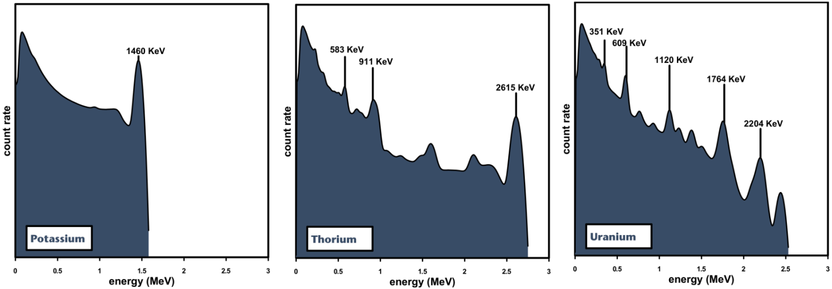
$$\begin{aligned} E_R &= \frac{p^2}{2mN} = \Gamma, \\ N &= \frac{p^2}{2m\Gamma} \approx \frac{2\times10^{-3}}{6\times10^{-9}} \approx 3.3\times10^5 \end{aligned} \tag{2}$$

which is many orders of magnitude smaller than the macroscopic scale 10^{23}.

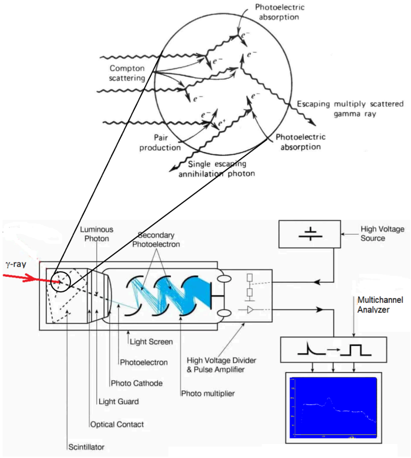
Homework

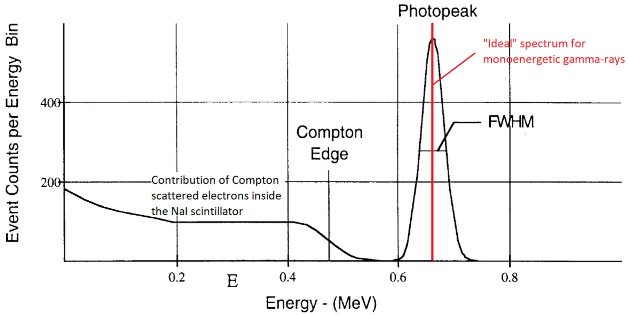
Write an article (3000 Chinese words) about Mössbauer effect and Mössbauer spectroscopy.

γ spectrum

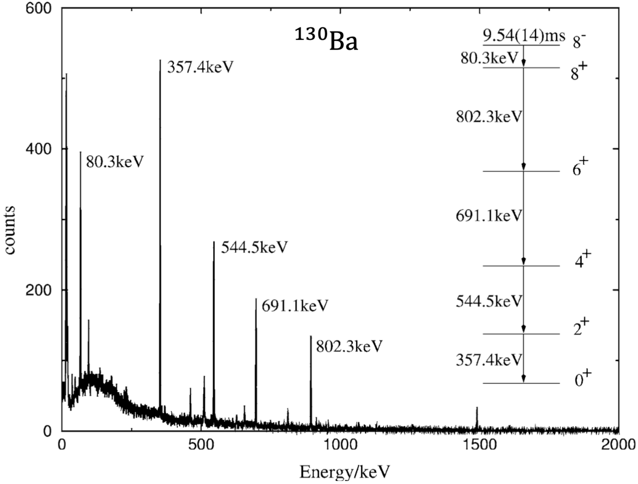


γ spectrum

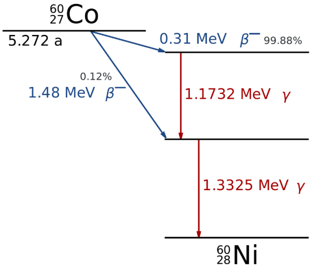

Photoelectric absorption
Compton scattering
Escaping multiply scattered gamma ray
Pair production
Photoelectric absorption
Single escaping annihilation photon
Luminous Photon
Secondary Photoelectron
High Voltage Source
γ-ray
Multichannel Analyzer
Light Screen
Photoelectron
High Voltage Divider & Pulse Amplifier
Photo Cathode
Photo multiplier
Light Guard
Optical Contact
Scintillator

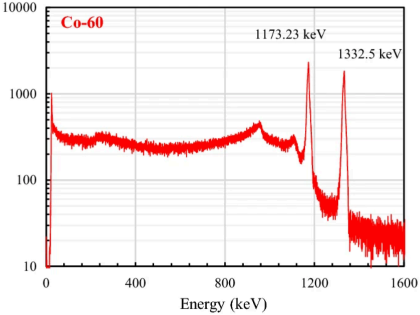

Photopeak
"Ideal" spectrum for monoenergetic gamma-rays
FWHM
Compton Edge
Contribution of Compton scattered electrons inside the NaI scintillator
Event Counts per Energy Bin
400
200
0.2
E
0.4
0.6
0.8
Energy - (MeV)

γ spectrum: example

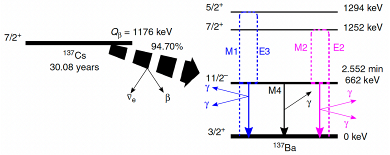


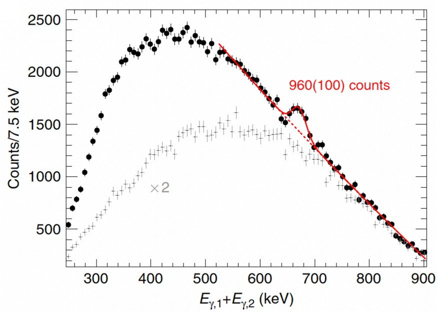
γ spectrum: example





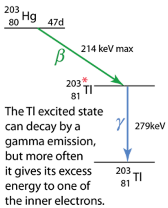
γ spectrum: example

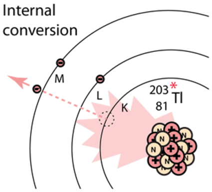


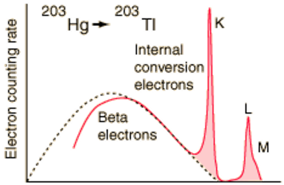


Internal conversion

- Internal conversion (IC): the excited nucleus interacts electromagnetically with one of the orbital electrons of an atom (via virtual photon), which causes the electron to be emitted from the atom (not from the nucleus)
- The ejected electron created a vacancy from the atom, which causes subsequent relaxation of the atom through X-ray emission and Auger electron emission (an atomic process similar to IC)
- Internal pair production: if the energy of the gamma exceeds the pair production limit $2m_e = 1.02\,\mathrm{MeV}$, an electron-positron pair may be emitted.







γ decay: electromagnetic interaction at work

Nuclei are many body systems of nucleons which interact via nuclear force or strong interaction. A nucleus can stay in some quantum states with specific energies. By external perturbation a nucleus can be excited to higher energetic states and can jump onto its ground state by radiating photons (for example, the nuclei normally stay in their excited states right after the α and β decay).

Electromagnetic interaction: QED

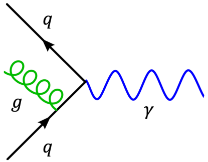
- On the fundamental level, gamma decay is induced by interaction between quarks and the photon, as described by quantum electrodynamics (QED)
- The interaction is described by the Lagrangian density,

$$\mathcal{L}_{\text{int}} = \sum_q e_q \bar{q} \gamma^\mu q A_\mu$$

where, $q = u, d$, $e_u = -1/3, e_u = +2/3$

- The electromagnetic current is a vector current,

$$J^\mu_{\text{em}} = \sum_q e_q \bar{q}\gamma^\mu q = +\frac{2}{3}\bar{u}\gamma^\mu u - \frac{1}{3}\bar{d}\gamma^\mu d$$



- Scattering amplitude:

$$i\mathcal{M}_{fi} = i\varepsilon^*_\mu(q,\lambda)\bar{u}_{s'}(p')i\gamma^\mu u_s(p)$$

Electromagnetic interaction: hadronic matrix element

- Moving from the quarks to nucleons, the scattering amplitude becomes,

$$i\mathcal{M}_{fi} = i\varepsilon^*_\mu(q,\lambda)\langle p', s'|J^\mu_{\rm em}(0)|p, s\rangle$$

  where, $\langle p', s'|J^\mu_{\rm em}(0)|p, s\rangle$ is known as the hadronic matrix element (HME)

- Lorentz decomposition of the HME,

$$\begin{aligned}\langle p', s'|J^\mu_{\rm em}(0)|p, s\rangle &= \bar{u}_{s'}(p')\Big[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\Big]u_s(p) \\ &= \bar{u}_{s'}(p')\Big[\frac{P^\mu}{M(1+\tau)}G_E(q^2) + \frac{i\epsilon^{\mu\nu\alpha\beta}q_\nu P_\alpha\gamma_\beta\gamma_5}{2M(1+\tau)}G_M(q^2)\Big]u_s(p)\end{aligned}$$

  where, $q = p' - p$, $p^2 = p'^2 = M^2$, $P = (p' + p)/2$, $\tau = -q^2/4M^2$. $F_{1,2}$ called the Dirac and Pauli form factors, respectively.

- It is useful to introduce the Sachs electric and magnetic form factors:

$$G_E = F_1 + \frac{q^2}{4M^2}F_2, \quad G_M = F_1 + F_2,$$

Elastic lepton-hadron scattering

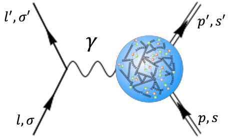
- HMEs are related to the amplitude of the elastic lepton-hadron scattering,

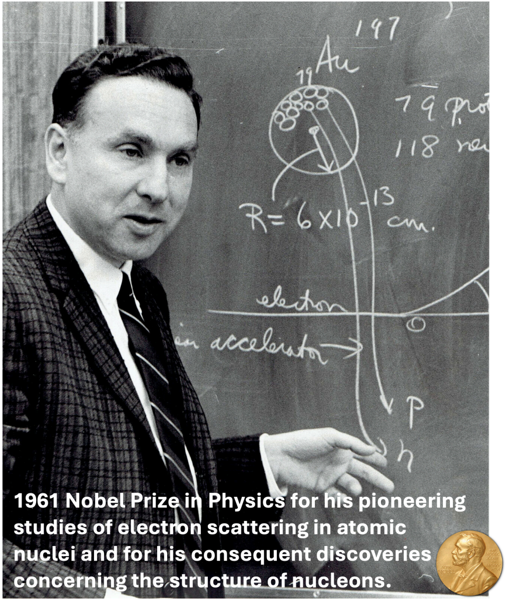
$$i\mathcal{M} = e^2 \bar{u}_{\sigma'}(l') i\gamma^\mu u_\sigma(l) \frac{-ig_{\mu\nu}}{q^2} \langle p', s' | iJ^\nu(0) | p, s \rangle$$

- Cross section (Rosenbluth formula),

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha_{\mathrm{em}}^2}{4E^2 \sin^4 \frac{\theta}{2}} \frac{E'}{E} \left(\frac{G_E^2 + \tau G_M^2}{1+\tau} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right) = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big|_{\text{Mott}} \frac{\tau}{\epsilon(1+\tau)} \left[G_M^2 + \frac{\epsilon}{\tau} G_E^2 \right]$$

where, $E'/E = (1 + \frac{2E}{M} \sin^2 \frac{\theta}{2})^{-1}$ is the target recoil factor; $\epsilon = (1 + 2(1+\tau) \tan^2 \frac{\theta}{2})^{-1}$.





1961 Nobel Prize in Physics for his pioneering studies of electron scattering in atomic nuclei and for his consequent discoveries concerning the structure of nucleons.

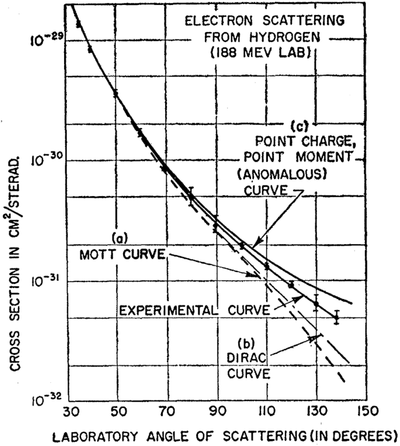


FIG. 24. Electron scattering from the proton at an incident energy of 188 Mev. The experimental points lie below the point-charge point-moment curve of Rosenbluth, indicating finite size effects.

Nucleon e.m. form factors

- LT separation:

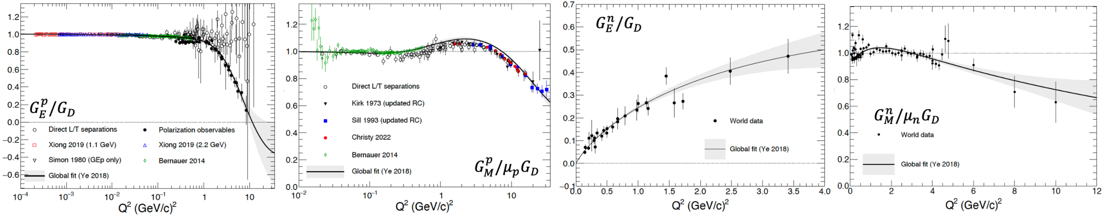
$$\sigma_R = \epsilon(1+\tau)\frac{(\mathrm{d}\sigma/\mathrm{d}\Omega)_{\text{exp}}}{(\mathrm{d}\sigma/\mathrm{d}\Omega)_{\text{Mott}}} = G_M^2 + \frac{\epsilon}{\tau}G_E^2$$

where, $\tau = Q^2/4M^2$, $\epsilon = \left(1+2(1+\tau)\tan^2\frac{\theta}{2}\right)^{-1}$ is the longitudinal virtual photon polarization.

- Polarization beam (PT): striking discrepancies due to two-photon exchange (TPE) contributions
- Dipole ansatz,

$$G_E^p(q^2) \approx G_D(-q^2),\ G_M^p(q^2) \approx \mu_p G_D(-q^2),\ G_E^n(q^2) \approx -\tau\mu_n G_D(-q^2),\ G_M^n(q^2) \approx \mu_n G_D(-q^2)$$

where, $\tau = -q^2/4M_N$, and $G_D(Q^2) = \left[1+Q^2/0.71\,\text{GeV}^2\right]^{-2}$ is the "standard dipole".



Electromagnetic charges of the nucleons

For gamma decay, the momentum transverse is much smaller than the mass of the nucleon $E_\gamma = |\vec{q}| \ll M_N$. Therefore, only the low energy constants, i.e. form factors at zero-momentum transfer, are relevant. They are related to the global charges of the particles,

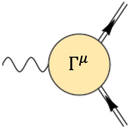
- Charge (monopole moment) $q = \pm e G_E(0)$. Charge conservation implies,

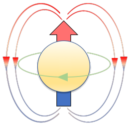
$$G_E(0) = F_1(0) = 1$$

- The magnetic dipole moment $\mu = g\mu_B$, where $\mu_B = q/2M$ and the Landé G-factor

$$G_M(0) = F_1(0) + F_2(0) = \frac{g}{2}$$

  - For the nucleons including the neutron, $\mu_B = |e|/2M_N$
  - $F_2(0) = (g-2)/2$ is known as the anomalous magnetic moment

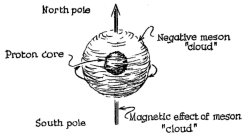




- The r.m.s. charge radius is defined as, $r_E^2 \equiv 6G'_E(0)$

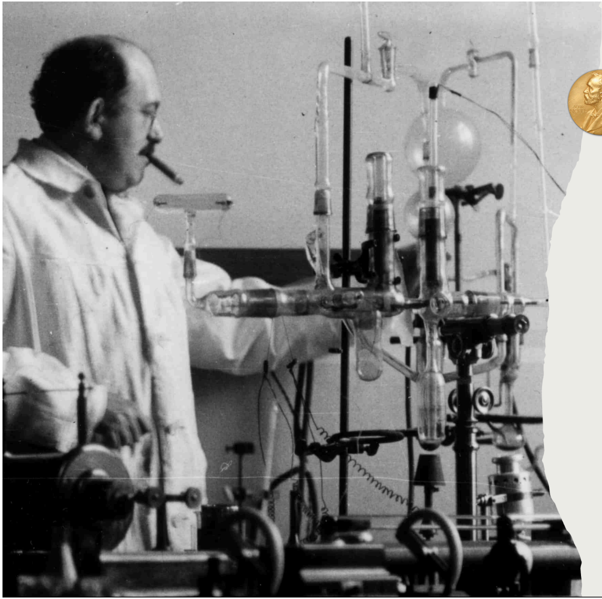
Electromagnetic charges of the nucleons

	charge $q/\|e\|$	Magnetic moment $g = \mu/\mu_B$	charge radius $\langle r_{\rm ch}^2 \rangle$
Electron	-1	$2 + \frac{\alpha_{\rm em}}{\pi} + O(\alpha_{\rm em}^2)$	∞
Proton	$+1$	2.79	$(0.84\ \mathrm{fm})^2$
Neutron	0	-1.91	$-0.12\ \mathrm{fm}^2$



$\mathrm{n} = \mathrm{p} + \pi^-$ cloud

taken from ref. [18]



Nobel prize 1943 for his development of the molecular beam method and discovery of proton magnetic moment

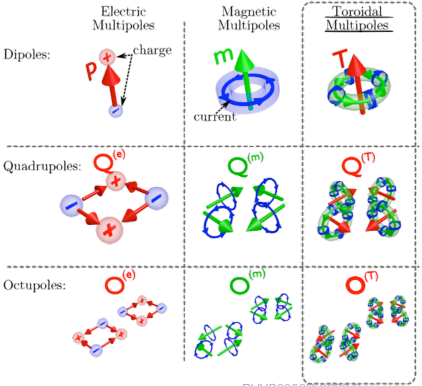
- Developed the molecular beam method
- Experimentally verified the Maxwell-Boltzmann distribution (1920)
- Demonstrated the quantization of the electron spin, later known as Stern-Gerlach experiment (1922)
- Experimentally tested the de Broglie relation for atoms and molecules (1926)
- Measurement of the proton magnetic moment (1933)

Nominated for Nobel prize for 82 times!

Fundamental symmetries

$$\langle p', s'|J^\mu(0)|p, s\rangle = \bar{u}_{s'}(p')\Big[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2) + \underbrace{i\varepsilon^{\mu\nu\rho\sigma}\frac{q_\nu\sigma_{\rho\sigma}}{4M}F_3(q^2)}_{\text{T odd}} + \underbrace{\frac{1}{2M}\Big(q^\mu - \frac{q^2}{2M}\gamma^\mu\Big)\gamma_5 F_4(q^2)}_{\text{P odd}}\Big]u_s(p)$$

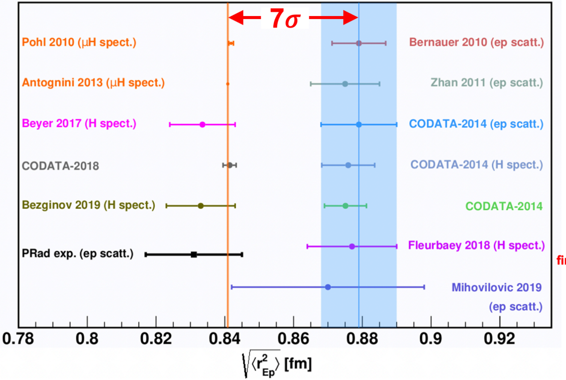
- $2J+1$ CPT conserving multipole moments for spin-J
- $q = eF_1(0)$ charge monopole moment
- $\mu = \frac{e}{M}[F_1(0) + F_2(0)]$ magnetic dipole moment
- $d = \frac{1}{2M}F_3(0)$ electric dipole moment (EDM)
- $a = F_4(0)$ Zel'dovich anapole moment/toroidal moment
- Neutron EDM: $< 10^{-26}e.\text{cm}$ (exp.), $\sim 10^{-32}e.\text{cm}$ (SM), sensitive to new physics
- A Majorana fermions has only the anapole moment

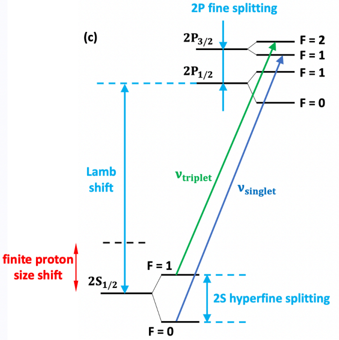


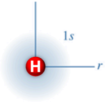
Proton radius puzzle: $r_E^2 \equiv 6G_E'(0)$

Experimental measurements:

- ep scattering at low-Q^2
- Atomic hydrogen spectroscopy $\Delta E_{\text{fin. size}} = \frac{2\alpha_{\text{em}}^4}{3n^3}\left(\frac{m_\ell M_p}{M_p+m_\ell}\right)^3 \langle r_{Ep}^2\rangle \delta_{l0}$







γ decay of nucleon resonance

- Δ^+ is an excited nucleon state with spin $\frac{3}{2}$ and parity $+1$. Its mass is 1.232 GeV
- It may decay into the nucleon and a photon

$$\Delta \to N\gamma$$

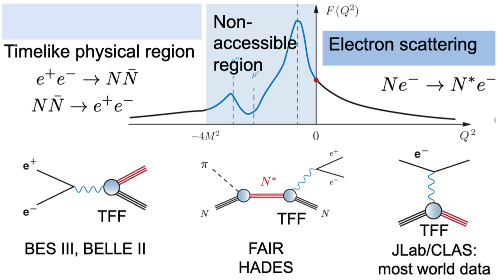
- The decay width

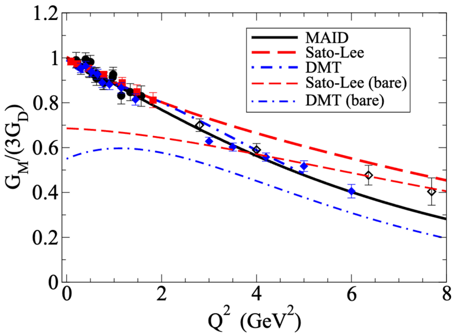
$$\Gamma_{\Delta\to\gamma N} = \frac{\alpha_{\rm em} M_N (M_\Delta^2 - M^2)}{(2J+1)M_\Delta^2} \Big\{ \big|\langle \Delta(+\tfrac{1}{2})|\epsilon_\mu^{+1} J^\mu | N(-\tfrac{1}{2})\rangle\big|^2 + \big|\langle \Delta(+\tfrac{3}{2})|\epsilon_\mu^{+1} J^\mu | N(+\tfrac{1}{2})\rangle\big|^2 \Big\}$$

which involves hadronic matrix element at $q^2 = 0$ (not zero-energy photon!)

- Hadronic matrix element of $\Delta \to N\gamma$

$$\langle\langle\Delta(p',s')|J^\mu|N(p,s)\rangle = \bar{u}_\alpha(p',s')\Big\{[q^\alpha\gamma^\mu - \not{q} g^{\alpha\mu}]G_1(q^2) + [q^\alpha p'^\mu - (p'\cdot q)g^{\alpha\mu}]G_2(q^2) + [q^\alpha q^\mu - q^2 g^{\alpha\mu}]G_3(q^2)\Big\}\gamma_5 u(p,s)$$



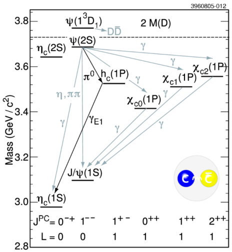


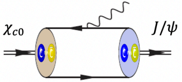
γ decay of charmonium

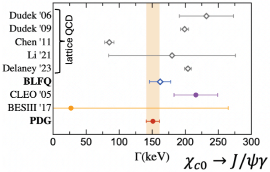
- Charmonia are bound states of charm and anti-charm quark
- Electromagnetic transitions provide important structure information of the system
- Decay width,

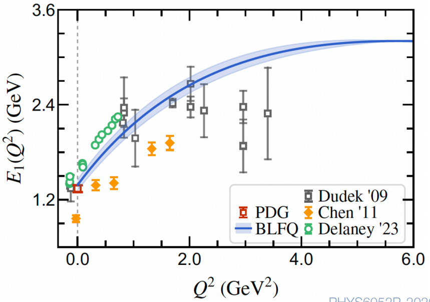
$$\Gamma_{\chi_{c0}\to\gamma J/\psi} = \frac{e_c^2\alpha_{\text{em}}}{2J_i+1}\frac{M_i^2-M_f^2}{2M_i^3}|E_1(0)|^2$$

$$\langle V(p',\lambda')|J^\mu(0)|S(p)\rangle = E_1(Q^2)\Big[e^{\mu*}_{\lambda'}(p') - \frac{e_{\lambda'}\cdot p}{(p\cdot p')^2 - M_S^2M_V^2}\big(p'^\mu(p\cdot p') - M_V^2 p^\mu\big)\Big]$$
$$+ C_1(Q^2)\frac{M_V}{Q(p\cdot p')^2 - M_S^2M_V^2}(e^*_{\lambda'}\cdot p)\Big[(p\cdot p')(p+p')^\mu - M_S^2p'^\mu - M_V^2p^\mu\Big]$$









Hadronic electromagnetic current

- Recall, the tree-level γ-nucleon scattering amplitude,

$$i\mathcal{M}_{fi} = i\varepsilon^*_\mu(q,\lambda)\bar{u}_{s'}(p')\Big[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\Big]u_s(p)$$

- Effective Lagrangian,

$$\begin{aligned}\mathcal{L}_{\text{int}} &= J^\mu(x)A_\mu(x) = J^\mu_E A_\mu + J^\mu_M A_\mu \\ &\equiv \frac{q_N}{2M_N}\bar{N}\overleftrightarrow{\partial}^\mu N A_\mu + \mu_N\partial_\nu(\bar{N}\sigma^{\mu\nu}N)A_\mu\end{aligned}$$

where, $f\overleftrightarrow{\partial}g = f\partial g - (\partial f)g$

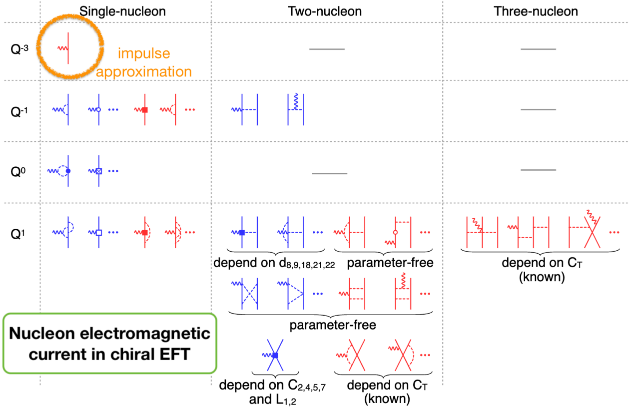
- Charge and current density operators in the non-relativistic limit,

$$\begin{aligned}J^0(r) &= \rho_E \equiv q_N\delta^3(r - \underline{r}_N(t)), \\ \vec{J}_E(r) &= q_N\vec{v}_N(t)\delta^3(r - \underline{r}_N(t)) \\ \vec{J}_M(r) &= \mu_N(\nabla\times\vec{s})\delta^3(r - \underline{r}_N(t))\end{aligned}$$

where, $\vec{s}$ is the spin of the particle. We use underscore $\underline{r}$ to indicate the position operator

Impulse approximation

Moving from the hadronic level to the nuclear level, we again adopt the impulse approximation and assume the current is a one-body operator, i.e. we ignore the mesonic degrees of freedom



Nuclear electromagnetic current

- Within the impulse approximation, the quantum many-body expressions of the nuclear electromagnetic current are,

$$J^0(r) = \rho_E \equiv \sum_{i=1}^{A} e_i \delta^3(r - \underline{r}_i),$$

$$\vec{J}(r) = \vec{J}_E + \vec{J}_M \equiv \sum_{i=1}^{A} e_i \vec{v}_i \delta^3(r - \underline{r}_i) + \sum_{i=1}^{A} \mu_i (\nabla \times \vec{s}_i) \delta^3(r - \underline{r}_i)$$

- Again, the underscore indicate the operator. These many-body operators can be converted to the second-quantized/quantum field operators following standard second quantization procedure

Electric multipoles

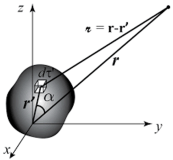
- Consider the classical electric potential away from the source (a collection of charges),

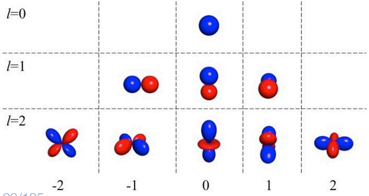
$$\varphi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \sum_{m=-l}^{l} \frac{O_E(l,m)}{r^{l+1}} Y_{lm}(\hat{r})$$

where, $O_E(l,m)$ is called the electric multipole moments, defined as,

$$O_E(l,m) = \int d^3r \rho_E(\vec{r}) r^l Y^*_{lm}(\hat{r})$$

- Spherical harmonics are eigen functions of the angular momentum, i.e. they rotate nicely (covariantly). For the point-charge systems with good rotation symmetry, the multipole expansion terminates up to the total orbital angular momentum L





Magnetization current and effective magnetic charge density

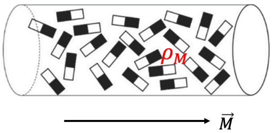
- The current $\vec{J}$ is related to the magnetization of the ``medium"

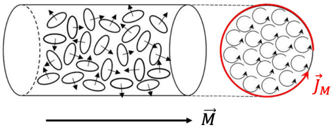
$$\vec{M} = \vec{m}\delta^3(r - r_N(t)) \quad \Rightarrow \quad \vec{J} = \nabla \times \vec{M}$$

where, $\vec{m} = \vec{m}_S + \vec{m}_L = \mu_N \vec{s} + \frac{1}{2} q_N (\vec{r} \times \vec{v})$ is the nucleon magnetic moment (in analogy to molecular magnetic moment in materials).

- $\vec{m}_S \equiv g_S \mu_B \vec{s} = \mu_N \vec{s}$ is the spin magnetic moment, and $\vec{m}_L = g_L \mu_B \vec{L} = \frac{1}{2} q_N (\vec{r} \times \vec{v})$ is the orbital magnetic moment. The Landé factors $g_S = \mu_N / \mu_B = 2G_M(0)$, and $g_L = 1$
- An equivalent view is to introduce effective magnetic charge distribution ρ_M. The magnetization $\vec{M}$ is induced by magnetic charge polarization of the medium, i.e. magnetic dipoles:

$$\rho_M = -\nabla \cdot \vec{M}$$





Magnetic multipoles

- Recall the electric multipole moments,

$$O_E(l,m) = \int d^3r \rho_E(\vec{r}) r^l Y^*_{lm}(\hat{r})$$

- In analogy to the electric multipoles, introduce the magnetic multipoles as

$$\begin{aligned}
O_M(l,m) &= \int d^3r \rho_M(\vec{r}) r^l Y^*_{lm}(\hat{r}) \\
&= -\frac{1}{l+1}\int d^3r \vec{M}\cdot\nabla r^l Y^*_{lm}(\hat{r}) \\
&= -\frac{1}{l+1}\int d^3r \vec{J}(\vec{r})\cdot(\vec{r}\times\nabla) r^l Y^*_{lm}(\hat{r})
\end{aligned}$$

- Here, we have used the identity:

$$\nabla\times(\vec{r}\times\nabla) r^l Y_{lm}(\hat{r}) = -(l+1)\nabla r^l Y_{lm}(\hat{r})$$

Electric and magnetic multipole operators

- Moving to the quantum theory, we promote the density and current to operators and define two sets of operators, the electric multipole operators and magnetic multipole operators:

$$\underline{O}_E(l,m) = \int d^3r \underline{\rho}_E(\vec{r}) r^l Y^*_{lm}(\hat{r}),$$

$$\underline{O}_M(l,m) = -\frac{1}{l+1}\int d^3r \underline{\vec{J}}(\vec{r}) \cdot (\vec{r}\times\nabla) r^l Y^*_{lm}(\hat{r})$$

- Recall, the quantum many-body operators of the electric density operator and the electric current are,

$$J^0(r) = \rho_E \equiv \sum_{i=1}^{A} e_i \delta^3(r - \underline{r}_i),$$

$$\vec{J}(r) = \vec{J}_E + \vec{J}_M \equiv \sum_{i=1}^{A} e_i \vec{v}_i \delta^3(r - \underline{r}_i) + \sum_{i=1}^{A} \mu_i (\nabla \times \vec{s}_i) \delta^3(r - \underline{r}_i)$$

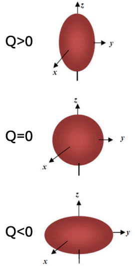
Electric and magnetic multipole moments

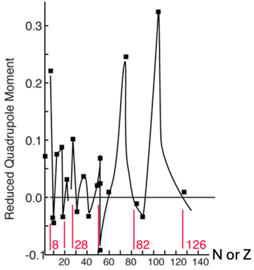
- Nuclear matrix elements:

$$\langle f|\underline{O}_{E,M}(l,m)|i\rangle$$

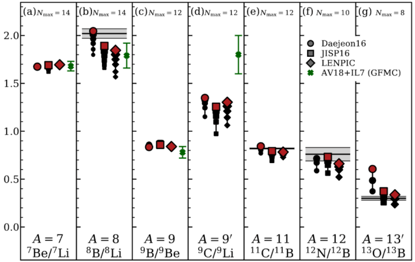
- The diagonal terms of the NMEs, i.e. $|i\rangle$ and $|f\rangle$ are the same nuclear state (allowing possible spin flips), give the (elastic) electric and magnetic multipole moments
- The off-diagonal terms of the NMEs, i.e. $|i\rangle$ and $|f\rangle$ are different nuclear states, gives the transition multipole moments, relevant for gamma decay

Electric quadrupole moments Q = $O_E(2, 0)$





electric quadrupole of odd-A nuclides



Parity

- Under the parity transformation,

$$\vec{r} \quad \rightarrow \quad -\vec{r}, \qquad\qquad t \quad \rightarrow \quad t$$

- Parities of the electric and magnetic multipoles,

$$P^{-1}O_{E,M}(l,m)P = \pi_{E,M}(l,m)O_{E,M}(l,m),$$
$$\Rightarrow \quad \pi_E(l,m) = (-1)^l, \quad \pi_M(l,m) = (-1)^{l+1}$$

- Consider the NMEs,

$$\langle f|O(l,m)|i\rangle = \langle f|PP^{-1}O(l,m)PP^{-1}|i\rangle = \pi(l,m)\pi_f\pi_i\langle f|O(l,m)|i\rangle,$$
$$\Rightarrow \quad \pi(l,m)\pi_f\pi_i = +1 \quad \text{(if NME is non-vanishing)}$$

where, $\pi_{i,f}$ are the parities of the initial and final nuclear states.

Parity selection rules

- For elastic nuclear multipoles, $\pi_f = \pi_i$ and only the even electric multipoles (monopole E0, quadrupole E2, hexadecapole E4, ...) and odd magnetic multipoles (dipole M1, octupole M3, ...) are present
- For electromagnetic transitions,

$$\Delta\pi \equiv \pi_i \pi_f = \begin{cases} -1, & \cancel{\mathrm{M0}}, \mathrm{E1}, \mathrm{M2}, \mathrm{E3}, \mathrm{M4}, \mathrm{E5}, \cdots \\ +1, & \cancel{\mathrm{E0}}, \mathrm{M1}, \mathrm{E2}, \mathrm{M3}, \mathrm{E4}, \mathrm{M5}, \cdots \end{cases}$$

- The E0 transition is allowed by the parity but excluded nevertheless, because it only couples to the longitudinal degree of freedom (d.o.f.) of the electromagnetic field. Real photons are transversely polarized and the longitudinal d.o.f., i.e. the static electric field, does not propagate.
- If the gamma photon is virtual, the longitudinal d.o.f. is dynamical and E0 transition is allowed, such as in the internal conversion process
- The M0 transition is also absent, due to the absence of magnetic monopoles in nature

Classical estimate of dipole radiation

- Dipole radiation power,

$$P_{E1} \equiv \frac{dE_\gamma}{dt} = \frac{1}{3} E_\gamma^4 p^2$$

where, $p = qd$ is the electric dipole moment, and d is the mean distance of the dipole

- The decay width,

$$\Gamma_{E1} \equiv \frac{1}{E_\gamma} \frac{dE_\gamma}{dt} = \frac{1}{3} E_\gamma^3 p^2$$

Interaction Hamiltonian

- Fermi's golden rule stated, the decay width,

$$\Gamma_\gamma = 2\pi\big|\langle f|H_{\text{int}}|i\rangle\big|^2 g(E_f)$$

- Recall, the interaction Lagrangian is,

$$\mathcal{L}_{\text{int}} = J_\mu(x)A^\mu(x)$$

- It is then non-trivial to show that the interaction Hamiltonian is

$$H_{\text{int}} = \int d^3r\, \vec{J}(r)\cdot\vec{A}(r).$$

Note that this expression is exact and relativistic, as we will show immediately.

Electromagnetic radiation

Classical electromagnetic radiation are described by Maxwell equations,

$$
\begin{aligned}
\nabla \cdot \vec{E} &= \rho, \\
\nabla \times \vec{B} &= \vec{J} + \frac{\partial \vec{E}}{\partial t}, \\
\nabla \cdot \vec{B} &= 0, \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}.
\end{aligned}
$$

From the last two equations, we can define scalar and vector potentials $\phi(t, \vec{x})$ and $\vec{A}(t, \vec{x})$,

$$
\begin{aligned}
\vec{B} &= \nabla \times \vec{A} \\
\vec{E} &= -\nabla\phi - \frac{\partial \vec{A}}{\partial t}
\end{aligned}
$$

Classical electrodynamics in covariant form

- Introduce the 4-vector potential $A^\mu = (\phi, \vec{A})$ and 4-vector current $J^\mu = (\rho, \vec{J})$
- The field strength tensor,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

  encodes the electric field $\vec{E}^i = F^{i0}$ and the magnetic field $\vec{B}^i = -\frac{1}{2}\epsilon^{ijk}F^{jk}$.
- Gauss-Faraday law:

$$\partial_\sigma \widetilde{F}^{\alpha\beta} = 0, \qquad (\widetilde{F}^{\alpha\beta} = \varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta})$$

  This is also known as the Bianchi identity.
- Coulomb-Ampere law:

$$\partial_\sigma F^{\alpha\beta} = J^\beta$$

- Lagrangian density of the electromagnetic field,

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}$$

Coupling to matter

- If matter field represents collection of pointlike particles, the coupling the electromagnetic field to matter is the minimal coupling, namely,

$$\begin{aligned} \partial_\mu &\to D_\mu \equiv \partial_\mu - ieA_\mu, \\ \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{EM}} &\to \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{int}} \end{aligned}$$

- We rewrite the interaction part as

$$\mathcal{L}_{\text{int}} = J_\mu A^\mu$$

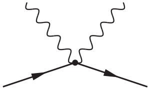
  to conform with the Maxwell equations.

- Note that J^μ may contain the vector field A^μ within it. For example, for the complex scalar field,

$$\mathcal{L} = \partial^\mu \varphi^\dagger \partial_\mu \varphi - m^2 |\varphi|^2$$

  there is an extra seagull term:

$$\mathcal{L}_{\text{sg}} = e^2 |\varphi|^2 A^\mu A_\mu$$



Canonical quantization

- Consider a system described by the Lagrangian,

$$L = L(q_i, \dot{q}_i)$$

  where, q_i are the generalized coordinates
- Conjugate momentum:

$$p^i = \frac{\partial L}{\partial \dot{q}_i}$$

- Hamiltonian is obtained from Legendre transformation,

$$H = p^i \dot{q}_i - L = H(q_i, p_i)$$

- Quantization: promote q_i, q_i to quantum operators and impose canonical commutation relations (CCRs):

$$[q_i, q_j] = [p_i, p_j] = 0, \quad [q_i, p_j] = i\delta_{ij}\hbar$$

(Naïve) quantization of the electromagnetic field

- The Lagrangian of the electromagnetic fields reads,

$$\begin{aligned}\mathcal{L}_{\rm EM} + \mathcal{L}_{\rm int} &= \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \rho\phi - \vec{J}\cdot\vec{A}, \\ &= \frac{1}{2}(\partial_t \vec{A})^2 - \frac{1}{2}\nabla_j A^i \nabla_j A^i + \frac{1}{2}\nabla_i A^j \nabla_j A^i + \partial_t \vec{A}\cdot\nabla\phi + \frac{1}{2}(\nabla\phi)^2 + \rho\phi - \vec{J}\cdot\vec{A}\end{aligned}$$

- Canonical momenta,

$$\Pi_i = \partial_t A_i, \quad \Pi_0 = 0$$

  Note that Π_0, the conjugate momentum of the electric potential $\phi = A^0$, vanishes. This is because its time derivative $\partial_t \phi$ is absent from the Lagrangian density. Therefore, the electric potential ϕ is non-dynamical, and could not be quantized.

- In principle, we can treat $\Pi_0 = 0$ as a constrain. Dirac proposed a method to quantize theories with constraints. Fortunately, we have an easy alternative.

Gauge symmetry and gauge fixing

- The underlying cause is that there exists a gauge redundancy (also called a gauge symmetry), i.e. the following gauge transformation,

$$\phi \to \phi' = \phi - \partial_t \Lambda, \quad \vec{A} \to \vec{A}' = \vec{A} + \nabla\Lambda$$

  where Λ is an arbitrary smooth function, does not change the e.m. fields $\vec{E}, \vec{B}$. This redundancy introduces unphysical d.o.f.'s.
- To eliminate the unphysical d.o.f.'s, we can adopt a particular gauge fixing condition. For radiation problems, a convenient choice is the Coulomb gauge (aka transverse gauge):

$$\nabla \cdot \vec{A} = 0$$

Table: Examples of gauge fixing conditions

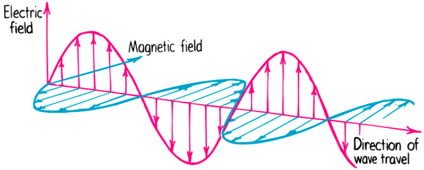
Weyl gauge	$A^0 = 0$	Lorenz gauge	$\partial_\mu A^\mu = 0$
light cone gauge	$A^0 + A^3 = 0$	Coulomb gauge	$\nabla \cdot \vec{A} = 0$
axial gauge	$A^3 = 0$	Poincaré gauge	$\vec{r} \times \vec{A} = 0$
R_ξ gauge		Fock–Schwinger gauge	$x \cdot A = 0$

Coulomb gauge

- In Coulomb gauge, the electric field $\vec{E} = \vec{E}_\parallel + \vec{E}_\perp$ can be split into a longitudinal part $\vec{E}_\parallel = -\nabla\phi$ and a transverse part $\vec{E}_\perp = -\partial_t \vec{A}$:

$$\nabla \cdot \vec{A} = 0 \quad \Rightarrow \quad \nabla \cdot \vec{E}_\perp = 0.$$

  The longitudinal field $\vec{E}_\parallel$ describes the static Coulomb field while the transverse field $\vec{E}_\perp$ describes the radiation field.
- For regions far away from charges, $\vec{E}_\parallel \to 0$ and we may adopt $\phi \to 0, \nabla \cdot \vec{A} = 0$, which is known as the radiation gauge.



Lagrangian in Coulomb gauge

$$\mathcal{L}_{\rm EM} + \mathcal{L}_{\rm int} = \frac{1}{2}\partial_t \vec{A}^2 - \frac{1}{2}\nabla_j A^i \nabla_j A^i - \vec{J}\cdot\vec{A} + \frac{1}{2}(\nabla\phi)^2 + \rho\phi = \mathcal{L}_{\rm R} + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm Coul}$$

where, $\mathcal{L}_{\rm EM} = \mathcal{L}_{\rm R} + \mathcal{L}_{\rm Coul}$. Note that now, ϕ and $\vec{A}$ decouple.

- ϕ is non-dynamical. It is simply a non-propagating classical field and can be obtained from Poisson equation,

  $$\nabla^2\phi = -\rho \quad \Rightarrow \quad \phi(t,\vec{r}) = \int d^3r' \frac{\rho(t,\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}$$

  The instantaneous nature of the Coulomb potential, at first sight, appears to violate causality. Note that this potential is not a physical observable.
- After eliminating ϕ, the Lagrangian becomes,

$$\mathcal{L}_{\rm R} + \mathcal{L}_{\rm int} + \mathcal{L}_{\rm Coul} = \frac{1}{2}\partial_t \vec{A}^2 - \frac{1}{2}\nabla_j A^i \nabla_j A^i - \vec{J}\cdot\vec{A} - \frac{1}{2}\int d^3r' \frac{\rho(\vec{r},t)\rho(\vec{r}\,',t)}{4\pi|\vec{r}-\vec{r}\,'|}$$

Quantization of the radiation fields in Coulomb gauge

$$\mathcal{L}_\mathrm{R} + \mathcal{L}_\mathrm{int} = \frac{1}{2}\partial_t \vec{A}^2 - \frac{1}{2}\nabla_j A^i \nabla_j A^i - \vec{J}\cdot\vec{A}$$

- The radiation field can be quantized by promoting $\vec{A}$ and its conjugate momentum $\vec{\Pi} = -\vec{E}_\perp$ to quantum operators and imposing canonical commutation relations (CCRs),

$$[A^i(t,\vec{r}), A^j(t,\vec{r}\,')] = [\Pi^i(t,\vec{r}), \Pi^j(t,\vec{r}\,')] = 0, \quad [A^i(t,\vec{r}), \Pi^j(t,\vec{r}\,')] = i\delta_\perp^{ij}\delta^3(r-r'),$$

where, $\delta_\perp^{ij}$ is the transverse Kronecker-δ defined as,

$$\delta_\perp^{ij}\delta^3(r-r') = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\Big(\delta^{ij} - \frac{k^i k^j}{\vec{k}^2}\Big) \equiv \Big(\delta^{ij} - \frac{\nabla^i\nabla^j}{\nabla^2}\Big)\delta^3(r-r')$$

$$= \delta^{ij}\delta^3(r-r') + \frac{1}{4\pi}\nabla^i\nabla^j\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{2}{3}\delta^{ij}\delta(r-r') - \frac{1}{4\pi}\Big(\frac{\delta^{ij}}{r^3} - 3\frac{r^i r^j}{r^5}\Big)$$

This factor appears to ensure the Coulomb gauge condition.

Hamiltonian

The Hamiltonian can be obtained from Legendre transformation,

$$
\begin{aligned}
H_R + H_{\text{int}} &= \int d^3r \left\{ \vec{\Pi} \cdot \partial_t \vec{A} - \mathcal{L} \right\} \\
&= \int d^3r \left\{ \frac{1}{2} \partial_t \vec{A}^2 + \frac{1}{2} \nabla_j A^i \nabla_j A^i + \vec{J} \cdot \vec{A} \right\} \\
&= \int d^3r \left\{ \frac{1}{2} (\vec{E}_T^2 + \vec{B}^2) + \vec{J} \cdot \vec{A} \right\}
\end{aligned}
$$

The full Hamiltonian reads,

$$
\begin{aligned}
H &= H_{\text{matter}} + H_{\text{EM}} + H_{\text{int}} \\
&= H_{\text{matter}} + H_{\text{Coul}} + H_{\text{R}} + H_{\text{int}} \\
&= H_{\text{matter}} + \frac{1}{2} \int d^3r \int d^3r' \frac{\rho(\vec{r}, t)\rho(\vec{r}', t)}{4\pi |\vec{r} - \vec{r}'|} + \frac{1}{2} \int d^3r \, (\vec{E}_\perp^2 + \vec{B}^2) + \int d^3r \, \vec{J} \cdot \vec{A}
\end{aligned}
$$

Spinor electrodynamics

- Matter part,

$$\mathcal{L}_{\text{spinor}} = \overline{\psi}(\gamma_\mu i\partial^\mu - m)\psi$$
$$\Rightarrow \quad H_{\text{spinor}} = \int d^3r \left\{\overline{\psi}(-i\vec{\gamma}\cdot\nabla + m)\psi\right.$$

where, ψ is a 4-by-1 spinor field, and $\overline{\psi} = \psi^\dagger\gamma^0$.

- Electromagnetic current

$$J^\mu = e\overline{\psi}\gamma^\mu\psi$$

- Hamiltonian

$$H = \int d^3r \left\{\overline{\psi}(-i\vec{\gamma}\cdot\nabla + m)\psi + \frac{1}{2}(\vec{E}_\perp^2 + \vec{B}^2)\right.$$
$$\left. + \frac{e^2}{2}\int d^3r' \frac{\left|\psi(\vec{r},t)\right|^2\left|\psi(\vec{r}',t)\right|^2}{4\pi|\vec{r}-\vec{r}'|} + e\overline{\psi}\vec{\gamma}\psi\cdot\vec{A}\right\}$$

Scalar electrodynamics

- Matter part,

$$\mathcal{L}_{\text{scalar}} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 |\varphi|^2$$

$$\Rightarrow \quad H_{\text{scalar}} = \int d^3 r \left\{ |\partial_t \varphi|^2 + |\nabla \varphi|^2 + m^2 |\varphi|^2 \right\}$$

where, φ is a complex scalar field.

- Electromagnetic current ($f \overleftrightarrow{\partial} g \equiv f \partial g - \partial f g$)

$$J^\mu = ie \varphi^\dagger \overleftrightarrow{\partial}^\mu \varphi + 2e^2 |\varphi|^2 A^\mu$$

- Hamiltonian

$$H = \int d^3 r \Big\{ |\partial_t \varphi|^2 + |\nabla \varphi|^2 + m^2 |\varphi|^2 + \frac{1}{2} (\vec{E}_\perp^2 + \vec{B}^2) + \frac{e^2}{2} \int d^3 r' \frac{\rho(\vec{r}, t) \rho(\vec{r}', t)}{4\pi |\vec{r} - \vec{r}'|} + ei \varphi^\dagger \overleftrightarrow{\nabla} \varphi \cdot \vec{A} + e^2 \vec{A}^2 |\varphi|^2 - ei \phi \varphi^\dagger \overleftrightarrow{\partial}_t \varphi \Big\}$$

Electrodynamics of point-like scalars

- Matter part,

$$L_{\rm part} = \sum_i \frac{1}{2} m_i \vec{v}_i^2 - U(\vec{r}_i) - \frac{1}{2}\sum_{i,j} V(\vec{r}_i - \vec{r}_j)$$

- Electromagnetic current

$$\rho(\vec{r}) = \sum_i e_i \delta^3(r - r_i), \quad \vec{J}(\vec{r}) = \sum_i e_i \vec{v}_i \delta^3(r - r_i)$$

- Hamiltonian

$$H = \sum_i \frac{(\vec{p}_i - e_i \vec{A}_i)^2}{2m_i} + U_i(\vec{r}_i) + \frac{1}{2}\sum_{i,j} V(\vec{r}_i - \vec{r}_j) + \int d^3 r \, \frac{1}{2}(\vec{E}_\perp^2 + \vec{B}^2) + \frac{1}{2}\sum_{i,j} \frac{e_i e_j}{4\pi|\vec{r}_i - \vec{r}_j|}$$

where, $\vec{A}_i = \vec{A}(\vec{r}_i)$ and the canonical momentum $\vec{p}_i = m_i \vec{v}_i + e_i \vec{A}_i$. This Hamiltonian can be generated from the minimal coupling $\vec{p}_i \to \vec{p}_i - e_i \vec{A}$

Electrodynamics of point-like spinors

- Matter part,

$$L_{\text{part}} = \sum_i \frac{1}{2} m_i \vec{v}_i^2 - U_i(\vec{r}_i) - \frac{1}{2}\sum_{i,j} V_{ij}(\vec{r}_i - \vec{r}_j)$$

- Electromagnetic current

$$\rho(\vec{r}) = \sum_i e_i \delta^3(r - r_i), \quad \vec{J}(\vec{r}) = \sum_i (e_i \vec{v}_i + \mu_i \nabla \times \vec{s}_i)\delta^3(r - r_i)$$

- Hamiltonian

$$H = \sum_i \frac{(\vec{p}_i - e_i \vec{A}_i)^2}{2m_i} - \mu_i \vec{s}_i \cdot \vec{B}_i + U_i(\vec{r}_i) + \frac{1}{2}\sum_{i,j} V_{ij}(\vec{r}_i - \vec{r}_j) + \int d^3 r\, \frac{1}{2}(\vec{E}_\perp^2 + \vec{B}^2) + \frac{1}{2}\sum_{i,j} \frac{e_i e_j}{4\pi |\vec{r}_i - \vec{r}_j|}$$

where, $\vec{A}_i = \vec{A}(\vec{r}_i)$, and $\vec{B}_i = \vec{B}(\vec{r}_i)$ and the canonical momentum $\vec{p}_i = m_i \vec{v}_i + e_i \vec{A}_i$.

Photon

- In gamma decay, what we detected in the detectors are the light quanta, photons. Photons are the eigenstate the QED Hamiltonian operator, away from the source $\vec{J} = 0$.
- Recall in classical theory, the ``free fields" are coupled harmonic oscillators. And the eigenstates are the normal modes $\xi(t, \vec{r}) \propto e^{i(\omega t - \vec{k}\cdot\vec{r})}$.
- In classical electromagnetism, the vector potential $\vec{A}$ in the region far away from charges satisfies

$$-\nabla^2\phi - \frac{\partial}{\partial t}\nabla \cdot \vec{A} = 0$$
$$\frac{\partial^2 \vec{A}}{\partial^2 t} - \nabla^2\vec{A} + \nabla(\nabla \cdot \vec{A}) + \frac{\partial}{\partial t}\nabla\phi = 0$$

- Imposing the radiation gauge (Coulomb gauge $\nabla \cdot \vec{A} = 0$ plus $\phi = 0$), the above equation becomes

$$\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2\vec{A} = 0. \qquad (\nabla \cdot \vec{A} = 0)$$

Wave equation

- We consider the normal-mode solution $\vec{A}(t,\vec{r}) \propto e^{i(\omega t - \vec{k}\cdot\vec{r})}$ to the wave equation. For definiteness, we restrict the radiation field within a finite box with volume $V = L^3$, and the momentum vector $\vec{k}$ are,

$$\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

where, $n_i = 0, 1, 2, \cdots$

- The most general solution reads,

$$\vec{A}(t,\vec{r}) = \sum_{\alpha=1,2} \frac{1}{V} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \Big\{ \vec{\epsilon}_\alpha(\vec{k}) a_\alpha(\vec{k}) e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \vec{\epsilon}^*_\alpha(\vec{k}) a^*_\alpha(\vec{k}) e^{+i\omega_k t - i\vec{k}\cdot\vec{r}} \Big\}$$

where, ω_k satisfies the dispersion relation $\omega_k = |\vec{k}|$.

Radiation energy

- Polarization vectors $\vec{\epsilon}_{1,2}(\vec{k})$ are two orthogonal transverse unit vectors as required by the Coulomb gauge condition:
$$\vec{\epsilon}_\alpha \cdot \vec{\epsilon}^*_\beta = \delta_{\alpha\beta}, \quad \vec{k}\cdot\vec{\epsilon}_\alpha = 0.$$
- The simplest choice of $\vec{\epsilon}_\alpha$ are linearly polarized modes satisfying,
$$\vec{\epsilon}_1 \times \vec{\epsilon}_2 = \hat{k}, \quad \vec{\epsilon}_2 \times \hat{k} = \vec{\epsilon}_1, \quad \hat{k} \times \vec{\epsilon}_2 = \vec{\epsilon}_2$$
- $a_\alpha(\vec{k})$ are arbitrary coefficients -- difference choice of $a_\alpha(\vec{k})$ gives different superposition of normal modes.
- One can show that the Hamiltonian can be rewritten as,
$$\begin{aligned} H_\mathrm{R} &= \frac{1}{2}\int d^3r \left(E_\perp^2 + \vec{B}^2\right) \\ &= \frac{1}{V}\sum_{\alpha,\vec{k}} \omega_k |a_\alpha(\vec{k})|^2 \end{aligned}$$

Second quantization of the radiation field

- Passing from the classical theory to quantum theory, we can expand the field operator A in terms of the normal modes, except treating the mode coefficient $a_\alpha(\vec{k})$ as a quantum operator,

$$\underline{\vec{A}}(t,\vec{r}) = \sum_{\alpha=1,2}\frac{1}{V}\sum_{\vec{k}}\frac{1}{\sqrt{2\omega_k}}\Big\{\vec{\epsilon}_\alpha(\vec{k})\underline{a}_\alpha(\vec{k})e^{-i\omega_k t+i\vec{k}\cdot\vec{r}} + \vec{\epsilon}^*_\alpha(\vec{k})\underline{a}^\dagger_\alpha(\vec{k})e^{+i\omega_k t-i\vec{k}\cdot\vec{r}}\Big\}$$

  where, ω_k satisfies the dispersion relation $\omega_k = |\vec{k}|$. We have used underscores to emphasize the quantum operators.

- The CCRs of the field operators imply the following commutation relations for the operators a and $a^\dagger$,

$$\big[a_s(\vec{k}), a_r(\vec{p})\big] = \big[a_s^\dagger(\vec{k}), a_r^\dagger(\vec{p})\big] = 0,$$
$$\big[a_s(\vec{k}), a_r^\dagger(\vec{p})\big] = \delta_{sr}\delta_{\vec{k}\vec{p}}$$

Second quantization of the radiation field

- The Hamiltonian operator can be written as,

$$H_R = \frac{1}{2V}\sum_{\alpha,\vec{k}} \omega_k \Big[a^\dagger_\alpha(\vec{k})a_\alpha(\vec{k}) + a_\alpha(\vec{k})a^\dagger_\alpha(\vec{k})\Big],$$
$$= \frac{1}{V}\sum_{\alpha,\vec{k}} \omega_k \Big[a^\dagger_\alpha(\vec{k})a_\alpha(\vec{k}) + \frac{1}{2}\Big]$$

- The momentum operator,

$$\vec{P} \equiv \int d^3r\, \vec{E}_\perp \times \vec{B}$$
$$= \frac{1}{V}\sum_{s,\vec{k}} \hbar\vec{k}\Big[a^\dagger_\alpha(\vec{k})a_\alpha(\vec{k}) + \frac{1}{2}\Big]$$

Second quantization of the radiation field

- The eigenstate of the radiation Hamiltonian is readily available:

$$|n;\vec{k},s\rangle = \frac{1}{\sqrt{n!}}\left(a_s^\dagger(\vec{k})\right)^n|0\rangle$$

where, $|0\rangle$ is the ground state of the Hamiltonian and a annihilates it: $a_s(\vec{k})|0\rangle = 0$.

$$\begin{aligned} a_s^\dagger(\vec{k})|n;\vec{k},s\rangle &= \sqrt{n+1}|n+1;\vec{k},s\rangle, \\ a_s(\vec{k})|n;\vec{k},s\rangle &= \sqrt{n}|n-1;\vec{k},s\rangle, \\ \Rightarrow \quad a_s^\dagger(\vec{k})a_s(\vec{k})|n;s,\vec{k}\rangle &= n|n;\vec{k},s\rangle, \end{aligned}$$

- Eigen-energy and momentum:

$$H_R|n;\vec{k},s\rangle = n\hbar\omega_k + E_0,$$

$$\vec{P}_R|n;\vec{k},s\rangle = n\hbar\vec{k} + \cancelto{0}{\vec{P}_0}$$

where, $E_0 = \frac{1}{2}\sum_{s,\vec{k}}\hbar\omega_k$, $\vec{P}_0 = \frac{1}{2}\sum_{s,\vec{k}}\hbar\vec{k} = 0$.

Second quantization of the radiation field

- Physically, $|n; \vec{k}, s\rangle$ represents n-photon state with identical polarization s and momentum $\vec{k}$. Factor $n!$ accounts for identical particle statistics. In particular, $|\vec{k}, s\rangle \equiv |1; \vec{k}, s\rangle = a_s^\dagger(\vec{k})|0\rangle$ represents the single-photon state with polarization s and momentum $\vec{k}$
- The the ground state $|0\rangle$ represents zero-photon state, i.e. the vacuum
- E_0 is the zero-point energy similar to quantum harmonic oscillators. In the continuum limit, $E_0 \to \infty$. However, it does not have any observable effects. We will simply ignore (aka. ``renormalize") it.

Second quantization of the radiation field

- Single-photon state is obtained by acting the creation operator $a^\dagger$ on the vacuum:

$$
\begin{aligned}
|\vec{k}, s\rangle &= a_s^\dagger(\vec{k})|0\rangle, \\
&= -i\vec{\epsilon}_\alpha(\vec{k}) \cdot \int d^3 r e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} \overleftrightarrow{\partial}_t \vec{A}(t, \vec{r})|0\rangle
\end{aligned}
$$

- In general, to create a photon with a wavepacket $f_s(\vec{k}) = \langle s, \vec{k}|f\rangle$,

$$
|f\rangle = \frac{1}{V}\sum_{s,\vec{k}} f_s(\vec{k}) a_s^\dagger(\vec{k})|0\rangle \equiv a_f^\dagger|0\rangle
$$

Homework

From the free field expansion of $\vec{A}$ and the definitions of H_R and $\vec{P}_R$,

$$H_{\mathrm{R}} = \frac{1}{2} \int d^3 r \left(E_\perp^2 + \vec{B}^2 \right)$$

$$\vec{P}_{\mathrm{R}} = \int d^3 r \vec{E}_\perp \times \vec{B}$$

derive the second quantization form of the radiation Hamiltonian and radiation momentum,

$$H_R = \frac{1}{V} \sum_{\alpha, \vec{k}} \omega_k \Big[a_\alpha^\dagger(\vec{k}) a_\alpha(\vec{k}) + \frac{1}{2} \Big],$$

$$\vec{P}_R = \frac{1}{V} \sum_{s, \vec{k}} \hbar \vec{k} \Big[a_\alpha^\dagger(\vec{k}) a_\alpha(\vec{k}) + \frac{1}{2} \Big]$$

Decay width

- Fermi's golden rule stated, the decay width,

$$\Gamma_\gamma = 2\pi \big|\langle f|H_{\rm int}|i\rangle\big|^2 g(E_f)$$

where,

$$H_{\rm int} = \int d^3r\, \vec{J}(r)\cdot\vec{A}(r).$$

- Initial state is a nuclear state $|I\rangle$with definite angular momentum J_i; final state is a nuclear state $|F\rangle$ with definite angular momentum J_f plus a photon $|\gamma\rangle = a_s^\dagger(\vec{k})|0\rangle$.
- If we adopt momentum basis,

$$g(E)dE = V\frac{d^3k}{(2\pi)^3}$$

and

$$\begin{aligned}\langle f|H_{\rm int}|i\rangle &= \int d^3r\langle F|a_s(\vec{k})\vec{J}(r)\cdot\vec{A}(r)|I\rangle,\\ &= \sum_i \frac{q_i}{m_i}\int d^3r\langle F|a_s(\vec{k})\vec{p}_i\cdot\vec{A}(r_i)|I\rangle + \mu_i\int d^3r\langle F|a_s(\vec{k})\vec{s}_i\cdot\vec{B}(r_i)|I\rangle\end{aligned}$$

Decay width

$$\nabla^2 \vec{A} - \partial_t^2 \vec{A} = 0$$

- Plane waves solution: modes with definite energy ω_k, polarization s and momentum $\vec{k}$

$$\underline{\vec{A}}(t, \vec{r}) = \sum_{\alpha=1,2} \frac{1}{V} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \Big\{ \vec{\epsilon}_\alpha(\vec{k}) \underline{a}_\alpha(\vec{k}) e^{-i\omega_k t + i\vec{k}\cdot\vec{r}} + \vec{\epsilon}^*_\alpha(\vec{k}) \underline{a}^\dagger_\alpha(\vec{k}) e^{+i\omega_k t - i\vec{k}\cdot\vec{r}} \Big\}$$

- Spherical waves: modes with definite energy $\omega_k = k$, angular momentum $\vec{L}$

$$\vec{A}^E_{LM} = -i \frac{1}{k\sqrt{L(L+1)}} (\nabla \times (\vec{r} \times \nabla)) j_L(kr) Y_{LM}(\hat{r}),$$

$$\vec{A}^M_{LM} = -i \frac{1}{\sqrt{L(L+1)}} (\vec{r} \times \nabla) j_L(kr) Y_{LM}(\hat{r}),$$

- Parities:

$$P Y_{lm}(\hat{r}) P = (-1)^l Y(\hat{r}) \quad \Rightarrow \pi_E = (-1)^L, \quad \pi_M (-1)^{L+1}$$

Transition

We calculate the transition amplitude of an atom or a nucleus between two energy states by absorption of emission of one photon,

$$a \to b + \gamma$$

through the coupling $\vec{A}_i \cdot \vec{p}_i$. The initial/final states for atom or nucleus and photons are as follows

$$\begin{aligned} \text{initial} &= |a\rangle \left|n(s,\vec{k})\right\rangle \\ \text{final} &= |b\rangle \left|n(s,\vec{k}) \pm 1\right\rangle \end{aligned} \tag{3}$$

The transition amplitude is then

$$\begin{aligned} M_{fi} &= \left\langle b, n(s,\vec{k}) \pm 1\right| H_I \left|a, n(s,\vec{k})\right\rangle \\ &= i \sum_i \frac{q_i}{m_i} \left\langle b, n(s,\vec{k}) \pm 1\right| \vec{A}_i \cdot \vec{\nabla}_i \left|a, n(s,\vec{k})\right\rangle \end{aligned} \tag{4}$$

Photon emission

First we consider the photon emission case. Inserting the qunatization form of $\vec{A}_i$ into the above equation, we obtain

$$\begin{aligned} M_{fi} &= i \sum_{\vec{k}_1, s_1} \left(\frac{2\pi}{V\omega_{k1}}\right)^{1/2} \left\langle n(s,\vec{k})+1 \middle| a^\dagger(s_1,\vec{k}_1) \middle| n(s,\vec{k}) \right\rangle \\ & \quad \times \boldsymbol{\epsilon}(s_1,\vec{k}_1) \cdot \langle b| \sum_i \frac{q_i}{m_i} e^{i\omega_{k1}t - i\vec{k}_1\cdot\vec{x}_i} \vec{\nabla}_i |a\rangle \\ &= i \left(\frac{2\pi}{V\omega_k}\right)^{1/2} \sqrt{n(s,\vec{k})+1} \\ & \quad \times \boldsymbol{\epsilon}(s,\vec{k}) \cdot \langle b| \sum_i \frac{q_i}{m_i} e^{i\omega_k t - i\vec{k}\cdot\vec{x}_i} \vec{\nabla}_i |a\rangle \end{aligned} \tag{5}$$

Photon emission

Following the Fermi golden rule, we obtain the differential transition rate,

$$
\begin{aligned}
d\lambda_{a\to b+\gamma} &= 2\pi\delta(E_b+\omega_k-E_a)\left|M_{fi}\right|^2\frac{Vd^3k}{(2\pi)^3} \\
&= \frac{1}{2\pi}d\Omega_k dk\delta(E_b+\omega_k-E_a)\omega_k\sum_{s=\pm}[n(s,\vec{k})+1] \\
&\quad\times\left|\boldsymbol{\epsilon}(s,\vec{k})\cdot\langle b|\sum_i\frac{q_i}{m_i}e^{-i\vec{k}\cdot\vec{x}_i}\vec{\nabla}_i\,|a\rangle\right|^2
\end{aligned}
\tag{6}
$$

Photon emission rate

After integrating over the photon energy, we arrive at

$$\begin{aligned}\lambda_{a\to b+\gamma} &= \frac{\omega_k}{2\pi}[\bar{n}(\vec{k})+1]\sum_{s=\pm 1}\int d\Omega_k \\ &\times\left|\boldsymbol{\epsilon}(s,\vec{k})\cdot\langle b|\sum_i \frac{q_i}{m_i}e^{-i\vec{k}\cdot\vec{x}_i}\vec{\nabla}_i\,|a\rangle\right|^2\end{aligned} \tag{7}$$

where we have used $\omega_k = E_a - E_b = |\vec{k}|$ and $\bar{n}(\vec{k}) = (1/2)\sum_{s=\pm} n(s,\vec{k})$.

Photon absorption

For photon absorption $b + \gamma \to a$, we have similar formula

$$\begin{aligned} M_{fi} &= i \sum_{\vec{k}_1, s_1} \left(\frac{2\pi}{V\omega_{k1}}\right)^{1/2} \left\langle n(s,\vec{k}) - 1 \middle| a(s_1,\vec{k}_1) \middle| n(s,\vec{k}) \right\rangle \\ &\quad \times \boldsymbol{\epsilon}(s_1,\vec{k}_1) \cdot \langle a| \sum_i \frac{q_i}{m_i} e^{-i\omega_{k1}t + i\vec{k}_1\cdot\vec{x}_i} \vec{\nabla}_i |b\rangle \\ &= i \left(\frac{2\pi}{V\omega_k}\right)^{1/2} \sqrt{n(s,\vec{k})} \\ &\quad \times \boldsymbol{\epsilon}(s,\vec{k}) \cdot \langle a| \sum_i \frac{q_i}{m_i} e^{-i\omega_k t + i\vec{k}\cdot\vec{x}_i} \vec{\nabla}_i |b\rangle \end{aligned} \tag{8}$$

Photon absorption

The differential transition rate reads,

$$\begin{aligned} d\lambda_{b+\gamma\to a} &= 2\pi\delta(E_b+\omega_k-E_a)\left|M_{fi}\right|^2\frac{Vd^3k}{(2\pi)^3} \\ &= \frac{1}{2\pi}d\Omega_k dk\delta(E_b+\omega_k-E_a)\omega_k\bar{n}(\vec{k}) \\ &\times\sum_{s=\pm1}\left|\boldsymbol{\epsilon}(s,\vec{k})\cdot\langle a|\sum_i\frac{q_i}{m_i}e^{i\vec{k}\cdot\vec{x}_i}\vec{\nabla}_i\,|b\rangle\right|^2 \end{aligned} \tag{9}$$

Photon absorption rate

The rate is then evaluated as

$$\lambda_{b+\gamma\to a} = \frac{\omega_k}{2\pi}\bar{n}(\vec{k}) \sum_{s=\pm 1} \int d\Omega_k \times \left| \boldsymbol{\epsilon}(s,\vec{k}) \cdot \langle a| \sum_i \frac{q_i}{m_i} e^{i\vec{k}\cdot\vec{x}_i} \vec{\nabla}_i |b\rangle \right|^2 \tag{10}$$

Relation between emission and absorption

The relation between emission and absorption for transition matrix element

$$\begin{aligned} & \left[\langle \mathrm{b} | \sum_\alpha \frac{q_\alpha}{m_\alpha} e^{-i\vec{k}\cdot\vec{x}_\alpha} \boldsymbol{\nabla}_\alpha | \mathrm{a} \rangle \right]^* \\ = \quad & \langle \mathrm{a} | \sum_\alpha \frac{q_\alpha}{m_\alpha} e^{i\vec{k}\cdot\vec{x}_\alpha} \boldsymbol{\nabla}^\dagger_\alpha | \mathrm{b} \rangle \\ = \quad & -\langle \mathrm{a} | \sum_\alpha \frac{q_\alpha}{m_\alpha} e^{i\vec{k}\cdot\vec{x}_\alpha} \boldsymbol{\nabla}_\alpha | \mathrm{b} \rangle \end{aligned} \tag{11}$$

Detailed balance

From Eq. (17) we further obtain

$$\frac{\lambda_{a\to b+\gamma}}{\lambda_{b+\gamma\to a}} = \frac{\bar{n}(\vec{k})+1}{\bar{n}(\vec{k})} \tag{12}$$

If we assume detailed balance,

$$N(a)\lambda_{a\to b} = N(b)\lambda_{b\to a} \tag{13}$$

then we have $N(a) \propto e^{-E_a/T}$ and $N(b) \propto e^{-E_b/T}$,

$$\frac{N(a)}{N(b)} = \frac{\bar{n}(\vec{k})}{\bar{n}(\vec{k})+1} = e^{-\omega_k/T}$$

$$\bar{n}(\vec{k}) = \frac{1}{e^{\omega_k/T}-1} \tag{14}$$

which is Bose-Einstein distribution.

Dipole transition

Now we look at the simplest case, the electric dipole radiation at the long wave length limit. At this limit the wavelength of radiation is much larger than the size of the atom or the nucleus, so we can approximate $\vec{k}\cdot\vec{x} \approx 0$ or $e^{-i\vec{k}\cdot\vec{x}} \approx 1$. Then the matrix element in Eq. (7) can be put into the form,

$$\begin{aligned} \langle b| \sum_i \frac{q_i}{m_i} \vec{\nabla}_i |a\rangle &= i \langle b| \sum_i q_i \frac{d\vec{x}_i}{dt} |a\rangle \\ &= \langle b| \sum_i q_i \left[\vec{x}_i, H\right] |a\rangle \\ &= \omega_k \langle b| \sum_i q_i \vec{x}_i |a\rangle \equiv \omega_k \vec{D}_{ba} \end{aligned} \tag{15}$$

where we have defined the electric dipole moment $\vec{D}_{ba} = \langle b| \sum_i q_i \vec{x}_i |a\rangle$.

Dipole transition

Here we have used

$$\vec{\nabla} \quad = \quad i\vec{p} = im\frac{d\vec{x}}{dt} = m[\vec{x}, H] \tag{16}$$

Then the photon emission/absorption rate in Eqs. (7,10) becomes

$$\begin{aligned} \lambda_{a\to b+\gamma} &= \frac{\omega_k^3}{2\pi}[\bar{n}(\vec{k})+1]\sum_{s=\pm 1}\int d\Omega_k \left|\boldsymbol{\epsilon}(s,\vec{k})\cdot\vec{D}_{ba}\right|^2 \\ \lambda_{b+\gamma\to a} &= \frac{\omega_k^3}{2\pi}\bar{n}(\vec{k})\sum_{s=\pm 1}\int d\Omega_k \left|\boldsymbol{\epsilon}(s,\vec{k})\cdot\vec{D}_{ab}\right|^2 \end{aligned} \tag{17}$$

Dipole transition

We observe that $\vec{D}_{ba}(a \to b+\gamma)$ and $\vec{D}_{ab}(b+\gamma \to a)$ are related

$$\begin{aligned} \vec{D}_{ab}(b+\gamma \to a) &= \sum_i q_i \int d^3x_i e^{i\vec{k}\cdot\vec{x}_i} \langle a|\, \vec{x}_i \,|b\rangle \\ &= (\sum_i q_i \int d^3x_i e^{-i\vec{k}\cdot\vec{x}_i} \langle b|\, \vec{x}_i \,|a\rangle)^* \\ &= \vec{D}^*_{ba}(a \to b+\gamma) \end{aligned} \tag{18}$$

Note that we have $|\vec{D}_{fi}|^2 = |\vec{D}^*_{fi}|^2$.

Dipole transition

If there is no other photons in the environment, we can set $\bar{n}(\vec{k}) = 0(1)$ for the process $a \to b + \gamma$ $(b + \gamma \to a)$, the above rates become

$$\lambda_{a\to b+\gamma} \quad = \quad \lambda_{b+\gamma\to a} = \frac{\omega_k^3}{2\pi} \sum_{s=\pm 1} \int d\Omega_k \left| \boldsymbol{\epsilon}(s, \vec{k}) \cdot \vec{D}_{ba} \right|^2 \tag{19}$$

The power of radiation is given by,

$$I_\omega = \lambda_{a\to b+\gamma} \omega_k = \frac{\omega_k^4}{2\pi} \sum_{s=\pm 1} \int d\Omega_k \left| \boldsymbol{\epsilon}(s, \vec{k}) \cdot \vec{D}_{ba} \right|^2 \tag{20}$$

Dipole transition

Suppose the photon is emitted along the z-axis ($\vec{k}$ is along the z-axis) and $\vec{D}_{ba}$ is in the plane of $\boldsymbol{\epsilon}(1, \vec{k})$ and $\vec{k}$, so $\vec{D}_{ba}$ is perpendicular to $\boldsymbol{\epsilon}(2, \vec{k})$. Then the above integral of matrix element becomes

$$\begin{aligned} \sum_{s=\pm 1} \int d\Omega_k \left|\boldsymbol{\epsilon}(s, \vec{k}) \cdot \vec{D}_{ba}\right|^2 &= \int d\Omega_k \left|\boldsymbol{\epsilon}(1, \vec{k}) \cdot \vec{D}_{ba}\right|^2 \\ &= \int d\Omega_k \left|\vec{D}_{ba}\right|^2 \sin^2\theta = \frac{8\pi}{3}\left|\vec{D}_{ba}\right|^2 \end{aligned} \tag{21}$$

where the angle between $\vec{D}_{ba}$ and $\vec{k}$ is θ. So the power of radiation turns out to be

$$I_\omega = \frac{4}{3}\omega_k^4 \left|\vec{D}_{ba}\right|^2 \tag{22}$$

The above power formula can be matched to the classical electrodynamics.

Quadrupole transition

We now look at the electric quadrupole radiation by considering the linear order term $\sim \vec{k} \cdot \vec{x}_j$ in the phase factor $e^{-i\vec{k}\cdot\vec{x}_j}$ in Eq. (7). For simplicity of notation, we suppress the subscript j which labels the charged particles, and we have

$$\frac{q}{m}(\vec{k} \cdot \vec{x})(\boldsymbol{\epsilon} \cdot \vec{p}) \quad = \quad C_+ + C_- \tag{23}$$

where $C_\pm$ is defined by

$$C_\pm \quad \equiv \quad \frac{q}{2m}[(\vec{k} \cdot \vec{x})(\boldsymbol{\epsilon} \cdot \vec{p}) \pm (\vec{k} \cdot \vec{p})(\boldsymbol{\epsilon} \cdot \vec{x})] \tag{24}$$

Quadrupole transition

We focus on C_+ which corresponds to quadrupole radiation,

$$
\begin{aligned}
C_+ &= \frac{q}{2m}[(\vec{k}\cdot\vec{x})(\boldsymbol{\epsilon}\cdot\vec{p}) + (\vec{k}\cdot\vec{p})(\boldsymbol{\epsilon}\cdot\vec{x})] \\
&= \frac{q}{2}[(\vec{k}\cdot\vec{x})\boldsymbol{\epsilon}\cdot\frac{d\vec{x}}{dt} + (\vec{k}\cdot\frac{d\vec{x}}{dt})(\boldsymbol{\epsilon}\cdot\vec{x})] \\
&= \frac{1}{6}q\frac{d}{dt}[3(\vec{k}\cdot\vec{x})(\boldsymbol{\epsilon}\cdot\vec{x}) - \vec{x}^2(\boldsymbol{\epsilon}\cdot\vec{k})] \\
&= \frac{1}{6}\epsilon_i k_j \frac{d}{dt}Q_{ij} = -i\frac{1}{6}\epsilon_i k_j [Q_{ij}, H_0]
\end{aligned}
\tag{25}
$$

where we used $\boldsymbol{\epsilon}\cdot\vec{k} = 0$ and we have defined $Q_{ij} = q(3x_i x_j - x^2\delta_{ij})$. We have also used the Schroedinger equation for the operator O, $i\dot{O} = [Q, H_0]$.

Quadrupole transition

Substituting Eq. (25) into Eq. (7) and setting $\bar{n}(\vec{k}) = 0$, we obtain the emission rate

$$\begin{aligned} \lambda_{a\to b+\gamma} &= \frac{\omega_k}{72\pi} \sum_{s=\pm 1} \int d\Omega_k \left| \epsilon_i(\vec{k}, s) k_j \langle b| \sum_n [Q_{ij}(n), H_0] |a\rangle \right|^2 \\ &= \frac{\omega_k^3}{72\pi} \sum_{s=\pm 1} \int d\Omega_k \left| \epsilon_i(\vec{k}, s) k_j \langle b| \sum_n Q_{ij}(n) |a\rangle \right|^2 \end{aligned} \qquad (26)$$

This gives the electric quadrupole transition rate. The absorption rate can be similarly derived from Eq. (10).

Magnetic dipole transition

Let us look at the C_- term in Eq. (24),

$$\begin{aligned} C_- &= \frac{q}{2m}[(\vec{k}\cdot\vec{x})(\boldsymbol{\epsilon}\cdot\vec{p}) - (\vec{k}\cdot\vec{p})(\boldsymbol{\epsilon}\cdot\vec{x})] = \frac{q}{2m}(\vec{k}\times\boldsymbol{\epsilon})\cdot(\vec{x}\times\vec{p}) \\ &= (\vec{k}\times\boldsymbol{\epsilon})\cdot\boldsymbol{\mu}_L \end{aligned}$$

Note that $\vec{k}\times\boldsymbol{\epsilon}$ comes from $\vec{B} = \nabla\times\vec{A}$, so $C_- \sim -\boldsymbol{\mu}_L\cdot\vec{B}$ is from the interaction of magnetic moment in magnetic field. So this term corresponds to the magnetic dipole transition. If the particle has spin, there is also a term $(\vec{k}\times\boldsymbol{\epsilon})\cdot\boldsymbol{\mu}_S$. So we can combine $\boldsymbol{\mu}_S$ and $\boldsymbol{\mu}_L$ and write the term as

$$C_- \quad\rightarrow\quad (\vec{k}\times\boldsymbol{\epsilon})\cdot(\boldsymbol{\mu}_L+\boldsymbol{\mu}_S) \tag{27}$$

Magnetic dipole transition

We then obtain the magnetic dipole transition rate from

$$\lambda_{a\to b+\gamma} = \frac{\omega_k^3}{2\pi}\sum_{s=\pm1}\int d\Omega_k \left|[\hat{\vec{k}}\times\boldsymbol{\epsilon}(\vec{k},s)]\cdot\langle b|\sum_i(\boldsymbol{\mu}_L+\boldsymbol{\mu}_S)_i\,|a\rangle\right|^2 \tag{28}$$

Multipole expansion

We can expand the phase factor

$$\begin{aligned} e^{-i\vec{k}\cdot\vec{x}} &= \sum_{L=0}(-i)^L(2L+1)j_L(kr)P_L(\hat{\vec{x}}\cdot\hat{\vec{k}}) \\ &= 4\pi\sum_{L=0}(-i)^L j_L(kr)\sum_{M=-L}^{L}Y_{LM}(\hat{\vec{x}})Y^*_{LM}(\hat{\vec{k}}) \\ &\approx 4\pi\sum_{L=0}\frac{(-i)^L}{(2L+1)!!}k^L r^L\sum_{M=-L}^{L}Y_{LM}(\hat{\vec{x}})Y^*_{LM}(\hat{\vec{k}}) \end{aligned} \tag{29}$$

After we carry out the integral over the photon direction

$$\sum_s\int d\Omega_k\boldsymbol{\epsilon}(s,\vec{k})Y^*_{LM}(\hat{\vec{k}}) \tag{30}$$

we are left with the matrix element involving $Y_{LM}(\hat{\vec{x}})$.

Electric multipole radiation

The transition rate from the electric multipole field (TM mode) is

$$\begin{aligned} \lambda_{(\mathrm{E})} &= 2\pi \int \frac{V d^3 k}{(2\pi)^3} \left|\langle b, 1 \,|H_I|\, a, 0\rangle\right|^2 \delta(E_\mathrm{b} + \omega_k - E_\mathrm{a}) \\ &= \frac{8\pi(L+1)}{L[(2L+1)!!]^2} k^{2L+1} \sum_{M_a, M_b, M} \frac{1}{2J_a+1} |Q_{1,LM} + Q_{2,LM}|^2 \end{aligned} \tag{31}$$

where we have taken average over the initial states and sum over the final states and

$$\begin{aligned} Q_{1,LM} &= \left\langle b \middle| e \sum_i q_i r_i^L Y^*_{LM}(i) \middle| a \right\rangle \\ Q_{2,LM} &= \frac{k}{L+1} \left\langle b \middle| \sum_i g_i \frac{e q_i}{2m_i} (\boldsymbol{\sigma}_i \cdot \hat{\vec{L}}_i)[r_i^L Y^*_{LM}(i)] \middle| a \right\rangle \end{aligned} \tag{32}$$

Electric multipole radiation

From spherical harmonics, $Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{3/8\pi}\sin\theta e^{\pm i\phi}$, $Y_{1,0}(\theta,\phi) = \sqrt{3/4\pi}\cos\theta$, we have

$$\begin{aligned} Q_{1;1,\pm1} &= \mp\sqrt{\frac{3}{8\pi}}\left\langle b \middle| e\sum_i q_i(x_i \pm i y_i) \middle| a \right\rangle \\ Q_{1;1,0} &= \sqrt{\frac{3}{4\pi}}\left\langle b \middle| e\sum_i q_i z_i \middle| a \right\rangle \end{aligned} \tag{33}$$

If we neglect $Q_{2,LM}$, Eq. (31) becomes

$$\begin{aligned} \lambda_{(\mathrm{E1})} &= \frac{4}{3}\omega_k^3 \sum_{M_a,M_b} \frac{1}{2J_a+1}\left|\vec{D}_{ba}(M_b,M_a)\right|^2 \\ I_{(\mathrm{E1})} &= \frac{4}{3}\omega_k^4 \sum_{M_a,M_b} \frac{1}{2J_a+1}\left|\vec{D}_{ba}(M_b,M_a)\right|^2 \end{aligned} \tag{34}$$

Magnetic multipole radiation

The transition rate from the magnetic multipole field (TE mode) is

$$\lambda_{(\mathrm{M})} = \frac{8\pi(L+1)}{L[(2L+1)!!]^2} k^{2L+1} \sum_{M_a, M_b, M} \frac{1}{2J_a+1} |M_{1,LM} + M_{2,LM}|^2$$

where $M_{1,LM}$ and $M_{2,LM}$ are defined by

$$\begin{aligned} M_{1,LM} &= \frac{1}{L+1} \sum_i \frac{eq_i}{m_i} \left\langle b \left| \hat{\vec{L}}_i \cdot \nabla_i [r_i^L Y_{LM}^*(i)] \right| a \right\rangle \\ &= \frac{1}{L+1} \sum_i \frac{eq_i}{m_i} \left\langle b \left| \{\nabla_i [r_i^L Y_{LM}^*(i)]\} \cdot \hat{\vec{L}}_i \right| a \right\rangle \\ M_{2,LM} &= \sum_i g_i \frac{eq_i}{2m_i} \langle b | \boldsymbol{\sigma}_i \cdot \nabla_i [r_i^L Y_{LM}^*(i)] | a \rangle \end{aligned} \tag{35}$$

Magnetic multipole radiation

Let us estimate the magnitude of the transition. For the electric multipole field,

$$
\begin{aligned}
Q_{1,LM} &= eq\left\langle b\left|r^L Y_{LM}^*\right|a\right\rangle \sim e\left\langle r^L\right\rangle \sim e\frac{\int drr^{L+2}}{\int drr^2} \sim e\frac{3}{L+3}R^L \\
Q_{2,LM} &= g\frac{eq}{2m}\frac{k}{L+1}\left\langle b\left|(\boldsymbol{\sigma}\cdot\hat{\vec{L}})(r^L Y_{LM}^*)\right|a\right\rangle \sim \frac{e\omega}{m}\frac{3}{2(L+1)(L+3)}R^L
\end{aligned}
\tag{36}
$$

For the nuclear γ decay, the ratio becomes

$$
\frac{Q_{2,LM}}{Q_{1,LM}} \sim \frac{\omega}{m} \sim 10^{-3} \tag{37}
$$

So we can neglect $Q_{2,LM}$ relative to $Q_{1,LM}$. For the magnetic multipole field, $M_{1,LM}$ and $M_{2,LM}$ are of the same order,

$$
M_{1,LM} \sim M_{2,LM} \sim \frac{e}{m}R^{L-1} \tag{38}
$$

Order of magnitude estimate

The transitions of the higher order are much suppressed relative to the lower order,

$$\frac{\lambda_{(\mathrm{E})}(L+1)}{\lambda_{(\mathrm{E})}(L)} \sim \frac{\lambda_{(\mathrm{M})}(L+1)}{\lambda_{(\mathrm{M})}(L)} \sim k^2 R^2 \sim (\mathrm{MeV} \cdot 10 \text{ fm})^2 \sim 2.5 \times 10^{-3} \tag{39}$$

The magnitude of the magnetic transition is suppressed relative to that of electric one of the same order,

$$\frac{\lambda_{(\mathrm{M})}(L)}{\lambda_{(\mathrm{E})}(L)} \sim \frac{1}{m^2 R^2} \sim \frac{1}{(1 \text{ GeV} \cdot 10 \text{ fm})^2} \sim 4 \times 10^{-4} \tag{40}$$

Then we can roughly have the relation $\lambda_{(\mathrm{E})}(L+1) \sim \lambda_{(\mathrm{M})}(L)$, i.e. $\mathrm{E(L{+}1)}$ radiation is comparable to ML one in magnitude.

Parity selection rule

Selection rules for parity are given as follows. The parity of the operators are know as

$$\begin{aligned} P(r^L Y^*_{LM}) &= (-1)^L \\ P[\vec{L}\cdot\nabla(r^L Y^*_{LM})] &= P[\boldsymbol{\sigma}\cdot\nabla(r^L Y^*_{LM})] = (-1)^{L-1} \end{aligned} \tag{41}$$

For EL and ML transitions

$$\begin{aligned} P_i P_f &= (-1)^L, \quad \text{EL} \\ P_i P_f &= (-1)^{L+1}, \quad \text{ML} \end{aligned} \tag{42}$$

where $P_{i,f}$ are parities for the initial and final states.

AM selection rule

Selection rules for angular momentum. Following the Wigner-Eckart theorem,

$$\langle J_f, M_f | T_{LM} | J_i, M_i \rangle \quad = \quad C^{J_f M_f}_{J_i M_i, LM} \langle J_f \| T_{LM} \| J_i \rangle \tag{43}$$

where

$$T_{LM} \quad = \quad r^L Y^*_{LM}, \; \hat{\vec{L}} \cdot \nabla(r^L Y^*_{LM}), \; \boldsymbol{\sigma} \cdot \nabla(r^L Y^*_{LM}) \tag{44}$$

and $L \neq 0$ obeys that $\vec{J}_f, \vec{J}_i, \vec{L}$ form a vector triangle,

$$\begin{gathered} \vec{J}_f = \vec{J}_i + \vec{L} \\ |J_i - J_f| \leq L \leq |J_i + J_f| \end{gathered} \tag{45}$$

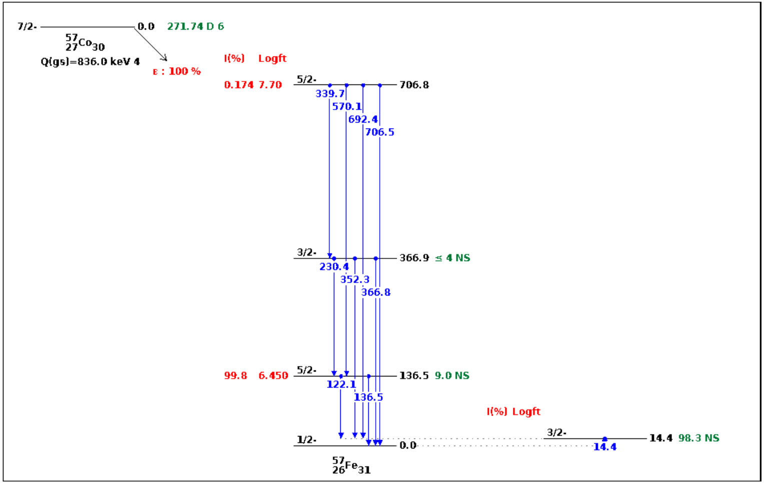
There is no transition with $J_i = J_f$ and $L = 0$. Note that the transition with $L = |J_i - J_f|$ is dominant.

AM selection rule

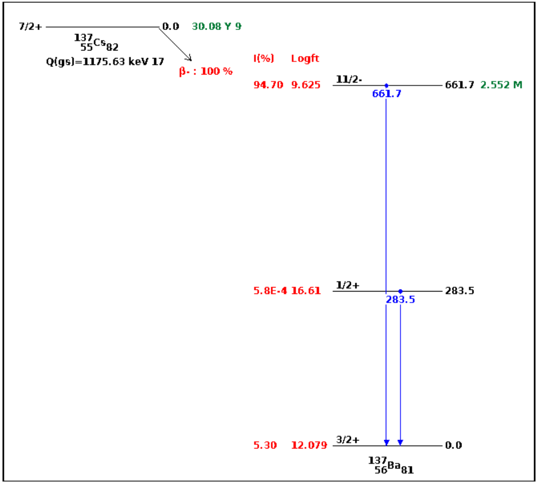
Table: Selection rules for parity and angular momentum.

$\lvert J_i - J_f \rvert$	0,1	2	3	4	5
$P_i P_f = +$	$M1(E2)$	$E2$	$M3(E4)$	$E4$	$M5(E6)$
$P_i P_f = -$	$E1$	$M2(E3)$	$E3$	$M4(E5)$	$E5$

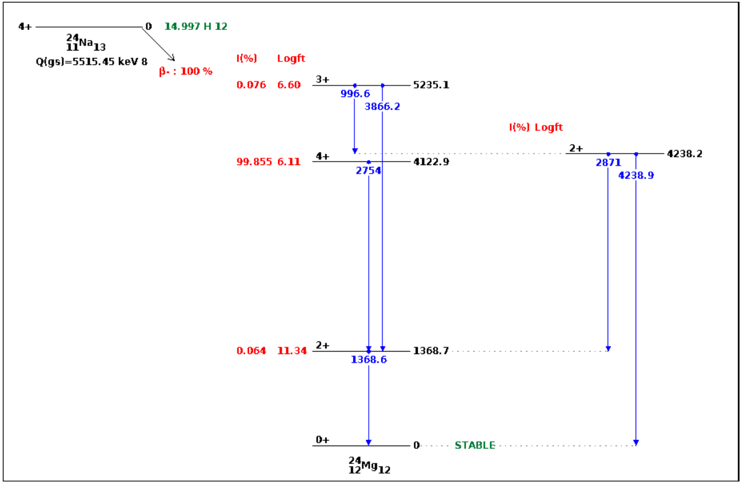
Example: Fe-57



Example: Ba-137



Example: Na-24

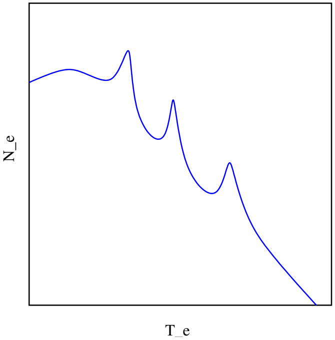


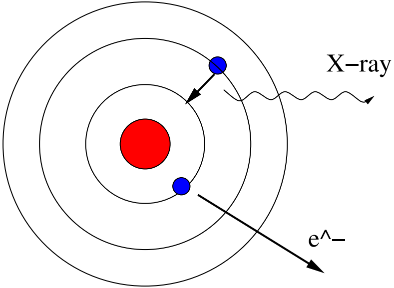
No monopole radiation

There are no $0^+ \to 0^+$ transition, since this would need a monopole radiation with $L = 0$ which does not exist. These decay processed can happen through internal conversion.

Internal conversion

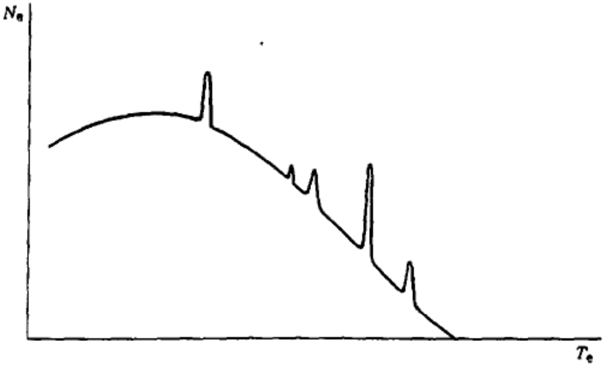
Figure: Energy spectra of electrons from beta decay and internal conversion of radioactive nuclei. The peaks on top of continuous the spectrum are from internal conversion.





Internal conversion

Figure: Electron energy spectra of β-decay and internal conversion. Taken from page 342 of Krane.



Internal conversion

As we have mentioned that the γ decay with $J_i = J_f = L = 0$ is forbidden, where the transition takes place as internal conversion. Internal conversion is a radioactive decay where a transition of an excited nucleus to its lower energy level takes place and the energy is transfered to an electron on the inner atomic shell which is emitted. Internal conversion is not photo-electric effect since there is no photon emission. The electrons emitted from internal conversion can be distinguished from those from the β-decay.

Internal conversion

The electron energy in internal conversion can be expressed by $T_e = E_\gamma - W$, where E_γ is the transition energy and W is the binding energy of the electron in the atomic shell. Normally an electron in the K-sehll $(n = 1)$ is kicked out, so from the K-shell binding energy W_K and the electron energy T_e we can determine the transition energy or the difference of the two energy levels. For the L-shell $(n = 2)$ electrons, there are atomic orbitals $2s_{1/2}$, $2p_{1/2}$ and $2p_{3/2}$, which are also called L_I, L_{II} and L_{III} shells. The vacancy left by the knocked out eletron is instantly filled by the electron from an outer shell. This results in accompanying X-ray.

Internal conversion coefficient

The double option for the decay of the excited state to a lower state by γ decay and internal conversion needs a quantity to measures the probability of one to the other,

$$\begin{aligned}\lambda_t &= \lambda_\gamma + \lambda_e \\ &= \lambda_\gamma(1+\alpha) \\ &= \lambda_\gamma(1+\alpha_K+\alpha_L+\alpha_M+\cdots)\end{aligned}$$

where α is called the internal conversion coefficient.

Theory for internal conversion

Let us consider the electric multipole transition with multipolarity L or EL transition. The standard multipole expansion of the electrostatic potential in the region outside the nucleus gives

$$H_C = \sum_{LM} \frac{4\pi e}{2L+1} \frac{1}{r^{L+1}} Y^*_{LM}(\hat{\vec{r}}) Q_{1,LM}$$

Let us estimate the probability of the internal conversion of the K-shell electron to a continuum state c.

$$d\lambda = 2(2\pi) \left| \langle f | H_C | i \rangle \right|^2 \delta(E_i - W_K - E_f - T_e) d\rho_e$$

where the initial and final states are defined as

$$|i\rangle = |e_K, i\rangle\,, \quad |f\rangle = |e_c, f\rangle$$

Theory for internal conversion

For the internal conversion with the EL transition we have

$$
\begin{aligned}
d\lambda \quad &= \quad 4\pi \left(\frac{4\pi e}{2L+1}\right)^2 \left|\sum_M \langle e_c| \frac{1}{r^{L+1}} Y^*_{LM}(\hat{\vec{r}}) \,|e_K\rangle\, Q_{1,LM}\right|^2 \\
&\quad\quad \times \delta(E_i - W_K - E_f - T_e) d\rho_e
\end{aligned}
$$

where the density of state for the electron is

$$
\begin{aligned}
d\rho_e &= \frac{V}{(2\pi)^3} d\Omega dk_e k_e^2 \\
&= \frac{V}{(2\pi)^3} d\Omega dE_e E_e k_e \approx \frac{V}{(2\pi)^3} d\Omega dE_e m_e k_e
\end{aligned}
$$

Theory for internal conversion

The wave function for the K-shell electron is

$$\psi_K = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

where

$$a_0 = \frac{a_B}{Z} = \frac{h^2}{m_e e^2 Z}$$

The wave function for the continuum electron is the plane wave

$$\psi_c = \frac{1}{\sqrt{V}} e^{i\vec{k}_e \cdot \vec{x}}$$

Matrix element for electron

We can expand the plane wave in terms of spherical Harmonics

$$
\begin{aligned}
e^{-i\vec{k}\cdot\vec{r}} &= \sum_{L=0}(-i)^L(2L+1)j_L(kr)P_L(\hat{\vec{r}}\cdot\hat{\vec{k}}) \\
&= 4\pi\sum_{L=0}(-i)^L j_L(kr)\sum_{M=-L}^{L} Y_{LM}(\hat{\vec{r}})Y^*_{LM}(\hat{\vec{k}})
\end{aligned}
$$

The matrix element for electron becomes

$$
\begin{aligned}
\langle e_c|\,\frac{1}{r^{L+1}}Y^*_{LM}(\hat{\vec{r}})\,|e_K\rangle &= \frac{4\pi}{\sqrt{V}}\frac{1}{\sqrt{\pi a_0^3}}\sum_{L_1=0}\sum_{M_1=-L_1}^{L_1}(-i)^{L_1}Y^*_{L_1M_1}(\hat{\vec{k}}_e) \\
&\times\int d^3\vec{r}\,\frac{1}{r^{L+1}}j_{L_1}(k_e r)e^{-r/r_0}Y^*_{LM}(\hat{\vec{r}})Y_{L_1M_1}(\hat{\vec{r}})
\end{aligned}
$$

Matrix element for electron

Using orthornormality condition for spherical Harmonics

$$\int d\Omega Y^*_{LM}(\hat{\vec{r}}) Y_{L_1 M_1}(\hat{\vec{r}}) = \delta_{L,L_1}\delta_{M,M_1}$$

The matrix element for electron becomes

$$\langle e_c | \frac{1}{r^{L+1}} Y^*_{LM}(\hat{\vec{r}}) | e_K \rangle = \frac{4\pi}{\sqrt{V}} \frac{1}{\sqrt{\pi a_0^3}} (-i)^L Y^*_{LM}(\hat{\vec{k}}_e) \times \int dr \frac{1}{r^{L-1}} j_L(k_e r) e^{-r/a_0}$$

Since ka_0 is larger than 1, so we consider $L < ka_0$, then $j_L(kr)$ suppresses the integrand before we reach the atomic radius a_0. In this case we can set $e^{-r/a_0} \sim 1$.

Matrix element for electron

We have

$$\int_0^\infty dr \frac{1}{r^{L-1}} j_L(kr) e^{-r/r_0} \approx \int_0^\infty dr \frac{1}{r^{L-1}} j_L(kr) = \frac{k^{L-2}}{(2L-1)!!}$$

Then we finally obtain

$$\begin{aligned}\lambda &= \frac{k^{2L-4}}{[(2L+1)!!]^2} \frac{128\pi}{a_0^3} m_e e^2 \\ &\times \int dE_e k_e \int d\Omega_e \left| \sum_M (-i)^L Y^*_{LM}(\hat{\vec{k}}_e) Q_{1,LM} \right|^2 \\ &\times \delta(E_i - W_K - E_f - T_e)\end{aligned}$$

where we have used $k_e = k$ and $E_e = m_e + T_e$.

Matrix element for electron

Then we finally obtain

$$\begin{aligned}\lambda &= \frac{k_e^{2L-3}}{[(2L+1)!!]^2}\frac{128\pi}{a_0^3}m_e e^2 \\ &\times \int d\Omega_e \sum_{M,M'} Y_{LM'}(\hat{\vec{k}}_e)Y^*_{LM}(\hat{\vec{k}}_e)Q^*_{1,LM'}Q_{1,LM} \\ &= 128\pi\frac{m_e e^2}{a_0^3}\frac{k_e^{2L-3}}{[(2L+1)!!]^2}\sum_M \left|Q_{1,LM}\right|^2 \\ &\to 128\pi\frac{m_e e^2}{a_0^3}\frac{k_e^{2L-3}}{[(2L+1)!!]^2}\sum_{M_i,M_f,M}\frac{1}{2J_i+1}\left|Q_{1,LM}\right|^2\end{aligned}$$

Coefficient of internal conversion

The coefficient of internal conversion can be obtain

$$\begin{aligned}\alpha_K &= 16\frac{L}{L+1}\frac{m_e e^2}{a_0^3}\frac{k_e^{2L-3}}{\omega^{2L+1}} \\ &\approx Z^3 e^8 \frac{L}{L+1}\left(\frac{2m_e}{\omega}\right)^{L+5/2}\end{aligned}$$

where we have used $\omega \approx k_e^2/(2m_e)$.

Example 1

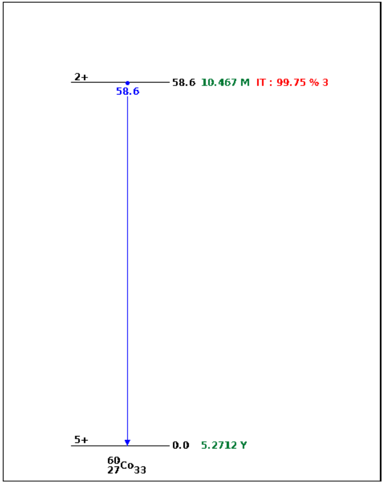
Authors: E. Browne, J. K. Tuli Citation:Nuclear Data Sheets 114, 1849 (2013)

Parent Nucleus	Parent E(level)	Parent Jπ	Parent $T_{1/2}$	Decay Mode	GS-GS Q-value (keV)	Daughter Nucleus	Decay Scheme	ENSDF file
$^{60}_{27}Co$	58.6037	2+	10.467 m *6*	IT		$^{60}_{27}Co$		

Electrons:

	Energy (keV)	Intensity (%)	Dose (MeV/Bq-s)
Auger L	0.75	128.5 % *3*	9.635E-4 *25*
Auger K	6.07	49.8 % *3*	0.003020 *20*
CE K	50.894 7	81.3 %	0.0414
CE L	57.677 7	14.1 %	0.00815
CE M	58.603 7	1.99 %	0.00117
CE N	58.603 7	0.0638 %	3.74E-5

Example 1



Example 2

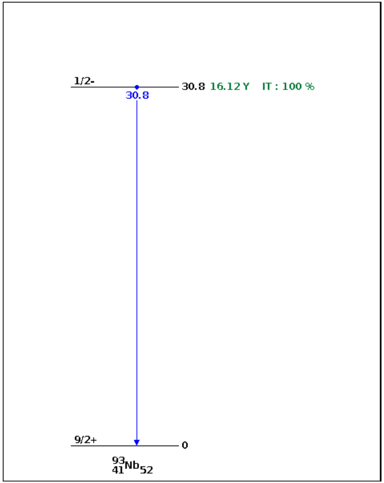
Author: Coral M. Baglin   Citation:Nuclear Data Sheets 112, 1163 (2011)

Parent Nucleus	Parent E(level)	Parent Jπ	Parent $T_{1/2}$	Decay Mode	GS-GS Q-value (keV)	Daughter Nucleus		
$^{93}_{41}$Nb	30.772	1/2-	16.12 y 12	IT: 100 %		$^{93}_{41}$Nb	Decay Scheme	ENSDF file

Electrons:

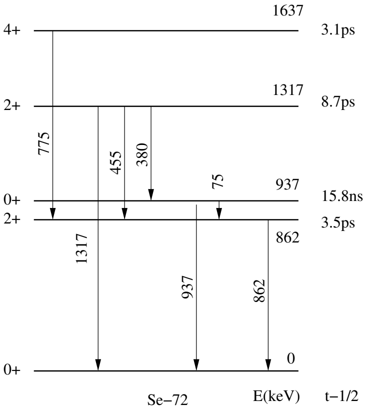
	Energy (keV)	Intensity (%)	Dose (MeV/Bq-s)
Auger L	2.15	81.1 % 3	0.001743 7
CE K	11.784 20	15.3 % 3	0.00180 4
Auger K	14.0	3.81 % 10	5.33E-4 14
CE L	28.072 20	68.0 % 8	0.01909 22
CE M	30.302 20	14.7 % 3	0.00445 9
CE N	30.712 20	1.91 % 4	5.87E-4 12

Example 2



Example 3

Figure: Energy level of Se-72.



Example

Let us take the γ-emission of Se-72 for an example. The enegry level of Se-72 is shown in Fig. 4. We notice that the 937-MeV transition must be an internal conversion.
Let us look at the energy level of 1317 keV. Its half life is is 8.7 ps corresponding to the total decay rate

$$\lambda_t = \frac{\ln 2}{t_{1/2}} = \frac{0.693}{8.7 \times 10^{-12} s} = 7.97 \times 10^{10}\, s^{-1} \tag{46}$$

The total decay rate is the sum of the rates for three transitions, 1317 keV, 455 keV and 380 keV,

$$\lambda_t \quad = \quad \lambda_{1317} + \lambda_{455} + \lambda_{380} \tag{47}$$

Example 3

The relative intensities for these γ-decays are

$$\lambda_{1317} : \lambda_{455} : \lambda_{380} = 51 : 39 : 10 \tag{48}$$

Then we obtain

$$\begin{aligned}
\lambda_{1317} &= 0.51\lambda_t = 4.1 \times 10^{10}\, s^{-1} \\
\lambda_{455} &= 0.39\lambda_t = 3.1 \times 10^{10}\, s^{-1} \\
\lambda_{380} &= 0.1\lambda_t = 0.8 \times 10^{10}\, s^{-1}
\end{aligned} \tag{49}$$

Example 3

We can compare these partial rates to the calculated results. Let us assume an E2 transition, then we get

$$\begin{aligned} \lambda_{\mathrm{E2,1317}} &\approx \frac{8\pi(L+1)}{L[(2L+1)!!]^2} k^{2L+1} e^2 \left(\frac{3}{L+3}\right)^2 R^{2L} \\ &\sim \frac{4\pi}{75\times 137} \times (1.317\times 5/197)^4 \times 0.36 \times 1.317/197\,\mathrm{fm}^{-1} \\ &\approx 3.67\times 10^{-12}\, c/\mathrm{fm} \sim 1.1\times 10^{12}\, s^{-1} \end{aligned} \tag{50}$$

where we have used $R \approx A^{1/3} r_0 \approx 5$ fm. We can obtain the rates of other E2 transitions as

$$\begin{aligned} \lambda_{\mathrm{E2,455}} &= (455/1317)^5 \lambda_{\mathrm{E2,1317}} = 5.4\times 10^9\, s^{-1} \\ \lambda_{\mathrm{E2,380}} &= (380/1317)^5 \lambda_{\mathrm{E2,1317}} = 2.2\times 10^9\, s^{-1} \end{aligned} \tag{51}$$