

Spinors on the light front

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These notes define a set of conventions in light-front quantum field theory. Similar conventions for light-front dynamics, can be found in, e.g.,

- A. Harindranath: *Light front QCD: lecture notes* (2005);
- M. Burkardt: *Light Front Quantization*, Adv. Nucl. Phys. **23**, 1 (2002) [arXiv:hep-ph/9505259];
- G. P. Lepage, S. J. Brodsky, *Exclusive processes in perturbative quantum chromodynamics*, Phys. Rev. D **22**, 2157 (1980);
- S. J. Brodsky, H.-C. Pauli, S. S. Pinsky: *Quantum chromodynamics and other field theories on the light cone*, Phys. Rep. **301**, 299 (1998);
- J. Carbonell, B. Desplanques, V. A. Karmanov and J.-F. Mathiot: Phys. Rep. **300**, 215 (1998).

Throughout the notes, we use natural units, $\hbar = c = 1$. Let $x = (x^0, x^1, x^2, x^3) = (t, \mathbf{x})$ be the standard space-time coordinates. The signature of *Minkowski* space metric tensor is $g_{\mu\nu} = \text{diag}\{+1, -1, -1, -1\}$.

1 Light-Front coordinates

The light-front coordinates are defined as (x^+, x^-, x^1, x^2) , where $x^+ = x^0 + x^3$ is the light-front time, $x^- = x^0 - x^3$ is the longitudinal coordinate, $\mathbf{x}^\perp = (x^1, x^2)$ are the transverse coordinates. The corresponding metric tensor and its inverse is,

$$g_{\mu\nu} = \begin{pmatrix} & \frac{1}{2} & & \\ \frac{1}{2} & & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} & 2 & & \\ 2 & & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad (1)$$

Note that $\sqrt{-\det g} = \frac{1}{2}$. The Levi-Civita tensor should be defined as

$$\varepsilon^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-\det g}} \begin{pmatrix} \mu & \nu & \rho & \sigma \\ - & + & 1 & 2 \end{pmatrix} = \begin{cases} +2 & \text{if } \mu, \nu, \rho, \sigma \text{ is an even permutation of } -, +, 1, 2 \\ -2 & \text{if } \mu, \nu, \rho, \sigma \text{ is an odd permutation of } -, +, 1, 2 \\ 0 & \text{other cases.} \end{cases} \quad (2)$$

Similarly, the light-front components of a 4-vector $v = (v^0, \mathbf{v})$ is $(v^+, v^-, \mathbf{v}^\perp)$, where $v^\pm = v^0 \pm v^3$ and $\mathbf{v}^\perp = (v^1, v^2)$. Sometimes it is also useful to introduce the complex representation for the transverse vector \mathbf{v}^\perp : $v^L = v^1 - iv^2$, and $v^R = v^1 + iv^2 = (v^L)^*$. The component of the contravariant 4-vector $v_\mu = g_{\mu\nu}v^\nu$ are: $(v_-, v_+, \mathbf{v}_\perp)$, where $v_\pm = \frac{1}{2}(v_0 \pm v_3) = \frac{1}{2}v^\mp$, $\mathbf{v}_\perp = -\mathbf{v}^\perp$.

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It is useful to introduce two vectors to symbolically restore the covariance: $\omega = (\omega^0, \boldsymbol{\omega}) = (1, 0, 0, -1)$, and $\eta = (\eta^0, \boldsymbol{\eta}) = (0, 1, 0, 0)$. They satisfy

$$\omega_\mu \omega^\mu = 0, \quad \eta_\mu \eta^\mu = -1, \quad \eta_\mu \omega^\mu = 0, \quad (\omega^2 = \eta^2 = 1). \quad (3)$$

Then, the longitudinal coordinate of a vector a can be written as $a^+ = \omega \cdot a$. Similarly, the transverse component of it becomes $\mathbf{a}^\perp = \mathbf{a} - \omega(\omega \cdot \mathbf{a})$.

2 Normalization

The coordinate space integration measure is defined as

$$\int d^3x \equiv \int dx_+ d^2x^\perp = \frac{1}{2} \int dx^- d^2x^\perp. \quad (4)$$

The full four-dimensional integration measure is,

$$\int d^4x = \int dx^0 dx^1 dx^2 dx^3 = \frac{1}{2} \int dx^+ dx^- d^2x^\perp = \int d^3x dx^+. \quad (5)$$

In the momentum space, we use the Lorentz invariant integration measure:

$$\int \frac{d^4p}{(2\pi)^4} \theta(p^+) 2\pi \delta(p^2 - m^2) = \int \frac{d^3p}{(2\pi)^3 2p^0} \theta(p^0) = \int \frac{d^2p_\perp dp^+}{(2\pi)^3 2p^+} \theta(p^+) = \int \frac{d^2p_\perp}{(2\pi)^3} \int_0^1 \frac{dx}{2x} \quad (6)$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$ is the on-shell energy and $x = p^+/P^+$ is the longitudinal momentum fraction. The corresponding normalization of the single-particle state is

$$\begin{aligned} \langle p, \sigma | p', \sigma' \rangle &= 2p^0 \theta(p^0) (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') \delta_{\sigma\sigma'} = (2\pi)^3 \delta^4(p - p') / \theta(p^0) \delta(p^2 - m^2) \\ &= 2p^+ \theta(p^+) (2\pi)^3 \delta^3(p - p') \delta_{\sigma\sigma'} = 2x (2\pi)^3 \delta(x - x') \delta^2(\mathbf{p}_\perp - \mathbf{p}'_\perp) \delta_{\sigma\sigma'}. \end{aligned} \quad (7)$$

Here the light-front delta function is defined as $\delta^3(p) = \delta^2(\mathbf{p}_\perp) \delta(p^+)$.

The transverse Fourier transformation and its inverse transformation are defined as,

$$\tilde{f}(\mathbf{r}_\perp) \equiv \int \frac{d^2p_\perp}{(2\pi)^2} e^{i\mathbf{p}_\perp \cdot \mathbf{r}_\perp} f(\mathbf{p}_\perp), \quad f(\mathbf{p}_\perp) \equiv \int d^2r_\perp e^{-i\mathbf{p}_\perp \cdot \mathbf{r}_\perp} \tilde{f}(\mathbf{r}_\perp). \quad (8)$$

2.1 Box regularization

In a typical box normalization, the system is confined in a box with length L :

$$-\frac{L}{2} \leq x \leq +\frac{L}{2}. \quad (9)$$

Here x is the coordinate of any one spatial dimension. Boundary conditions, such as periodic boundary condition or anti-periodic boundary condition, may apply. The conjugate moment is discretized. In the case of the periodic boundary condition, it becomes, $p = 2\pi n/L$, ($n = 0, \pm 1, \pm 2, \dots$). The conversion of the integrations and δ -functions are listed in Table 1. For example,

$$\begin{aligned} \int_{-\infty}^{+\infty} dx e^{-i(p-p')x} &= 2\pi \delta(p - p'), \quad \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx e^{-i(p-p')x} = L \delta_{p,p'}. \\ \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{ip(x-x')} &= \delta(x - x'), \quad \frac{1}{L} \sum_p e^{ip(x-x')} = \delta(x - x'). \end{aligned} \quad (10)$$

In light-front dynamics, the spectral condition requires that the longitudinal momentum is always positive. The box regularization of the longitudinal momentum can be implemented using the same

Table 1: Conversion formula for the box regularization.

box regularization	$\xrightarrow{L \rightarrow \infty}$	continuum
$\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} dx$		$\int_{-\infty}^{+\infty} dx$
$\frac{1}{L} \sum_p$	$\xrightarrow{L \rightarrow \infty}$	$\int_{-\infty}^{+\infty} \frac{dp}{2\pi}$
$\delta(x - x')$		$\delta(x - x')$
$L\delta_{p,p'}$		$2\pi\delta(p - p')$

method above but with an addition Heaviside θ -function to impose the positivity of the longitudinal momentum: $\theta(p^+)$. Note also that the longitudinal momentum is conjugate to $x_+ = \frac{1}{2}x^-$. For example,

$$\begin{aligned} \frac{1}{2} \int_{-\infty}^{+\infty} dx^- e^{-\frac{i}{2}(p^+ - p'^+)x^-} &= 2\pi\delta(p^+ - p'^+), & \frac{1}{2} \int_{-L}^{+L} dx^- e^{-\frac{i}{2}(p^+ - p'^+)x^-} &= L\delta_{p^+, p'^+}. \\ \int_{-\infty}^{+\infty} \frac{dp^+}{2\pi} \theta(p^+) e^{\frac{i}{2}p^+(x^- - x'^-)} &= 2\delta(x^- - x'^-), & \frac{1}{L} \sum_{p^+} \theta(p^+) e^{\frac{i}{2}p^+(x^- - x'^-)} &= 2\delta(x^- - x'^-). \end{aligned} \quad (11)$$

2.2 Longitudinal propagators

$$\frac{1}{\partial^+} f(x^-) \equiv \frac{1}{4} \int_{-\infty}^{+\infty} dy^- [\theta(x^- - y^-) - \theta(y^- - x^-)] f(y^-). \quad (12)$$

This propagator is the inverse of the derivative operator and satisfies the anti-periodic boundary condition in the longitudinal direction:

$$\partial^+ \left(\frac{1}{\partial^+} f \right) (x^-) = f(x^-), \quad \left(\frac{1}{\partial^+} f \right) (-\infty) = - \left(\frac{1}{\partial^+} f \right) (+\infty). \quad (13)$$

$$\frac{1}{(\partial^+)^2} f(x^-) \equiv \frac{1}{8} \int_{-\infty}^{+\infty} dy^- |x^- - y^-| f(y^-). \quad (14)$$

This propagator is the inverse of the double-derivative operator and satisfies the anti-periodic boundary condition in the longitudinal direction:

$$(\partial^+)^2 \left(\frac{1}{(\partial^+)^2} f \right) (x^-) = f(x^-), \quad \left(\frac{1}{(\partial^+)^2} f \right) (-\infty) = - \left(\frac{1}{(\partial^+)^2} f \right) (+\infty). \quad (15)$$

3 Kinematics

Two-Body kinematics Let $\mathbf{P}_\perp = \mathbf{p}_{1\perp} + \mathbf{p}_{2\perp}$, $P^+ = p_1^+ + p_2^+$ be the c.m. momentum of two on-shell particles with 4-momentum p_1, p_2 , respectively ($p_a^2 = m_a^2$). Define the longitudinal momentum fraction $x_a = p_a^+/P^+$, ($x_1 + x_2 = 1$), and relative transverse momentum $\mathbf{p}_\perp = \mathbf{p}_{1\perp} - x_1 \mathbf{P}_\perp$ ($-\mathbf{p}_\perp = \mathbf{p}_{2\perp} - x_2 \mathbf{P}_\perp$). Then, the momentum space integration measure admits a factorization:

$$\int \frac{d^2 p_1^\perp dp_1^+}{(2\pi)^3 2p_1^+} \int \frac{d^2 p_2^\perp dp_2^+}{(2\pi)^3 2p_2^+} = \int \frac{d^2 p_1^\perp dx_1}{(2\pi)^3 2x_1} \int \frac{d^2 p_2^\perp dx_2}{(2\pi)^3 2x_2} = \int \frac{d^2 P^\perp dP^+}{(2\pi)^3 2P^+} \int \frac{d^2 p^\perp dx}{(2\pi)^3 2x(1-x)}. \quad (16)$$

where $x = x_1$. Similarly, the two-body Fock state

$$\begin{aligned}\langle p'_1, p'_2 | p_1, p_2 \rangle &= 2x_1 \theta(p_1^+) (2\pi)^3 \delta^2(p_1^\perp - p_1'^\perp) \delta(x_1 - x'_1) 2x_2 \theta(p_2^+) (2\pi)^3 \delta^2(p_2^\perp - p_2'^\perp) \delta(x_2 - x'_2) \\ &= 2P^+ \theta(P^+) (2\pi)^3 \delta^3(P - P') 2x(1-x) (2\pi)^3 \delta^2(p_\perp - p'_\perp) \delta(x - x') \\ &= \langle P'; \mathbf{p}'^\perp, x' | P; \mathbf{p}^\perp, x \rangle.\end{aligned}\quad (17)$$

Furthermore, the two-body invariant mass squared is,

$$s_2 \equiv (p_1 + p_2)^2 = \frac{\mathbf{p}_{1\perp}^2 + m_1^2}{x_1} + \frac{\mathbf{p}_{2\perp}^2 + m_2^2}{x_2} - \mathbf{P}_\perp^2 = \frac{\mathbf{p}_\perp^2 + m_1^2}{x} + \frac{\mathbf{p}_\perp^2 + m_2^2}{1-x}.\quad (18)$$

here m_a is the a -th particle's mass.

Few-Body kinematics Define the few-body c.m. momentum $\mathbf{P}_\perp = \sum_a \mathbf{p}_{a\perp}$, $P^+ = \sum_a p_a^+$. Introduce the momentum fraction and the relative transverse momentum:

$$x_a = p_a^+ / P^+, \quad \mathbf{k}_{a\perp} = \mathbf{p}_{a\perp} - x_a \mathbf{P}_\perp\quad (19)$$

Then, it is clear that $\sum_a x_a = 1$, and $\sum_a \mathbf{k}_{a\perp} = 0$. The few-body momentum space integration measure admits a factorization of the c.m. momentum:

$$\prod_a \int \frac{d^2 p_a^\perp d p_a^+}{(2\pi)^3 2 p_a^+} \theta(p_a^+) = \int \frac{d^2 P^\perp d P^+}{(2\pi)^3 2 P^+} \theta(P^+) \prod_a \int \frac{d^2 k_a^\perp d x_a}{(2\pi)^3 2 x_a} \times 2(2\pi)^3 \delta^2\left(\sum_a \mathbf{k}_{a\perp}\right) \delta\left(\sum_a x_a - 1\right)\quad (20)$$

The few-body invariant mass squared is,

$$s_n \equiv \left(\sum_a p_a\right)^2 = \sum_a \frac{\mathbf{k}_{a\perp}^2 + m_a^2}{x_a}\quad (21)$$

Lemma *cluster decomposition of s_n :*

Let $(x_a, \mathbf{k}_{a\perp})$, $(a = 1, 2, \dots, n)$ be n relative momenta, i.e. $\sum_a x_a = 1$, $\sum_a \mathbf{k}_{a\perp} = 0$. Define new relative momenta with respect to the cluster without the n -th particle: $\zeta_a = x_a / (1 - x_n)$, $\boldsymbol{\kappa}_{a\perp} = \mathbf{k}_{a\perp} + \zeta_a \mathbf{k}_{n\perp}$. Then, the n -body invariant mass squared can be written as,

$$(1 - x_n) \left(\sum_{a=1}^n \frac{\mathbf{k}_{a\perp}^2 + m_a^2}{x_a} - M^2 \right) = \sum_{a=1}^{n-1} \frac{\boldsymbol{\kappa}_{a\perp}^2 + m_a^2}{\zeta_a} - m_n^2 + (1 - x_n) \left(\frac{\mathbf{k}_{n\perp}^2 + m_n^2}{x_n(1 - x_n)} - M^2 \right),\quad (22)$$

or in short form,

$$(1 - x_n)(s_n - M^2) = s_{n-1}^r - (m_n^2 - (1 - x_n)(s_2 - M^2)).\quad (23)$$

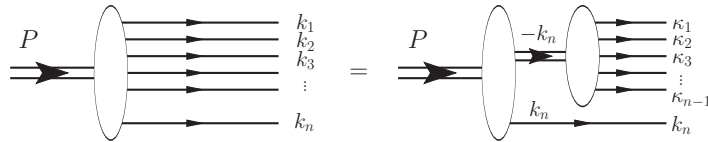


Figure 1: Cluster decomposition of the few-body invariant mass squared.

Lemma *cluster decomposition of s :*

Let $(x_i, \mathbf{k}_{i\perp})$ be relative momenta ($i = 1, 2, \dots$), i.e. $\sum_i x_i = 1$, $\sum_i \mathbf{k}_{i\perp} = 0$. Partition the system into two clusters A and B . Let $x_A = \sum_{i \in A} x_i$, $\mathbf{k}_{A\perp} = \sum_{i \in A} \mathbf{k}_{i\perp}$, $x_B = \sum_{i \in B} x_i$, $\mathbf{k}_{B\perp} = \sum_{i \in B} \mathbf{k}_{i\perp}$. Obviously, $\mathbf{k}_{A\perp} + \mathbf{k}_{B\perp} = 0$, $x_A + x_B = 1$.

Define new relative momenta with respect to the cluster: $\zeta_{iA} = x_i/x_A$, $\boldsymbol{\kappa}_{iA\perp} = \mathbf{k}_{i\perp} - \zeta_{iA}\mathbf{k}_{A\perp}$. $\zeta_{iB} = x_i/x_B$, $\boldsymbol{\kappa}_{iB\perp} = \mathbf{k}_{i\perp} - \zeta_{iB}\mathbf{k}_{B\perp}$. Introduce the invariant masses $s_A = \sum_{i \in A} \frac{\boldsymbol{\kappa}_{iA\perp}^2 + m_i^2}{\zeta_{iA}}$, $s_B = \sum_{i \in B} \frac{\boldsymbol{\kappa}_{iB\perp}^2 + m_i^2}{\zeta_{iB}}$.

Then, the total invariant mass squared $s \equiv s_{A+B} = \sum_i \frac{\mathbf{k}_{i\perp}^2 + m_i^2}{x_i}$ can be written as,

$$\sum_i \frac{\mathbf{k}_{i\perp}^2 + m_i^2}{x_i} = \frac{\mathbf{k}_{A\perp}^2 + s_A}{x_A} + \frac{\mathbf{k}_{B\perp}^2 + s_B}{1 - x_A} \quad (24)$$

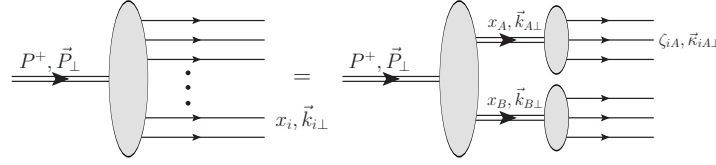


Figure 2: Cluster decomposition of the few-body invariant mass squared.

4 gamma matrices

In this convention, the 4-by-4 gamma matrices are defined as (cf. Dirac and chiral representation):

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix} \quad (25)$$

where $\sigma = (1, \boldsymbol{\sigma})$ are the standard Pauli matrices,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (26)$$

The γ -matrices defined here furnish a representation of the Clifford algebra $\text{Cl}_{1,3}(\mathbb{R})$:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}. \quad (27)$$

Then, $S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ furnishes a spinorial representation of the Lorentz group. It is convenient to introduce the following 4-by-4 matrices,

- front-form: $\gamma^\pm \equiv \gamma^0 \pm \gamma^3$, $\boldsymbol{\gamma}^\perp \equiv (\gamma^1, \gamma^2)$, $\gamma^L \equiv \gamma^1 - i\gamma^2$, $\gamma^R \equiv \gamma^1 + i\gamma^2$, $\not{p} = p_\mu \gamma^\mu = \frac{1}{2}p^+ \gamma^- + \frac{1}{2}p^- \gamma^+ - \boldsymbol{p}_\perp \cdot \boldsymbol{\gamma}_\perp$
corollaries: $\gamma^+ \gamma^+ = 0$; $\gamma^- \gamma^- = 0$; $\gamma^+ \gamma^- \gamma^+ = 4\gamma^+$, $\gamma^- \gamma^+ \gamma^- = 4\gamma^-$; $\gamma^0 \gamma^\pm = \gamma^\mp \gamma^0$. The matrix form are:

$$\gamma^+ = \begin{pmatrix} 0 & 0 \\ 2i & 0 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} 0 & -2i \\ 0 & 0 \end{pmatrix}, \quad \gamma^L = \begin{pmatrix} \sigma^L & 0 \\ 0 & -\sigma^L \end{pmatrix}, \quad \gamma^R = \begin{pmatrix} -\sigma^R & 0 \\ 0 & \sigma^R \end{pmatrix},$$

where,

$$\sigma^L = \sigma^1 - i\sigma^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad \sigma^R = \sigma^1 + i\sigma^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

- projections: $\Lambda^\pm \equiv \Lambda_\pm \equiv \frac{1}{2}\gamma^0\gamma^\pm$;
corollaries: $\Lambda_\pm^2 = \Lambda_\pm$, $\Lambda^+\Lambda^- = 0$, $\Lambda^-\Lambda^+ = 0$, $\Lambda^+ + \Lambda^- = 1$.
 $\Lambda_\pm^\dagger = \Lambda_\pm$, $\bar{\Lambda}_\pm = \Lambda_\mp$, $\frac{1}{4}\gamma^+\gamma^- = \Lambda^-$, $\frac{1}{4}\gamma^-\gamma^+ = \Lambda^+$.

Under the convention we use, the projections are diagonal and simple:

$$\Lambda^+ = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \quad \Lambda^- = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$$

- parity matrix: $\beta = \gamma^0$; charge conjugation matrix: $\mathcal{C} = -i\gamma^2$; time reversal matrix: $T = \gamma^1\gamma^3 = \mathcal{C}\gamma_5$
- chiral matrix: $\gamma^5 = \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$ is diagonal:

$$\gamma^5 = \begin{pmatrix} \sigma^3 & \\ & -\sigma^3 \end{pmatrix}, \quad P_L = \frac{1}{2} \begin{pmatrix} \sigma^- & \\ & \sigma^+ \end{pmatrix}, \quad P_R = \frac{1}{2} \begin{pmatrix} \sigma^+ & \\ & \sigma^- \end{pmatrix},$$

where $P_L = \frac{1}{2}(1 - \gamma_5) = \text{diag}\{0, 1, 1, 0\}$, $P_R = \frac{1}{2}(1 + \gamma_5) = \text{diag}\{1, 0, 0, 1\}$ are the two chiral projections, also diagonal. It is easy to see $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L P_R = P_R P_L = 0$, $P_L + P_R = 1$.
 σ -identity:

$$iS^{\mu\nu}\gamma_5 = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}S_{\rho\sigma} \quad (28)$$

- $\bar{\psi} \equiv \psi^\dagger\beta$ (for spinorial vector) and $\bar{A} = \beta A^\dagger\beta$ (for spinorial matrix).

Overbar identities:

$$\overline{\gamma^\mu} = \gamma^\mu, \quad \overline{S^{\mu\nu}} = S^{\mu\nu}, \quad \overline{i\gamma_5} = i\gamma_5, \quad \overline{\gamma^\mu\gamma_5} = \gamma^\mu\gamma_5, \quad \overline{i\gamma_5 S^{\mu\nu}} = i\gamma_5 S^{\mu\nu}$$

In other words, the spinorial representation is real.

- spin projection matrix S_z :

$$S_z = S^{12} = \frac{i}{2}\gamma^1\gamma^2 = \frac{1}{2} \begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix}, \quad S^i = \frac{1}{2}\epsilon^{ijk}S^{jk} = \frac{1}{2} \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}$$

The gamma matrix identities Because gamma matrices satisfy anti-commutation relations, the trace of a string of gamma matrices follows the *Wick theorem* (see, e.g., S. Weinberg, *The quantum theory of fields*, Vol. 1, 2005):

The trace of the product of gamma matrices equals the sum of all possible contractions with the corresponding permutation signatures included.

A contraction of any two gamma matrices γ_μ, γ_ν gives a factor $4g_{\mu\nu}$. If the two contracted gamma matrices are not adjacent, there would be a sign $(-1)^n$, where n is the number of exchange operations needed to makes them adjacent (but keeping their relative order).

Frequently used identities in $D = 4$ dimensions:

- $\text{tr}\{\text{product of odd number of } \gamma\text{'s}\} = \text{tr}\{\gamma_5 \cdot \text{product of odd number of } \gamma\text{'s}\} = 0$
- $\text{tr } 1 = 4, \quad \text{tr } \gamma_5 = 0$
- $\text{tr}\{\not{a}\not{b}\} = 4(a \cdot b), \quad \text{tr}\{\gamma_5\not{a}\not{b}\} = 0$
- $\text{tr}\{\not{a}\not{b}\not{c}\not{d}\} = 4((a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)), \quad \text{tr}\{\gamma_5\not{a}\not{b}\not{c}\not{d}\} = -4i\varepsilon^{\mu\nu\rho\sigma}a_\mu b_\nu c_\rho d_\sigma$
Note that, when $\mu, \nu, \rho, \sigma = +, -, 1, 2$, $\varepsilon^{\mu\nu\rho\sigma} = \epsilon(\mu, \nu, \rho, \sigma)/\sqrt{-\det g}$
- $\gamma_\mu\gamma^\mu = 4, \quad \gamma_\mu\not{a}\gamma^\mu = (2-D)\not{a}, \quad \gamma_\mu\not{a}\not{b}\gamma^\mu = 4(a \cdot b) - (4-D)\not{a}\not{b}, \quad \gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{a}\not{b} + (4-D)\not{a}\not{b}\not{c},$
 $\gamma_\mu\not{a}\not{b}\not{c}\not{d}\gamma^\mu = 2(\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d}) - (4-D)\not{a}\not{b}\not{c}\not{d};$
- $\not{a}\not{a} = a^2, \quad \not{a}\not{b} = -\not{b}\not{a} + 2(a \cdot b)$

5 Spinors

The u, v spinors are defined as,

$$\begin{aligned} u_s(p) &= \frac{1}{2\sqrt{p^+}}(\not{p} + m)\gamma^+\chi_s = \frac{1}{\sqrt{p^+}}(\not{p} + m)\beta\chi_s = \frac{1}{\sqrt{p^+}}(p^+ + \boldsymbol{\alpha}^\perp \cdot \mathbf{p}^\perp + \beta m)\chi_s; \\ v_s(p) &= \frac{1}{2\sqrt{p^+}}(\not{p} - m)\gamma^+\chi_{-s} = \frac{1}{\sqrt{p^+}}(\not{p} - m)\beta\chi_{-s} = \frac{1}{\sqrt{p^+}}(p^+ + \boldsymbol{\alpha}^\perp \cdot \mathbf{p}^\perp - \beta m)\chi_{-s}; \end{aligned} \quad (29)$$

where $\chi_+ = (1, 0, 0, 0)^\top, \chi_- = (0, 1, 0, 0)^\top$ are the basis of the two-component spinors (the dynamical spinors on the light front) and satisfy:

$$\Lambda_+\chi_s = \chi_s, \quad \Lambda_-\chi_s = 0, \quad \chi_s^\dagger\chi_{s'} = \delta_{ss'}, \quad S_z\chi_\pm = \pm\frac{1}{2}\chi_\pm. \quad (30)$$

The u, v spinors are polarized in the longitudinal direction:

$$S_z u_\pm(p^+, \mathbf{p}_\perp = 0) = \pm\frac{1}{2}u_\pm(p^+, \mathbf{p}_\perp = 0), \quad S_z v_\pm(p^+, \mathbf{p}_\perp = 0) = \mp\frac{1}{2}v_\pm(p^+, \mathbf{p}_\perp = 0). \quad (31)$$

and following the standard normalization scheme:

$$\bar{u}_s(p)u_{s'}(p) = 2m\delta_{ss'}, \quad \bar{v}_s(p)v_{s'}(p) = -2m\delta_{ss'}, \quad \bar{u}_s(p)v_{s'}(p) = \bar{v}_s(p)u_{s'}(p) = 0. \quad (32)$$

The spinor identities:

- Dirac equation:

$$(\not{p} - m)u_\sigma(p) = 0, \quad (\not{p} + m)v_\sigma(p) = 0; \quad (33)$$

- normalization:

$$\bar{u}_s(p)u_{s'}(p) = 2m\delta_{ss'}, \quad \bar{v}_s(p)v_{s'}(p) = -2m\delta_{ss'}, \quad \bar{u}_s(p)v_{s'}(p) = 0; \quad (34)$$

- spin sum:

$$\sum_{s=\pm} u_s(p)\bar{u}_s(p) = \not{p} + m, \quad \sum_{s=\pm} v_s(p)\bar{v}_s(p) = \not{p} - m; \quad (35)$$

- crossing symmetry:

$$u_s(p) = \sqrt{-1}v_{-s}(-p), \quad \bar{u}_s(p) = \sqrt{-1}\bar{v}_{-s}(-p), \quad v_s(p) = \sqrt{-1}u_{-s}(-p), \quad \bar{v}_s(p) = \sqrt{-1}\bar{u}_{-s}(-p); \quad (36)$$

Note that $p \rightarrow -p$ flips the sign of all four components of the momentum, including the light-front energy and the longitudinal momentum.

The crossing symmetry is clearer if we define $w_s(p) = \frac{1}{2p^+}(\not{p} + m)\gamma^+\chi_s = \frac{1}{\sqrt{p^+}}u_s(p)$, and $z_s(p) = \frac{1}{2p^+}(\not{p} - m)\gamma^+\chi_{-s} = \frac{1}{\sqrt{p^+}}v_s(p)$. Then the cross symmetry between w and z is $z_s(p) = w_{-s}(-p)$, $\bar{z}_s(p) = \bar{w}_{-s}(-p)$.

- Gordon identities:

$$2m\bar{u}_{s'}(p')\gamma^\mu u_s(p) = \bar{u}_{s'}(p')[(p + p')^\mu + 2iS^{\mu\nu}(p' - p)_\nu]u_s(p); \quad (37)$$

$$-2m\bar{v}_{s'}(p')\gamma^\mu v_s(p) = \bar{v}_{s'}(p')[(p + p')^\mu + 2iS^{\mu\nu}(p' - p)_\nu]v_s(p); \quad (38)$$

$$2m\bar{u}_{s'}(p')\gamma^\mu v_s(p) = \bar{u}_{s'}(p')[(p' - p)^\mu + 2iS^{\mu\nu}(p' + p)_\nu]v_s(p); \quad (39)$$

$$2m\bar{u}_{s'}(p')\gamma^\mu\gamma_5 u_s(p) = \bar{u}_{s'}(p')[(p - p')^\mu\gamma_5 + 2iS^{\mu\nu}(p' + p)_\nu\gamma_5]u_s(p); \quad (40)$$

$$0 = \bar{u}_{s'}(p')[(p' - p)^\mu + 2iS^{\mu\nu}(p' + p)_\nu]u_s(p); \quad (41)$$

$$0 = \bar{u}_{s'}(p')[(p + p')^\mu\gamma_5 + 2iS^{\mu\nu}(p' - p)_\nu\gamma_5]u_s(p). \quad (42)$$

- other useful identities:

$$\begin{aligned}
\bar{u}_s(p)\gamma^\mu u_{s'}(p) &= 2p^\mu \delta_{ss'}, \quad \bar{v}_s(p)\gamma^\mu v_{s'}(p) = 2p^\mu \delta_{ss'}; \\
\bar{u}_{s'}(p')\gamma^+\gamma_5 u_s(p) &= 2\sqrt{p^+p'^+} \delta_{ss'} \text{sign}(s); \\
\bar{u}_s(p)\gamma^+ u_{s'}(p') &= \bar{v}_s(p)\gamma^+ v_{s'}(p') = 2\sqrt{p^+p'^+} \delta_{ss'} \\
\bar{u}_s(\mathbf{p})\gamma^0 v_{s'}(-\mathbf{p}) &= 0
\end{aligned} \tag{43}$$

Spinor vertices In general, the spinor vertex can be written as,

$$\begin{aligned}
V_n &= \bar{u}_{s'}(p')\phi_1\phi_2\cdots\phi_n u(p) \\
&= \frac{1}{4\sqrt{p^+p'^+}} \chi_{s'}^\dagger \gamma^0 \gamma^+(p' + m) \phi_1\phi_2\cdots\phi_n (\not{p} + m) \gamma^+ \chi_s \\
&= \frac{1}{2\sqrt{p^+p'^+}} \chi_{s'}^\dagger \Lambda_+(p' + m) \phi_1\phi_2\cdots\phi_n (\not{p} + m) \gamma^+ \chi_s \\
&= \frac{1}{4\sqrt{p^+p'^+}} \text{tr}[(\not{p}' + m) \phi_1\phi_2\cdots\phi_n (\not{p} + m) \gamma^+ \chi_{ss'}]
\end{aligned} \tag{44}$$

Now the spinor vertex is turned into the trace of a string of gamma matrices, and

$$\chi_s \chi_{s'}^\dagger \triangleq \frac{1}{2} \chi_{ss'} = \frac{1}{2} \begin{cases} 1 + \gamma_5, & s = +, s' = + \\ 1 - \gamma_5, & s = -, s' = - \\ -\gamma^R = -\gamma^1 - i\gamma^2, & s = +, s' = - \\ \gamma^L = \gamma^1 - i\gamma^2, & s = -, s' = + \end{cases} \tag{45}$$

- scalar vertex: ($p^R = p^1 + ip^2$, $p^L = p^1 - ip^2$, $p_\mu p^\mu = m^2$)

$$\bar{u}_{s'}(p') u_s(p) = -\bar{v}_s(p) v_{s'}(p') = \sqrt{p^+p'^+} \begin{cases} m\left(\frac{1}{p^+} + \frac{1}{p'^+}\right), & s, s' = ++, -- \\ \frac{p^R}{p^+} - \frac{p'^R}{p'^+}, & s, s' = +, - \\ \frac{p'^L}{p'^+} - \frac{p^L}{p^+}, & s, s' = -, + \end{cases} \tag{46}$$

- pseudo scalar vertex: ($p^R = p^1 + ip^2$, $p^L = p^1 - ip^2$, $p_\mu p^\mu = m^2$)

$$\bar{u}_{s'}(p') \gamma_5 u_s(p) = -\bar{v}_s(p) \gamma_5 v_{s'}(p') = \sqrt{p^+p'^+} \begin{cases} m\left(\frac{1}{p^+} - \frac{1}{p'^+}\right), & s, s' = ++ \\ m\left(\frac{1}{p^+} - \frac{1}{p'^+}\right), & s, s' = -- \\ \frac{p^R}{p^+} - \frac{p'^R}{p'^+}, & s, s' = +, - \\ \frac{p^L}{p^+} - \frac{p'^L}{p'^+}, & s, s' = -, + \end{cases} \tag{47}$$

- vector vertex: ($p^R = p^1 + ip^2$, $p^L = p^1 - ip^2$, $p_\mu p^\mu = m^2$)

$$\begin{aligned}
\bar{u}_{s'}(p')\gamma^+u_s(p) &= \bar{v}_s(p)\gamma^+v_{s'}(p') = 2\sqrt{p^+p'^+}\delta_{ss'} \\
\bar{u}_{s'}(p')\gamma^-u_s(p) &= \bar{v}_s(p)\gamma^-v_{s'}(p') = \frac{2}{\sqrt{p^+p'^+}} \begin{cases} m^2 + p^R p'^L, & s, s' = +, + \\ m^2 + p^L p'^R, & s, s' = -, - \\ m(p^R - p'^R), & s, s' = +, - \\ m(p'^L - p^L), & s, s' = -, + \end{cases} \\
\bar{u}_{s'}(p')\gamma^L u_s(p) &= \bar{v}_s(p)\gamma^L v_{s'}(p') = 2\sqrt{p^+p'^+} \begin{cases} \frac{p'^L}{p'^+}, & s, s' = +, + \\ \frac{p^L}{p^+}, & s, s' = -, - \\ m\left(\frac{1}{p'^+} - \frac{1}{p^+}\right), & s, s' = +, - \\ 0, & s, s' = -, + \end{cases} \\
\bar{u}_{s'}(p')\gamma^R u_s(p) &= \bar{v}_s(p)\gamma^R v_{s'}(p') = 2\sqrt{p^+p'^+} \begin{cases} \frac{p^R}{p^+}, & s, s' = +, + \\ \frac{p'^R}{p'^+}, & s, s' = -, - \\ 0, & s, s' = +, - \\ m\left(\frac{1}{p^+} - \frac{1}{p'^+}\right), & s, s' = -, + \end{cases}
\end{aligned} \tag{48}$$

- pseudo vector:

$$\begin{aligned}
\bar{u}_{s'}(p')\gamma^+\gamma_5 u_s(p) &= -\bar{v}_s(p)\gamma^+\gamma_5 v_{s'}(p') = 2\sqrt{p^+p'^+}\delta_{ss'}\text{sign}(s) \\
\bar{u}_{s'}(p')\gamma^-\gamma_5 u_s(p) &= -\bar{v}_s(p)\gamma^-\gamma_5 v_{s'}(p') = \frac{2}{\sqrt{p^+p'^+}} \begin{cases} -m^2 + p'^L p^R, & s, s' = +, + \\ m^2 - p'^R p^L, & s, s' = -, - \\ m(p^R + p'^R), & s, s' = +, - \\ m(p^L + p'^L), & s, s' = -, + \end{cases} \\
\bar{u}_{s'}(p')\gamma^L \gamma_5 u_s(p) &= -\bar{v}_s(p)\gamma^L \gamma_5 v_{s'}(p') = 2\sqrt{p^+p'^+} \begin{cases} \frac{p'^L}{p'^+}, & s, s' = +, + \\ -\frac{p^L}{p^+}, & s, s' = -, - \\ m\left(\frac{1}{p^+} + \frac{1}{p'^+}\right), & s, s' = +, - \\ 0, & s, s' = -, + \end{cases} \\
\bar{u}_{s'}(p')\gamma^R \gamma_5 u_s(p) &= -\bar{v}_s(p)\gamma^R \gamma_5 v_{s'}(p') = 2\sqrt{p^+p'^+} \begin{cases} \frac{p^R}{p^+}, & s, s' = +, + \\ -\frac{p'^R}{p'^+}, & s, s' = -, - \\ 0, & s, s' = +, - \\ m\left(\frac{1}{p^+} + \frac{1}{p'^+}\right), & s, s' = -, + \end{cases}
\end{aligned} \tag{49}$$

6 Polarization of massless vector bosons

Define the polarization vector:

$$\varepsilon_\lambda^\mu(k) = (\varepsilon_\lambda^+, \varepsilon_\lambda^-, \varepsilon_\lambda^\perp) = (0, 2\frac{\epsilon_\lambda^\perp \cdot k^\perp}{k^+}, \epsilon_\lambda^\perp), \quad (\lambda = \pm 1) \tag{50}$$

where $\epsilon_\pm^\perp = \frac{1}{\sqrt{2}}(1, \pm i)$. In fact, $\varepsilon_\lambda^L = \sqrt{2}\delta_{\lambda,+}$, $\varepsilon_\lambda^R = \sqrt{2}\delta_{\lambda,-}$. This definition satisfies the light-cone gauge $\omega \cdot A = A^+ = 0$ and Lorenz condition $\partial_\mu A^\mu = 0$,

$$\omega_\mu \varepsilon_\lambda^\mu(k) = \varepsilon_\lambda^+(k) = 0, \quad k_\mu \varepsilon_\lambda^\mu(k) = 0. \tag{51}$$

There is another set of conventions that define $\epsilon_\pm^\perp = -\frac{1}{\sqrt{2}}(1, \pm i)$.

Polarization identities:

- orthogonality:

$$\varepsilon_\lambda^\mu(k) \varepsilon_{\lambda'\mu}^*(k) = -\delta_{\lambda,\lambda'}; \quad (52)$$

- helicity sum:

$$\sum_{\lambda=\pm} \varepsilon_\lambda^{i*}(k) \varepsilon_\lambda^j(k) = \delta^{ij}, \quad d^{\mu\nu} \equiv \sum_{\lambda=\pm} \varepsilon_\lambda^{\mu*}(k) \varepsilon_\lambda^\nu(k) = -g^{\mu\nu} + \frac{\omega^\mu k^\nu + \omega^\nu k^\mu}{\omega \cdot k} - \omega^\mu \omega^\nu \frac{k^2}{(\omega \cdot k)^2}. \quad (53)$$

In particular, if k is on-shell, i.e. $k^2 = 0$, the second identity is reduced to

$$\sum_{\lambda=\pm} \varepsilon_\lambda^{\mu*}(k) \varepsilon_\lambda^\nu(k) = -g^{\mu\nu} + \frac{\omega^\mu k^\nu + \omega^\nu k^\mu}{\omega \cdot k}. \quad (54)$$

- crossing symmetry:

$$\varepsilon_\lambda^{\mu*}(k) = \varepsilon_{-\lambda}^\mu(k) = \varepsilon_{-\lambda}^\mu(-k) \quad (55)$$

where $\omega^\mu = (1, 0, 0, -1)$, is the null normal vector of light-front, $\omega \cdot \omega = 0$, $\omega \cdot v = v^+$.

7 Spin vector of massive vector bosons

Define the spin vector for the massive vector bosons:

$$e_\lambda(k) = (e_\lambda^+(k), e_\lambda^-(k), e_\lambda^\perp(k)) = \begin{cases} \left(\frac{k^+}{m}, \frac{k_\perp^2 - m^2}{mk^+}, \frac{k^\perp}{m} \right), & \lambda = 0 \\ \left(0, 2\frac{\epsilon_\lambda^\perp \cdot k^\perp}{k^+}, \epsilon_\lambda^\perp \right), & \lambda = \pm 1 \end{cases} \quad (56)$$

where $\epsilon_\pm^\perp = \frac{1}{\sqrt{2}}(1, \pm i)$, and $\epsilon_\pm^{\perp*} = \epsilon_\mp^\perp$, $m^2 = k_\mu k^\mu \triangleq k^2$ is the mass of the particle¹.

Spin identities:

- Proca equation, $k_\mu e_\lambda^\mu(k) = 0$.

- orthogonality:

$$e_\lambda^\mu(k) e_{\lambda'\mu}^*(k) = -\delta_{\lambda,\lambda'}; \quad (57)$$

- spin sum:

$$K^{\mu\nu} \equiv \sum_{\lambda=-1}^{+1} e_\lambda^{\mu*}(k) e_\lambda^\nu(k) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}. \quad (58)$$

$$k^\mu K_{\mu\nu}(k) = k^\mu k^\nu K_{\mu\nu}(k) = 0.$$

- crossing symmetry:

$$e_\lambda^{\mu*}(k) = e_{-\lambda}^\mu(k), \quad e_\lambda^\mu(-k) = (-1)^{\lambda+1} e_\lambda^\mu(k); \quad (59)$$

¹There is another set of conventions that define $\epsilon_\pm^\perp = -\frac{1}{\sqrt{2}}(1, \pm i)$. One should be careful about the consistency of the conventions one chooses. For example, under the definition of $\epsilon_\pm^\perp = -\frac{1}{\sqrt{2}}(1, \pm i)$ the eigenvalue of the mirror parity (light-front parity) operator $\hat{m}_P = \hat{R}_x(\pi)\hat{P}$ of a vector state is $\hat{m}_P|p^+, p^1, p^2, j, m_j\rangle = (-i)^{2j} P|p^+, -p^1, p^2, j, -m_j\rangle$. But if one use our definition $\epsilon_\pm^\perp = \frac{1}{\sqrt{2}}(1, \pm i)$, there will be an extra minus sign, i.e., $\hat{m}_P|p^+, p^1, p^2, j, m_j\rangle = -(-i)^{2j} P|p^+, -p^1, p^2, j, -m_j\rangle$.