Spinors on the light front

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These notes define a set of conventions in light-front quantum field theory. Similar conventions for light-front dynamics, can be found in, e.g.,

- A. Harindranath: *Light front QCD: lecture notes* (2005);
- M. Burkardt: *Light Front Quantization*, Adv. Nucl. Phys. **23**, 1 (2002) [arXiv:hep-ph/9505259];
- G. P. Lepage, S. J. Brodsky, *Exclusive processes in perturbative quantum chromodynamics*, Phys. Rev. D **22**, 2157 (1980);
- S. J. Brodsky, H.-C. Pauli , S. S. Pinsky: *Quantum chromodynamics and other field theories on the light cone*, Phys. Rep. **301**, 299 (1998);
- J. Carbonell , B. Desplanques , V. A. Karmanov and J.-F. Mathiot: Phys. Rep. **300**, 215 (1998).

Throughout the notes, we use natural units, $\hbar = c = 1$. Let $x = (x^0, x^1, x^2, x^3) = (t, \boldsymbol{x})$ be the standard space-time coordinates. The signature of *Minkowski* space metric tensor is $g_{\mu\nu} = \mathrm{diag}\{+1, -1, -1, -1\}$.

1 Light-Front coordinates

The light-front coordinates are defined as (x^+, x^-, x^1, x^2), where $x^+ = x^0 + x^3$ is the light-front time, $x^- = x^0 - x^3$ is the longitudinal coordinate, $\boldsymbol{x}^\perp = (x^1, x^2)$ are the transverse coordinates. The corresponding metric tensor and its inverse is,

$$g_{\mu\nu} = \begin{pmatrix} & \frac{1}{2} & & \\ \frac{1}{2} & & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} & 2 & & \\ 2 & & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \tag{1}$$

Note that $\sqrt{-\det g} = \frac{1}{2}$. The Levi-Civita tensor should be defined as

$$\varepsilon^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-\det g}} \begin{pmatrix} \mu & \nu & \rho & \sigma \\ - & + & 1 & 2 \end{pmatrix} = \begin{cases} +2 & \text{if } \mu,\nu,\rho,\sigma \text{ is an even permutation of } -,+,1,2 \\ -2 & \text{if } \mu,\nu,\rho,\sigma \text{ is an odd permutation of } -,+,1,2 \\ 0 & \text{other cases.} \end{cases} \tag{2}$$

Similarly, the light-front components of a 4-vector $v = (v^0, \boldsymbol{v})$ is $(v^+, v^-, \boldsymbol{v}^\perp)$, where $v^\pm = v^0 \pm v^3$ and $\boldsymbol{v}^\perp = (v^1, v^2)$. Sometimes it is also useful to introduce the complex representation for the transverse vector $\boldsymbol{v}^\perp$: $v^L = v^1 - \mathrm{i}v^2$, and $v^R = v^1 + \mathrm{i}v^2 = (v^L)^*$. The component of the contravariant 4-vector $v_\mu = g_{\mu\nu}v^\nu$ are: $(v_-, v_+, \boldsymbol{v}_\perp)$, where $v_\pm = \frac{1}{2}(v_0 \pm v_3) = \frac{1}{2}v^\mp$, $\boldsymbol{v}_\perp = -\boldsymbol{v}^\perp$.

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It is useful to introduce two vectors to symbolically restore the covariance: $\omega = (\omega^0, \boldsymbol{\omega}) = (1, 0, 0, -1)$, and $\eta = (\eta^0, \boldsymbol{\eta}) = (0, 1, 0, 0)$. They satisfy

$$\omega_\mu \omega^\mu = 0,\ \eta_\mu \eta^\mu = -1,\ \eta_\mu \omega^\mu = 0, \quad (\boldsymbol{\omega}^2 = \boldsymbol{\eta}^2 = 1). \tag{3}$$

Then, the longitudinal coordinate of a vector a can be written as $a^+ = \omega \cdot a$. Similarly, the transverse component of it becomes $\boldsymbol{a}^\perp = \boldsymbol{a} - \boldsymbol{\omega}(\boldsymbol{\omega} \cdot \boldsymbol{a})$.

2 Normalization

The coordinate space integration measure is defined as

$$\int \mathrm{d}^3 x \equiv \int \mathrm{d}x_+ \mathrm{d}^2 x^\perp = \frac{1}{2} \int \mathrm{d}x^- \mathrm{d}^2 x^\perp. \tag{4}$$

The full four-dimensional integration measure is,

$$\int \mathrm{d}^4 x = \int \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 = \frac{1}{2} \int \mathrm{d}x^+ \mathrm{d}x^- \mathrm{d}^2 x^\perp = \int \mathrm{d}^3 x\, \mathrm{d}x^+. \tag{5}$$

In the momentum space, we use the Lorentz invariant integration measure:

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \theta(p^+) 2\pi \delta(p^2 - m^2) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0} \theta(p^0) = \int \frac{\mathrm{d}^2 p_\perp \mathrm{d}p^+}{(2\pi)^3 2p^+} \theta(p^+) = \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^3} \int_0^1 \frac{\mathrm{d}x}{2x} \tag{6}$$

where $p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$ is the on-shell energy and $x = p^+/P^+$ is the longitudinal momentum fraction. The corresponding normalization of the single-particle state is

$$\begin{aligned} \langle p, \sigma | p', \sigma' \rangle &= 2p^0 \theta(p^0) (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p}') \delta_{\sigma\sigma'} = (2\pi)^3 \delta^4(p - p')/\theta(p^0)\delta(p^2 - m^2) \\ &= 2p^+ \theta(p^+) (2\pi)^3 \delta^3(p - p') \delta_{\sigma\sigma'} = 2x (2\pi)^3 \delta(x - x') \delta^2(\boldsymbol{p}_\perp - \boldsymbol{p}'_\perp) \delta_{\sigma\sigma'}. \end{aligned} \tag{7}$$

Here the light-front delta function is defined as $\delta^3(p) = \delta^2(\boldsymbol{p}_\perp)\delta(p^+)$.

The transverse Fourier transformation and its inverse transformation are defined as,

$$\widetilde{f}(\boldsymbol{r}_\perp) \equiv \int \frac{\mathrm{d}^2 p_\perp}{(2\pi)^2} e^{i\boldsymbol{p}_\perp \cdot \boldsymbol{r}_\perp} f(\boldsymbol{p}_\perp), \quad f(\boldsymbol{p}_\perp) \equiv \int \mathrm{d}^2 r_\perp\, e^{-i\boldsymbol{p}_\perp \cdot \boldsymbol{r}_\perp} \widetilde{f}(\boldsymbol{r}_\perp). \tag{8}$$

2.1 Box regularization

In a typical box normalization, the system is confined in a box with length L:

$$-\tfrac{L}{2} \le x \le +\tfrac{L}{2}. \tag{9}$$

Here x is the coordinate of any one spatial dimension. Boundary conditions, such as periodic boundary condition or anti-periodic boundary condition, may apply. The conjugate moment is discretized. In the case of the periodic boundary condition, it becomes, $p = 2\pi n/L$, $(n = 0, \pm 1, \pm 2, \cdots)$. The conversion of the integrations and δ-functions are listed in Table 1. For example,

$$\begin{aligned} \int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-i(p-p')x} &= 2\pi\delta(p - p'), \quad \int_{-\frac{1}{2}L}^{+\frac{1}{2}L} \mathrm{d}x\, e^{-i(p-p')x} = L\delta_{p,p'}. \\ \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} e^{ip(x-x')} &= \delta(x - x'), \quad \frac{1}{L} \sum_p e^{ip(x-x')} = \delta(x - x'). \end{aligned} \tag{10}$$

In light-front dynamics, the spectral condition requires that the longitudinal momentum is always positive. The box regularization of the longitudinal momentum can be implemented using the same

Table 1: Conversion formula for the box regularization.

box regularization	$\stackrel{L\to\infty}{\longrightarrow}$	continuum
$\int_{-\frac{1}{2}L}^{+\frac{1}{2}L} \mathrm{d}x$		$\int_{-\infty}^{+\infty} \mathrm{d}x$
$\frac{1}{L}\sum_p$	$\stackrel{L\to\infty}{\longrightarrow}$	$\int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi}$
$\delta(x-x')$		$\delta(x-x')$
$L\delta_{p,p'}$		$2\pi\delta(p-p')$

method above but with an addition Heaviside θ-function to impose the positivity of the longitudinal momentum: $\theta(p^+)$. Note also that the longitudinal momentum is conjugate to $x_+ = \frac{1}{2}x^-$. For example,

$$\begin{aligned} \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}x^- \, e^{-\frac{i}{2}(p^+-p'^+)x^-} &= 2\pi\delta(p^+-p'^+), \quad \frac{1}{2}\int_{-L}^{+L} \mathrm{d}x^- \, e^{-\frac{i}{2}(p^+-p'^+)x} = L\delta_{p^+,p'^+}. \\ \int_{-\infty}^{+\infty} \frac{\mathrm{d}p^+}{2\pi} \theta(p^+) e^{\frac{i}{2}p^+(x^- - x'^-)} &= 2\delta(x^- - x'^-), \quad \frac{1}{L}\sum_{p^+} \theta(p^+) e^{\frac{i}{2}p^+(x^- - x'^-)} = 2\delta(x^- - x'^-). \end{aligned} \tag{11}$$

2.2 Longitudinal propagators

$$\frac{1}{\partial^+} f(x^-) \equiv \frac{1}{4}\int_{-\infty}^{+\infty} \mathrm{d}y^- \, \big[\theta(x^- - y^-) - \theta(y^- - x^-)\big] f(y^-). \tag{12}$$

This propagator is the inverse of the derivative operator and satisfies the anti-periodic boundary condition in the longitudinal direction:

$$\partial^+ \left(\frac{1}{\partial^+} f\right)(x^-) = f(x^-), \quad \left(\frac{1}{\partial^+} f\right)(-\infty) = -\left(\frac{1}{\partial^+} f\right)(+\infty). \tag{13}$$

$$\frac{1}{(\partial^+)^2} f(x^-) \equiv \frac{1}{8}\int_{-\infty}^{+\infty} \mathrm{d}y^- \, \big|x^- - y^-\big| f(y^-). \tag{14}$$

This propagator is the inverse of the double-derivative operator and satisfies the anti-periodic boundary condition in the longitudinal direction:

$$\big(\partial^+\big)^2 \left(\frac{1}{(\partial^+)^2} f\right)(x^-) = f(x^-), \quad \left(\frac{1}{(\partial^+)^2} f\right)(-\infty) = -\left(\frac{1}{(\partial^+)^2} f\right)(+\infty). \tag{15}$$

3 Kinematics

Two-Body kinematics Let $\boldsymbol{P}_\perp = \boldsymbol{p}_{1\perp} + \boldsymbol{p}_{2\perp}$, $P^+ = p_1^+ + p_2^+$ be the c.m. momentum of two on-shell particles with 4-momentum p_1, p_2, respectively ($p_a^2 = m_a^2$). Define the longitudinal momentum fraction $x_a = p_a^+/P^+$, ($x_1 + x_2 = 1$), and relative transverse momentum $\boldsymbol{p}_\perp = \boldsymbol{p}_{1\perp} - x_1 \boldsymbol{P}_\perp$ ($-\boldsymbol{p}_\perp = \boldsymbol{p}_{2\perp} - x_2\boldsymbol{P}_\perp$). Then, the momentum space integration measure admits a factorization:

$$\int \frac{\mathrm{d}^2 p_1^\perp \mathrm{d}p_1^+}{(2\pi)^3 2p_1^+} \int \frac{\mathrm{d}^2 p_2^\perp \mathrm{d}p_2^+}{(2\pi)^3 2p_2^+} = \int \frac{\mathrm{d}^2 p_1^\perp \mathrm{d}x_1}{(2\pi)^3 2x_1} \int \frac{\mathrm{d}^2 p_2^\perp \mathrm{d}x_2}{(2\pi)^3 2x_2} = \int \frac{\mathrm{d}^2 P^\perp \mathrm{d}P^+}{(2\pi)^3 2P^+} \int \frac{\mathrm{d}^2 p^\perp \mathrm{d}x}{(2\pi)^3 2x(1-x)}. \tag{16}$$

where $x = x_1$. Similarly, the two-body Fock state

$$\begin{aligned}\langle p_1', p_2' | p_1, p_2\rangle =& 2x_1\theta(p_1^+)(2\pi)^3\delta^2(p_1^\perp - p_1'^\perp)\delta(x_1 - x_1')2x_2\theta(p_2^+)(2\pi)^3\delta^2(p_2^\perp - p_2'^\perp)\delta(x_2 - x_2') \\ =& 2P^+\theta(P^+)(2\pi)^3\delta^3(P - P')2x(1-x)(2\pi)^3\delta^2(p_\perp - p'_\perp)\delta(x - x') \\ =& \langle P'; \boldsymbol{p}'^\perp, x' | P; \boldsymbol{p}^\perp, x\rangle.\end{aligned} \tag{17}$$

Furthermore, the two-body invariant mass squared is,

$$s_2 \equiv (p_1 + p_2)^2 = \frac{\boldsymbol{p}_{1\perp}^2 + m_1^2}{x_1} + \frac{\boldsymbol{p}_{2\perp}^2 + m_2^2}{x_2} - \boldsymbol{P}_\perp^2 = \frac{\boldsymbol{p}_\perp^2 + m_1^2}{x} + \frac{\boldsymbol{p}_\perp^2 + m_2^2}{1-x}. \tag{18}$$

here m_a is the a-th particle's mass.

Few-Body kinematics Define the few-body c.m. momentum $\boldsymbol{P}_\perp = \sum_a \boldsymbol{p}_{a\perp}$, $P^+ = \sum_a p_a^+$. Introduce the momentum fraction and the relative transverse momentum:

$$x_a = p_a^+/P^+, \quad \boldsymbol{k}_{a\perp} = \boldsymbol{p}_{a\perp} - x_a\boldsymbol{P}_\perp \tag{19}$$

Then, it is clear that $\sum_a x_a = 1$, and $\sum_a \boldsymbol{k}_{a\perp} = 0$. The few-body momentum space integration measure admits a factorization of the c.m. momentum:

$$\prod_a \int \frac{\mathrm{d}^2p_a^\perp \mathrm{d}p_a^+}{(2\pi)^3 2p_a^+}\theta(p_a^+) = \int \frac{\mathrm{d}^2P^\perp \mathrm{d}P^+}{(2\pi)^3 2P^+}\theta(P^+)\prod_a \int \frac{\mathrm{d}^2k_a^\perp \mathrm{d}x_a}{(2\pi)^3 2x_a} \times 2(2\pi)^3\delta^2\Big(\sum_a \boldsymbol{k}_{a\perp}\Big)\delta\Big(\sum_a x_a - 1\Big) \tag{20}$$

The few-body invariant mass squared is,

$$s_n \equiv \Big(\sum_a p_a\Big)^2 = \sum_a \frac{\boldsymbol{k}_{a\perp}^2 + m_a^2}{x_a} \tag{21}$$

Lemma *cluster decomposition of s_n:*

Let $(x_a, \boldsymbol{k}_{a\perp})$, $(a = 1, 2\cdots, n)$ be n relative momenta, i.e. $\sum_a x_a = 1$, $\sum_a \boldsymbol{k}_{a\perp} = 0$. Define new relative momenta with respect to the cluster without the n-th particle: $\zeta_a = x_a/(1-x_n)$, $\boldsymbol{\kappa}_{a\perp} = \boldsymbol{k}_{a\perp} + \zeta_a\boldsymbol{k}_{n\perp}$. Then, the n-body invariant mass squared can be written as,

$$(1-x_n)\Big(\sum_{a=1}^{n} \frac{\boldsymbol{k}_{a\perp}^2 + m_a^2}{x_a} - M^2\Big) = \sum_{a=1}^{n-1} \frac{\boldsymbol{\kappa}_{a\perp}^2 + m_a^2}{\zeta_a} - m_n^2 + (1-x_n)\Big(\frac{\boldsymbol{k}_{n\perp}^2 + m_n^2}{x_n(1-x_n)} - M^2\Big), \tag{22}$$

or in short form,

$$(1-x_n)(s_n - M^2) = s_{n-1}^r - \big(m_n^2 - (1-x_n)(s_2 - M^2)\big). \tag{23}$$

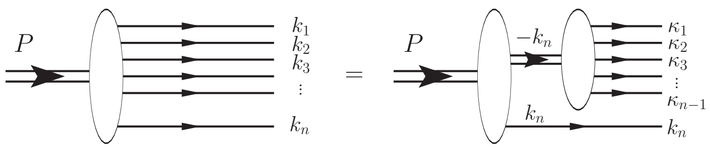

Figure 1: Cluster decomposition of the few-body invariant mass squared.

Lemma *cluster decomposition of s*:

Let $(x_i, \boldsymbol{k}_{i\perp})$ be relative momenta $(i = 1, 2\cdots)$, i.e. $\sum_i x_i = 1$, $\sum_i \boldsymbol{k}_{i\perp} = 0$. Partition the system into two clusters A and B. Let $x_A = \sum_{i\in A} x_i, \boldsymbol{k}_{A\perp} = \sum_{i\in A} \boldsymbol{k}_{i\perp}, x_B = \sum_{i\in B} x_i, \boldsymbol{k}_{B\perp} = \sum_{i\in B} \boldsymbol{k}_{i\perp}$. Obviously, $\boldsymbol{k}_{A\perp} + \boldsymbol{k}_{B\perp} = 0$, $x_A + x_B = 1$.

Define new relative momenta with respect to the cluster: $\zeta_{iA} = x_i/x_A$, $\boldsymbol{\kappa}_{iA\perp} = \boldsymbol{k}_{i\perp} - \zeta_{iA}\boldsymbol{k}_{A\perp}$. $\zeta_{iB} = x_i/x_B$, $\boldsymbol{\kappa}_{iB\perp} = \boldsymbol{k}_{i\perp} - \zeta_{iB}\boldsymbol{k}_{B\perp}$. Introduce the invariant masses $s_A = \sum_{i\in A} \frac{\boldsymbol{\kappa}^2_{iA\perp}+m_i^2}{\zeta_{iA}}$, $s_B = \sum_{i\in B} \frac{\boldsymbol{\kappa}^2_{iB\perp}+m_i^2}{\zeta_{iB}}$.

Then, the total invariant mass squared $s \equiv s_{A+B} = \sum_i \frac{\boldsymbol{k}^2_{i\perp}+m_i^2}{x_i}$ can be written as,

$$\sum_i \frac{\boldsymbol{k}^2_{i\perp} + m_i^2}{x_i} = \frac{\boldsymbol{k}^2_{A\perp} + s_A}{x_A} + \frac{\boldsymbol{k}^2_{A\perp} + s_B}{1 - x_A} \tag{24}$$

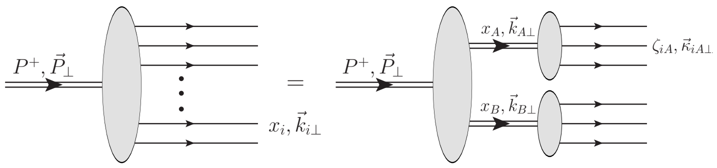

Figure 2: Cluster decomposition of the few-body invariant mass squared.

4 gamma matrices

In this convention, the 4-by-4 gamma matrices are defined as (cf. Dirac and chiral representation):

$$\gamma^0 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} -\mathrm{i}\sigma^2 & 0 \\ 0 & \mathrm{i}\sigma^2 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} \mathrm{i}\sigma^1 & 0 \\ 0 & -\mathrm{i}\sigma^1 \end{pmatrix} \tag{25}$$

where $\sigma = (1, \boldsymbol{\sigma})$ are the standard Pauli matrices,

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{26}$$

The γ-matrices defined here furnish a representation of the Clifford algebra $\mathrm{C}\ell_{1,3}(\mathbb{R})$:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{27}$$

Then, $S^{\mu\nu} \equiv \frac{\mathrm{i}}{4}[\gamma^\mu, \gamma^\nu]$ furnishes a spinorial representation of the Lorentz group. It is convenient to introduce the following 4-by-4 matrices,

- front-form: $\gamma^\pm \equiv \gamma^0 \pm \gamma^3$, $\boldsymbol{\gamma}^\perp \equiv (\gamma^1, \gamma^2)$, $\gamma^L \equiv \gamma^1 - \mathrm{i}\gamma^2$, $\gamma^R \equiv \gamma^1 + \mathrm{i}\gamma^2$, $\not{p} = p_\mu\gamma^\mu = \frac{1}{2}p^+\gamma^- + \frac{1}{2}p^-\gamma^+ - \boldsymbol{p}_\perp \cdot \boldsymbol{\gamma}_\perp$

  corollaries: $\gamma^+\gamma^+ = 0; \gamma^-\gamma^- = 0; \gamma^+\gamma^-\gamma^+ = 4\gamma^+, \gamma^-\gamma^+\gamma^- = 4\gamma^-; \gamma^0\gamma^\pm = \gamma^\mp\gamma^0$. The matrix form are:

  $$\gamma^+ = \begin{pmatrix} 0 & 0 \\ 2\mathrm{i} & 0 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} 0 & -2\mathrm{i} \\ 0 & 0 \end{pmatrix}, \quad \gamma^L = \begin{pmatrix} \sigma^L & 0 \\ 0 & -\sigma^L \end{pmatrix}, \quad \gamma^R = \begin{pmatrix} -\sigma^R & 0 \\ 0 & \sigma^R \end{pmatrix},$$

  where,

  $$\sigma^L = \sigma^1 - \mathrm{i}\sigma^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \quad \sigma^R = \sigma^1 + \mathrm{i}\sigma^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

- projections: $\Lambda^{\pm} \equiv \Lambda_{\pm} \equiv \frac{1}{2}\gamma^0\gamma^{\pm}$;
  corollaries: $\Lambda_{\pm}^2 = \Lambda_{\pm}$, $\Lambda^+\Lambda^- = 0$, $\Lambda^-\Lambda^+ = 0$, $\Lambda^+ + \Lambda^- = 1$.
  $\Lambda_{\pm}^{\dagger} = \Lambda_{\pm}$, $\overline{\Lambda}_{\pm} = \Lambda_{\mp}$, $\frac{1}{4}\gamma^+\gamma^- = \Lambda^-$, $\frac{1}{4}\gamma^-\gamma^+ = \Lambda^+$.
  Under the convention we use, the projections are diagonal and simple:
  $$\Lambda^+ = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \quad \Lambda^- = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$$
- parity matrix: $\beta = \gamma^0$; charge conjugation matrix: $\mathcal{C} = -\mathrm{i}\gamma^2$; time reversal matrix: $T = \gamma^1\gamma^3 = \mathcal{C}\gamma_5$
- chiral matrix: $\gamma^5 = \gamma_5 \equiv \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{\mathrm{i}}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$ is diagonal:
  $$\gamma^5 = \begin{pmatrix} \sigma^3 & \\ & -\sigma^3 \end{pmatrix}, \quad P_L = \frac{1}{2}\begin{pmatrix} \sigma^- & \\ & \sigma^+ \end{pmatrix} \quad P_R = \frac{1}{2}\begin{pmatrix} \sigma^+ & \\ & \sigma^- \end{pmatrix},$$
  where $P_L = \frac{1}{2}(1-\gamma_5) = \mathrm{diag}\{0,1,1,0\}$, $P_R = \frac{1}{2}(1+\gamma_5) = \mathrm{diag}\{1,0,0,1\}$ are the two chiral projections, also diagonal. It is easy to see $P_L^2 = P_L$, $P_R^2 = P_R$, $P_L P_R = P_R P_L = 0$, $P_L + P_R = 1$.
  σ-identity:
  $$iS^{\mu\nu}\gamma_5 = -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}S_{\rho\sigma} \tag{28}$$
- $\bar{\psi} \equiv \psi^\dagger\beta$ (for spinorial vector) and $\bar{A} = \beta A^\dagger \beta$ (for spinorial matrix).
  Overbar identities:
  $$\overline{\gamma^\mu} = \gamma^\mu, \quad \overline{S^{\mu\nu}} = S^{\mu\nu}, \quad \overline{\mathrm{i}\gamma_5} = i\gamma_5, \quad \overline{\gamma^\mu\gamma_5} = \gamma^\mu\gamma_5, \quad \overline{\mathrm{i}\gamma_5 S^{\mu\nu}} = \mathrm{i}\gamma_5 S^{\mu\nu}$$
  In other words, the spinorial representation is real.
- spin projection matrix S_z:
  $$S_z = S^{12} = \frac{\mathrm{i}}{2}\gamma^1\gamma^2 = \frac{1}{2}\begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix}, \quad S^i = \frac{1}{2}\epsilon^{ijk}S^{jk} = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i}\sigma^i \\ \mathrm{i}\sigma^i & \end{pmatrix}$$

The gamma matrix identities Because gamma matrices satisfy anti-commutation relations, the trace of a string of gamma matrices follows the *Wick theorem* (see, e.g., S. Weinberg, *The quantum theory of fields*, Vol. 1, 2005):

> The trace of the product of gamma matrices equals the sum of all possible contractions with the corresponding permutation signatures included.

A contraction of any two gamma matrices γ_μ, γ_ν gives a factor $4g_{\mu\nu}$. If the two contracted gamma matrices are not adjacent, there would be a sign $(-1)^n$, where n is the number of exchange operations needed to makes them adjacent (but keeping their relative order).

Frequently used identities in $D = 4$ dimensions:

- $\mathrm{tr}\{\text{product of odd number of } \gamma\text{'s}\} = \mathrm{tr}\{\gamma_5 \cdot \text{ product of odd number of } \gamma\text{'s}\} = 0$
- $\mathrm{tr}\,1 = 4, \quad \mathrm{tr}\,\gamma_5 = 0$
- $\mathrm{tr}\{\not{a}\not{b}\} = 4(a\cdot b), \quad \mathrm{tr}\{\gamma_5\not{a}\not{b}\} = 0$
- $\mathrm{tr}\{\not{a}\not{b}\not{c}\not{d}\} = 4\left((a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)\right), \quad \mathrm{tr}\{\gamma_5\not{a}\not{b}\not{c}\not{d}\} = -4\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}a_\mu b_\nu c_\rho d_\sigma$
  Note that, when $\mu,\nu,\rho,\sigma = +,-,1,2$, $\varepsilon^{\mu\nu\rho\sigma} = \epsilon(\mu,\nu,\rho,\sigma)/\sqrt{-\det g}$
- $\gamma_\mu\gamma^\mu = 4, \quad \gamma_\mu\not{a}\gamma^\mu = (2-D)\not{a}, \quad \gamma_\mu\not{a}\not{b}\gamma^\mu = 4(a\cdot b) - (4-D)\not{a}\not{b}, \quad \gamma_\mu\not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a} + (4-D)\not{a}\not{b}\not{c}$,
  $\gamma_\mu\not{a}\not{b}\not{c}\not{d}\gamma^\mu = 2\big(\not{d}\not{a}\not{b}\not{c} + \not{c}\not{b}\not{a}\not{d}\big) - (4-D)\not{a}\not{b}\not{c}\not{d}$;
- $\not{a}\not{a} = a^2, \quad \not{a}\not{b} = -\not{b}\not{a} + 2(a\cdot b)$

5 Spinors

The u, v spinors are defined as,

$$\begin{aligned} u_s(p) &= \frac{1}{2\sqrt{p^+}}(\not{p}+m)\gamma^+\chi_s = \frac{1}{\sqrt{p^+}}(\not{p}+m)\beta\chi_s = \frac{1}{\sqrt{p^+}}(p^+ + \boldsymbol{\alpha}^\perp\cdot\boldsymbol{p}^\perp + \beta m)\chi_s; \\ v_s(p) &= \frac{1}{2\sqrt{p^+}}(\not{p}-m)\gamma^+\chi_{-s} = \frac{1}{\sqrt{p^+}}(\not{p}-m)\beta\chi_{-s} = \frac{1}{\sqrt{p^+}}(p^+ + \boldsymbol{\alpha}^\perp\cdot\boldsymbol{p}^\perp - \beta m)\chi_{-s}; \end{aligned} \tag{29}$$

where $\chi_+ = (1,0,0,0)^\intercal, \chi_- = (0,1,0,0)^\intercal$ are the basis of the two-component spinors (the dynamical spinors on the light front) and satisfy:

$$\Lambda_+\chi_s = \chi_s, \quad \Lambda_-\chi_s = 0, \quad \chi_s^\dagger\chi_{s'} = \delta_{ss'}, \quad S_z\,\chi_\pm = \pm\frac{1}{2}\chi_\pm. \tag{30}$$

The u, v spinors are polarized in the longitudinal direction:

$$S_z u_\pm(p^+,\boldsymbol{p}_\perp=0) = \pm\tfrac{1}{2}u_\pm(p^+,\boldsymbol{p}_\perp=0), \quad S_z v_\pm(p^+,\boldsymbol{p}_\perp=0) = \mp\tfrac{1}{2}v_\pm(p^+,\boldsymbol{p}_\perp=0). \tag{31}$$

and following the standard normalization scheme:

$$\bar{u}_s(p)u_{s'}(p) = 2m\delta_{ss'}, \quad \bar{v}_s(p)v_{s'}(p) = -2m\delta_{ss'}, \quad \bar{u}_s(p)v_{s'}(p) = \bar{v}_s(p)u_{s'}(p) = 0. \tag{32}$$

The spinor identities:

- Dirac equation:

$$(\not{p}-m)u_\sigma(p) = 0, \quad (\not{p}+m)v_\sigma(p) = 0; \tag{33}$$

- normalization:

$$\bar{u}_s(p)u_{s'}(p) = 2m\delta_{ss'}, \quad \bar{v}_s(p)v_{s'}(p) = -2m\delta_{ss'}, \quad \bar{u}_s(p)v_{s'}(p) = 0; \tag{34}$$

- spin sum:

$$\sum_{s=\pm} u_s(p)\bar{u}_s(p) = \not{p}+m, \quad \sum_{s=\pm} v_s(p)\bar{v}_s(p) = \not{p}-m; \tag{35}$$

- crossing symmetry:

$$u_s(p) = \sqrt{-1}v_{-s}(-p),\ \bar{u}_s(p) = \sqrt{-1}\bar{v}_{-s}(-p),\ v_s(p) = \sqrt{-1}u_{-s}(-p),\ \bar{v}_s(p) = \sqrt{-1}\bar{u}_{-s}(-p); \tag{36}$$

  Note that $p \to -p$ flips the sign of all four components of the momentum, including the light-front energy and the longitudinal momentum.
  The crossing symmetry is clearer if we define $w_s(p) = \frac{1}{2p^+}(\not{p}+m)\gamma^+\chi_s = \frac{1}{\sqrt{p^+}}u_s(p)$, and $z_s(p) = \frac{1}{2p^+}(\not{p}-m)\gamma^+\chi_{-s} = \frac{1}{\sqrt{p^+}}v_s(p)$. Then the cross symmetry between w and z is $z_s(p) = w_{-s}(-p), \bar{z}_s(p) = \bar{w}_{-s}(-p)$.

- Gordon identities:

$$2m\bar{u}_{s'}(p')\gamma^\mu u_s(p) = \bar{u}_{s'}(p')\big[(p+p')^\mu + 2\mathrm{i}S^{\mu\nu}(p'-p)_\nu\big]u_s(p); \tag{37}$$

$$-2m\bar{v}_{s'}(p')\gamma^\mu v_s(p) = \bar{v}_{s'}(p')\big[(p+p')^\mu + 2\mathrm{i}S^{\mu\nu}(p'-p)_\nu\big]v_s(p); \tag{38}$$

$$2m\bar{u}_{s'}(p')\gamma^\mu v_s(p) = \bar{u}_{s'}(p')\big[(p'-p)^\mu + 2\mathrm{i}S^{\mu\nu}(p'+p)_\nu\big]v_s(p); \tag{39}$$

$$2m\bar{u}_{s'}(p')\gamma^\mu\gamma_5 u_s(p) = \bar{u}_{s'}(p')\big[(p-p')^\mu\gamma_5 + 2\mathrm{i}S^{\mu\nu}(p'+p)_\nu\gamma_5\big]u_s(p); \tag{40}$$

$$0 = \bar{u}_{s'}(p')\big[(p'-p)^\mu + 2\mathrm{i}S^{\mu\nu}(p'+p)_\nu\big]u_s(p); \tag{41}$$

$$0 = \bar{u}_{s'}(p')\big[(p+p')^\mu\gamma_5 + 2\mathrm{i}S^{\mu\nu}(p'-p)_\nu\gamma_5\big]u_s(p). \tag{42}$$

- other useful identities:

$$
\begin{aligned}
&\bar{u}_s(p)\gamma^\mu u_{s'}(p) = 2p^\mu \delta_{ss'}, \quad \bar{v}_s(p)\gamma^\mu v_{s'}(p) = 2p^\mu \delta_{ss'};\\
&\bar{u}_{s'}(p')\gamma^+\gamma_5 u_s(p) = 2\sqrt{p^+p'^+}\,\delta_{ss'}\mathrm{sign}(s);\\
&\bar{u}_s(p)\gamma^+ u_{s'}(p') = \bar{v}_s(p)\gamma^+ v_{s'}(p') = 2\sqrt{p^+p'^+}\,\delta_{ss'}\\
&\bar{u}_s(\boldsymbol{p})\gamma^0 v_{s'}(-\boldsymbol{p}) = 0
\end{aligned}
\tag{43}
$$

Spinor vertices In general, the spinor vertex can be written as,

$$
\begin{aligned}
V_n =&\bar{u}_{s'}(p')\not{\epsilon}_1\not{\epsilon}_2\cdots\not{\epsilon}_n u_(p)\\
=&\frac{1}{4\sqrt{p^+p'^+}}\chi^\dagger_{s'}\gamma^0\gamma^+(p'+m)\not{\epsilon}_1\not{\epsilon}_2\cdots\not{\epsilon}_n(\not{p}+m)\gamma^+\chi_s\\
=&\frac{1}{2\sqrt{p^+p'^+}}\chi^\dagger_{s'}\Lambda_+(p'+m)\not{\epsilon}_1\not{\epsilon}_2\cdots\not{\epsilon}_n(\not{p}+m)\gamma^+\chi_s\\
=&\frac{1}{4\sqrt{p^+p'^+}}\mathrm{tr}\big[(\not{p}'+m)\not{\epsilon}_1\not{\epsilon}_2\cdots\not{\epsilon}_n(\not{p}+m)\gamma^+\chi_{ss'}\big]
\end{aligned}
\tag{44}
$$

Now the spinor vertex is turned into the trace of a string of gamma matrices, and

$$
\chi_{ss'} = \begin{cases} 1+\gamma_5, & s=+, s'=+\\ 1-\gamma_5, & s=-, s'=-\\ -\gamma^R = -\gamma^1 - \mathrm{i}\gamma^2, & s=+, s'=-\\ \gamma^L = \gamma^1 - \mathrm{i}\gamma^2, & s=-, s'=+ \end{cases}
\tag{45}
$$

$\chi_s\chi^\dagger_{s'} = \frac{1}{2}\Lambda_+\chi_{ss'}$.

- scalar vertex: ($p^R = p^1 + \mathrm{i}p^2$, $p^L = p^1 - \mathrm{i}p^2$, $p_\mu p^\mu = m^2$)

$$
\bar{u}_{s'}(p')u_s(p) = -\bar{v}_s(p)v_{s'}(p') = \sqrt{p^+p'^+}\begin{cases} m\big(\frac{1}{p^+}+\frac{1}{p'^+}\big), & s,s' = ++,--\\ \frac{p^R}{p^+}-\frac{p'^R}{p'^+}, & s,s'=+,-\\ \frac{p'^L}{p'^+}-\frac{p^L}{p^+}, & s,s'=-,+ \end{cases}
\tag{46}
$$

- pseudo scalar vertex: ($p^R = p^1 + \mathrm{i}p^2$, $p^L = p^1 - \mathrm{i}p^2$, $p_\mu p^\mu = m^2$)

$$
\bar{u}_{s'}(p')\gamma_5 u_s(p) = -\bar{v}_s(p)\gamma_5 v_{s'}(p') = \sqrt{p^+p'^+}\begin{cases} m\big(\frac{1}{p'^+}-\frac{1}{p^+}\big), & s,s' = ++\\ m\big(\frac{1}{p^+}-\frac{1}{p'^+}\big), & s,s' = --\\ \frac{p^R}{p^+}-\frac{p'^R}{p'^+}, & s,s'=+,-\\ \frac{p^L}{p^+}-\frac{p'^L}{p'^+}, & s,s'=-,+ \end{cases}
\tag{47}
$$

- vector vertex: ($p^R = p^1 + \mathrm{i}p^2$, $p^L = p^1 - \mathrm{i}p^2$, $p_\mu p^\mu = m^2$)

$$
\begin{aligned}
\bar{u}_{s'}(p')\gamma^+ u_s(p) = \bar{v}_s(p)\gamma^+ v_{s'}(p') =& 2\sqrt{p^+p'^+}\,\delta_{ss'} \\
\bar{u}_{s'}(p')\gamma^- u_s(p) = \bar{v}_s(p)\gamma^- v_{s'}(p') =& \frac{2}{\sqrt{p^+p'^+}} \begin{cases} m^2 + p^R p'^L, & s,s' = +,+ \\ m^2 + p^L p'^R, & s,s' = -,- \\ m(p^R - p'^R), & s,s' = +,- \\ m(p'^L - p^L), & s,s' = -,+ \end{cases} \\
\bar{u}_{s'}(p')\gamma^L u_s(p) = \bar{v}_s(p)\gamma^L v_{s'}(p') =& 2\sqrt{p^+p'^+} \begin{cases} \frac{p'^L}{p'^+}, & s,s' = +,+ \\ \frac{p^L}{p^+}, & s,s' = -,- \\ m\big(\frac{1}{p'^+} - \frac{1}{p^+}\big), & s,s' = +,- \\ 0, & s,s' = -,+ \end{cases} \\
\bar{u}_{s'}(p')\gamma^R u_s(p) = \bar{v}_s(p)\gamma^R v_{s'}(p') =& 2\sqrt{p^+p'^+} \begin{cases} \frac{p^R}{p^+}, & s,s' = +,+ \\ \frac{p'^R}{p'^+}, & s,s' = -,- \\ 0, & s,s' = +,- \\ m\big(\frac{1}{p^+} - \frac{1}{p'^+}\big), & s,s' = -,+ \end{cases}
\end{aligned} \tag{48}
$$

- pseudo vector:

$$
\begin{aligned}
\bar{u}_{s'}(p')\gamma^+\gamma_5 u_s(p) = -\bar{v}_s(p)\gamma^+\gamma_5 v_{s'}(p') =& 2\sqrt{p^+p'^+}\delta_{ss'}\mathrm{sign}(s) \\
\bar{u}_{s'}(p')\gamma^-\gamma_5 u_s(p) = -\bar{v}_s(p)\gamma^-\gamma_5 v_{s'}(p') =& \frac{2}{\sqrt{p^+p'^+}} \begin{cases} -m^2 + p'^L p^R, & s,s' = +,+ \\ m^2 - p'^R p^L, & s,s' = -,- \\ m(p^R + p'^R), & s,s' = +,- \\ m(p^L + p'^L), & s,s' = -,+ \end{cases} \\
\bar{u}_{s'}(p')\gamma^L\gamma_5 u_s(p) = -\bar{v}_s(p)\gamma^L\gamma_5 v_{s'}(p') =& 2\sqrt{p^+p'^+} \begin{cases} \frac{p'^L}{p'^+}, & s,s' = +,+ \\ -\frac{p^L}{p^+}, & s,s' = -,- \\ m\big(\frac{1}{p^+} + \frac{1}{p'^+}\big), & s,s' = +,- \\ 0, & s,s' = -,+ \end{cases} \\
\bar{u}_{s'}(p')\gamma^R\gamma_5 u_s(p) = -\bar{v}_s(p)\gamma^R\gamma_5 v_{s'}(p') =& 2\sqrt{p^+p'^+} \begin{cases} \frac{p^R}{p^+}, & s,s' = +,+ \\ -\frac{p'^R}{p'^+}, & s,s' = -,- \\ 0, & s,s = +,- \\ m\big(\frac{1}{p^+} + \frac{1}{p'^+}\big), & s,s = -,+ \end{cases}
\end{aligned} \tag{49}
$$

6 Polarization of massless vector bosons

Define the polarization vector:

$$\varepsilon^\mu_\lambda(k) = (\varepsilon^+_\lambda, \varepsilon^-_\lambda, \boldsymbol{\varepsilon}^\perp_\lambda) = (0, 2\frac{\boldsymbol{\epsilon}^\perp_\lambda\cdot\boldsymbol{k}^\perp}{k^+}, \boldsymbol{\epsilon}^\perp_\lambda), \quad (\lambda = \pm 1) \tag{50}$$

where $\boldsymbol{\epsilon}^\perp_\pm = \frac{1}{\sqrt{2}}(1, \pm\mathrm{i})$. In fact, $\varepsilon^L_\lambda = \sqrt{2}\delta_{\lambda,+}, \varepsilon^R_\lambda = \sqrt{2}\delta_{\lambda,-}$. This definition satisfies the light-cone gauge $\omega\cdot A = A^+ = 0$ and Lorenz condition $\partial_\mu A^\mu = 0$,

$$\omega_\mu\varepsilon^\mu_\lambda(k) = \varepsilon^+_\lambda(k) = 0, \quad k_\mu\varepsilon^\mu_\lambda(k) = 0. \tag{51}$$

There is another set of conventions that define $\boldsymbol{\epsilon}^\perp_\pm = -\frac{1}{\sqrt{2}}(1, \pm\mathrm{i})$.

Polarization identities:

- orthogonality:

$$\varepsilon^{\mu}_{\lambda}(k)\varepsilon^{*}_{\lambda'\mu}(k) = -\delta_{\lambda,\lambda'}; \tag{52}$$

- helicity sum:

$$\sum_{\lambda=\pm}\varepsilon^{i*}_{\lambda}(k)\varepsilon^{j}_{\lambda}(k) = \delta^{ij}, \quad d^{\mu\nu} \equiv \sum_{\lambda=\pm}\varepsilon^{\mu*}_{\lambda}(k)\varepsilon^{\nu}_{\lambda}(k) = -g^{\mu\nu} + \frac{\omega^{\mu}k^{\nu}+\omega^{\nu}k^{\mu}}{\omega\cdot k} - \omega^{\mu}\omega^{\nu}\frac{k^2}{(\omega\cdot k)^2}. \tag{53}$$

In particular, if k is on-shell, i.e. $k^2 = 0$, the second identity is reduced to

$$\sum_{\lambda=\pm}\varepsilon^{\mu*}_{\lambda}(k)\varepsilon^{\nu}_{\lambda}(k) = -g^{\mu\nu} + \frac{\omega^{\mu}k^{\nu}+\omega^{\nu}k^{\mu}}{\omega\cdot k}. \tag{54}$$

- crossing symmetry:

$$\varepsilon^{\mu*}_{\lambda}(k) = \varepsilon^{\mu}_{-\lambda}(k) = \varepsilon^{\mu}_{-\lambda}(-k) \tag{55}$$

where $\omega^{\mu} = (1, 0, 0, -1)$, is the null normal vector of light-front, $\omega\cdot\omega = 0$, $\omega\cdot v = v^{+}$.

7 Spin vector of massive vector bosons

Define the spin vector for the massive vector bosons:

$$e_{\lambda}(k) = (e^{+}_{\lambda}(k), e^{-}_{\lambda}(k), \boldsymbol{e}^{\perp}_{\lambda}(k)) = \begin{cases} \big(\frac{k^+}{m}, \frac{\boldsymbol{k}^2_{\perp}-m^2}{mk^+}, \frac{\boldsymbol{k}^{\perp}}{m}\big), & \lambda = 0 \\ \big(0, 2\frac{\boldsymbol{\epsilon}^{\perp}_{\lambda}\cdot\boldsymbol{k}^{\perp}}{k^+}, \boldsymbol{\epsilon}^{\perp}_{\lambda}\big), & \lambda = \pm 1 \end{cases} \tag{56}$$

where where $\boldsymbol{\epsilon}^{\perp}_{\pm} = \frac{1}{\sqrt{2}}(1, \pm\mathrm{i})$, and $\boldsymbol{\epsilon}^{\perp *}_{+} = \boldsymbol{\epsilon}^{\perp}_{-}$, $m^2 = k_{\mu}k^{\mu} \triangleq k^2$ is the mass of the particle[1].

Spin identities:

- Proca equation, $k_{\mu}e^{\mu}_{\lambda}(k) = 0$.
- orthogonality:

$$e^{\mu}_{\lambda}(k)e^{*}_{\lambda'\mu}(k) = -\delta_{\lambda,\lambda'}; \tag{57}$$

- spin sum:

$$K^{\mu\nu} \equiv \sum_{\lambda=-1}^{+1} e^{\mu*}_{\lambda}(k)e^{\nu}_{\lambda}(k) = -g^{\mu\nu} + \frac{k^{\mu}k^{\nu}}{k^2}. \tag{58}$$

$k^{\mu}K_{\mu\nu}(k) = k^{\mu}k^{\nu}K_{\mu\nu}(k) = 0.$

- crossing symmetry:

$$e^{\mu*}_{\lambda}(k) = e^{\mu}_{-\lambda}(k), \quad e^{\mu}_{\lambda}(-k) = (-1)^{\lambda+1}e^{\mu}_{\lambda}(k); \tag{59}$$

[1] There is another set of conventions that define $\boldsymbol{\epsilon}^{\perp}_{\pm} = -\frac{1}{\sqrt{2}}(1, \pm\mathrm{i})$. One should be careful about the consistency of the conventions one chooses. For example, under the definition of $\boldsymbol{\epsilon}^{\perp}_{\pm} = -\frac{1}{\sqrt{2}}(1, \pm\mathrm{i})$ the eigenvalue of the mirror parity (light-front parity) operator $\hat{m}_P = \hat{R}_x(\pi)\hat{P}$ of a vector state is $\hat{m}_P|p^+, p^1, p^2, j, m_j\rangle = (-i)^{2j}P|p^+, -p^1, p^2, j, -m_j\rangle$. But if one use our definition $\boldsymbol{\epsilon}^{\perp}_{\pm} = \frac{1}{\sqrt{2}}(1, \pm\mathrm{i})$, there will be an extra minus sign, i.e., $\hat{m}_P|p^+, p^1, p^2, j, m_j\rangle = -(-i)^{2j}P|p^+, -p^1, p^2, j, -m_j\rangle$.