Revisiting hadron structures in 3D

From electromagnetic to mechanical densities

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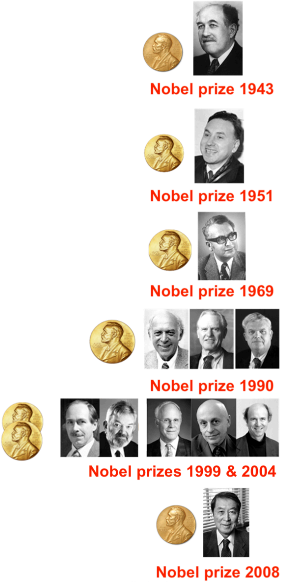
Jefferson Lab

A brief (and biased) history of the proton structure

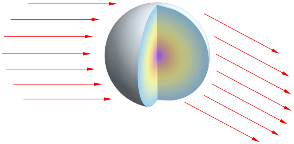
- Proton magnetic moment (1930s)
- Elastic scattering of proton (1950s)
- Quark model (early 1960s)
- Chiral symmetry breaking (1960s)
- Deep inelastic scattering (late 1960s)
- Quantum chromodynamics (1970s)
- ......

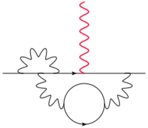
Tremendous progress, but many puzzles remain. See, F. Gross and E. Klempt (eds.), *50 Years of quantum chromodynamics*, EPJC, 2023



Electron structure in history

- Thomson discovered the electron in 1897 in cathode rays. Classical view of the electron: a spherical corpuscles of the size $r_e \sim 10^{-15}\,\mathrm{m}$ moving in high speeds
- Minkowski (1908), and Einstein & Laub (1908) established the correct theory of macroscopic electromagnetism of moving bodies based on Einstein's electrodynamic theory of moving bodies (1905), i.e. special relativity [W. Pauli. Theory of Relativity (Oxford Univ. Press, 1958)]
  - Modern view of the electron: a pointlike particle $r_e \leq 10^{-24}\,\mathrm{m}$, first verified by Sam Ting in 1967
  - Relativity as a byproduct -- a right theory (indeed, a great theory) for a wrong problem
  - Modern applications: cosmology, black hole merger, inertial fusion, quark-gluon plasma (hydro), ...





Minkowski-Einstein-Laub theory

$$\begin{aligned}\partial_\alpha F^{\alpha\beta} &= j^\beta \\ \partial_\alpha \widetilde{F}^{\alpha\beta} &= 0\end{aligned} \quad \xrightarrow{j^\beta = j^\beta_\mathrm{f} + \partial_\alpha M^{\alpha\beta}} \quad \begin{aligned}\partial_\alpha H^{\alpha\beta} &= j^\beta_\mathrm{f} \\ \partial_\alpha \widetilde{F}^{\alpha\beta} &= 0\end{aligned}$$

where, j^α is the full current, and $j^\alpha_f = \varrho u^\alpha + j^\alpha_\mathrm{Ohm}$ is the free current, and $H^{\alpha\beta} \equiv F^{\alpha\beta} - M^{\alpha\beta}$

- The theory appears identical to Maxwell's theory of macroscopic electromagnetism, but now applicable to media in motion -- rediscovered several times in applied physics!
- Co-moving decomposition of the medium polarization tensor:

$$M^{\alpha\beta} = u^\alpha \mathcal{P}^\beta - u^\beta \mathcal{P}^\alpha + \varepsilon^{\alpha\beta\kappa\lambda} u_\kappa \mathcal{M}_\lambda$$

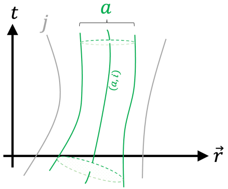
  where, u^α is the velocity vector, $\mathcal{P}^\alpha$ and $\mathcal{M}^\alpha$ are co-moving polarization & magnetization vectors
  - Co-moving polarization & effective magnetic charge densities: $\varrho_\mathrm{pol} = -\partial_\alpha \mathcal{P}^\alpha$, $\varrho_\mathrm{mag} = -\partial_\alpha \mathcal{M}^\alpha$
  - Dependance on the choice of u^α, e.g. Landau-Lifshitz vs. Eckart frames [Eckart:1940te]
- Need the microscopic theory to obtain the full current as developed by Max Born in 1909

[H. Minkowski Nachr. Ges. Wiss. Gött., Math.-Phys. Kl. 53 (1908);]

[H. Minkowski, Math.Ann. 68 (1910) 472;]

[A. Einstein, J. Laub, Annals Phys. 331 (1908) 532]

Classical many-body theory of Born



$$j^\mu(x) = \sum_{a,i} e_{ai} \int \mathrm{d}\tau_{ai} \dot{X}^\mu_{ai}(\tau_{ai}) \delta^4(X_{ai}(\tau_{ai}) - x)$$

where a and i enumerate the ``atoms" and their constituents, respectively.

- Born introduced a privileged worldline $X_a(\tau_a)$ for atom a and define $r_{ai} = X_{ai} - X_a$:

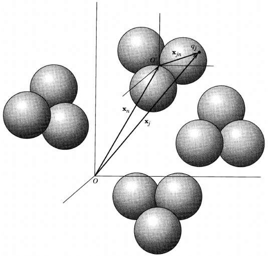
$$j^\mu(x) = \sum_{n=0}^{\infty} \sum_{a\in A} \int \tau_a \sum_{i\in a} e_{ai} \Big[\dot{X}^\mu_a(\tau_a) + \dot{r}^\mu_{ai}(\tau_a)\Big] (r_{ai}\cdot\partial)^n \delta^4(X_a(\tau_a) - x).$$

  - Multipole structure, and the monopole term $(n = 0)$ defines the free current

$$j^\mu_f(x) = \sum_a \int \mathrm{d}\tau_a e_a \dot{X}^\mu_a(\tau_a)\delta^4(X_a(\tau_a) - x)$$

  - If atomic motion $\dot{X}_a$ neligible, factorization of the atomic distribution and the internal density within each atom

$$j^\mu(x) = \int \mathrm{d}^3R\, \mathcal{J}^\mu(\vec{x} - \vec{R})\rho_a(\vec{R}, t) + O(\dot{X}_a)$$



- Weyl quantization, Wigner-Newton position operator & particle localization

Quantum many-body theory and nuclear structure

- One-body density (OBD):

$$\rho(\vec{r}) = \langle\Psi|\sum_i e_i \delta^3(r - \underline{r_i})|\Psi\rangle$$

- Factorization of c.m. motion $|\Psi\rangle = |\Phi_{\text{cm}}\rangle|\psi_{\text{rel}}\rangle$ and translationally-invariant OBD:

$$\rho(\vec{r}) = \int \mathrm{d}^3 R\, \rho_{\text{cm}}(\vec{R})\varrho(\vec{r} - \vec{R})$$

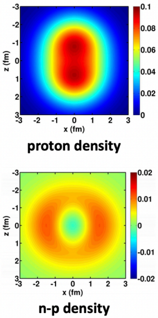
- Form factor: F.T. of OBD $\varrho(r)$

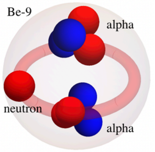
$$F(q^2) = \int \mathrm{d}^3 r\, e^{i\vec{q}\cdot\vec{r}}\varrho(\vec{r})$$

  - Root-mean-square (r.m.s.) radius:

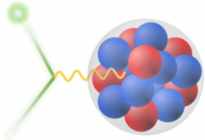
$$r^2 \equiv \int \mathrm{d}^3 r\, r^2 \varrho(\vec{r}) = -6F'(0)$$

- Elastic eA scattering $\sigma_{\text{el.}} = \sigma_{\text{Mott}}\left|F(\vec{q}^2)\right|^2$





[Maris:2012ee, Cockrell:2012hfi]

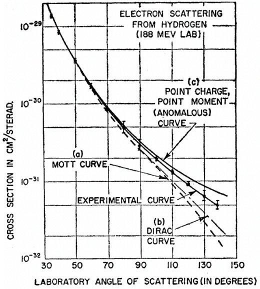


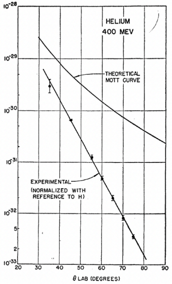
Nucleus is at rest (no recoil)

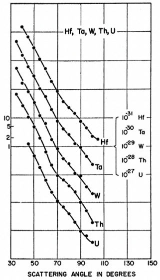
Nuclear structure within eA scattering

[Hofstadter:1956qs]

Hofstadter



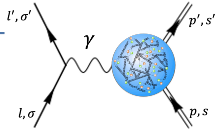




	r	M	rM
pion	0.67 fm	0.14 GeV	0.5
charmonium	0.25--0.4 fm	3.0 GeV	3.8--6
proton	0.87 fm	0.94 GeV	4
nuclei	1.3 $A^{\frac{1}{3}}$ fm	0.94A GeV	$6A^{\frac{4}{3}}$

- Hofstadter et al. systematically investigated nuclear structure using electron scattering
- For the proton, the nucleus of hydrogen, nucleus recoil is non-negligible: $E_\gamma \gtrsim M_p c^2$ and full relativistic description is needed.
  - Protons are intrinsically relativistic, since their radius $r_p \sim \lambda_C = M_p^{-1}$
  - In order to resolve the proton, the photon wavelength $\lambda_\gamma \ll r \Rightarrow E_\gamma \gg M_p$
  - Similar to other hadrons, e.g. neutron, pion

Proton form factors

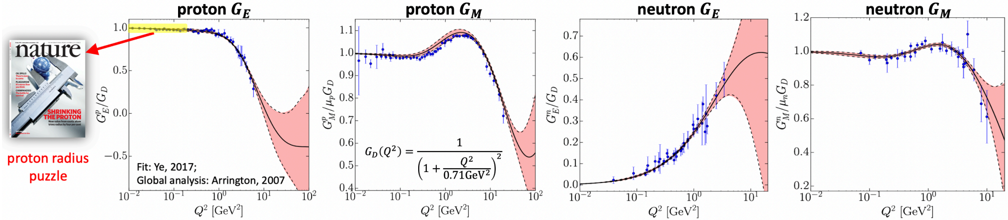


$$\frac{d\sigma_{\text{el.}}}{d\Omega} = \frac{d\sigma}{d\Omega}\Big|_{\text{Mott}}\left[\frac{G_E^2(Q^2)+\tau G_M^2(Q^2)}{1+\tau} + 2\tau\tan^2\frac{\theta}{2}G_M^2(Q^2)\right]$$

where, $\tau = Q^2/4M^2$, and G_E and G_M parametrize the Lorentz covariant structures of the current hadronic matrix elements,

$$\begin{aligned}\langle p', s'|J^\mu(0)|p, s\rangle &= \bar{u}_{s'}(p')\left[\frac{P^\mu}{M}G_E(q^2) + \frac{i\varepsilon^{\mu\nu\rho\sigma}q_\nu P_\rho \gamma_\sigma\gamma_5}{2M^2}G_M(q^2)\right]u_s(p) \\ &= \bar{u}_{s'}(p')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_2(q^2)\right]u_s(p)\end{aligned}$$

Here, $P = \frac{1}{2}(p+p')$, $q = p'-p$. $G_{E,M}$ are called the Sachs form factors. $F_{1,2}$ are called the Dirac and Pauli form factors



Proton charge density: Sachs/Breit-frame density

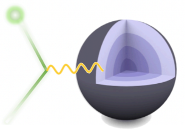
[Ernst:1960zza, Sachs:1962zzc]

The Sachs densities are defined as the F.T. of the hadronic matrix elements within the Breit frame

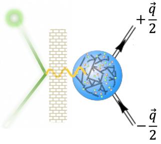
$$\rho_E^{(\text{Sach})}(\vec{r}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2E_q} e^{-i\vec{q}\cdot\vec{r}} \langle +\tfrac{1}{2}\vec{q} | J^0(0) | -\tfrac{1}{2}\vec{q} \rangle = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} G_E(-\vec{q}^2) e^{-i\vec{q}\cdot\vec{r}}$$

- Ambiguities G_E vs F_1 vs $G_E/\sqrt{1+\tau}$ [Lorce:2020onh]
- Frame dependence: the proton is not at rest in the Breit frame. Densities in other frames?
- Lack of local probabilistic interpretation $J^0(x) = \overline{\Psi}\gamma^0\Psi \neq \sum_i e_i \delta^3(x - X_i)$ [Miller:2018ybm]
- Underlying assumption: proton as a rigid ball -- in contradiction with relativity [Jaffe:2020ebz]

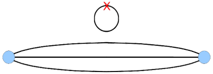
Is it possible to generalize the non-relativistic quantum many-body OBD to relativistic quantum theory?



$\lambda_\gamma \sim r_{\text{nucl}} \gg \lambda_{\text{Comp}} = M_{\text{nucl}}^{-1}$



$\lambda_\gamma \sim r_N \sim \lambda_{\text{Comp}} = M_N^{-1}$



Relativistic quantum many-body theory on the light front

- In relativistic quantum theory, the position operator does not exist*
- Fortunately, in light-front dynamics, transverse position operator exist, which defines the OBD on the 2D transverse plane:

$$\rho_T(\vec{r}_\perp) = \langle \Psi | \sum_i e_i \delta^2(r_\perp - r_{i\perp}) | \Psi \rangle$$

where, $\Sigma^+(\vec{r}_\perp) \equiv \sum_i e_i \delta^2(r_\perp - r_{i\perp})$ is the transverse charge density operator.

- Factorization of c.m. motion $|\Psi\rangle = |\Phi_{\text{cm}}\rangle |\psi_{\text{rel}}\rangle$ thanks to Galilean covariance in LFD
- Indeed, F.T. of $\rho_T(\vec{r}_\perp)$ is given by the celebrated Drell-Yan-West formula [Drell:1969km, West:1970av]

$$\rho_T(\vec{r}_\perp) \quad \xrightarrow{\text{F.T.}} \quad F(\vec{q}_\perp) = \sum_n \int [\mathrm{d}x_i \mathrm{d}^2 k_{i\perp}]_n \sum_j e_j \psi_n^*(\{x_i, \vec{k}'_{i\perp}\}) \psi_n(\{x_i, \vec{k}_{i\perp}\})$$

where, Drell-Yan frame $q^+ = 0$ is equivalent to the longitudinal compactification $\Sigma^+ = (1/2) \int \mathrm{d}x^- J^+(x)$

light-cone coordinates: $x^\pm = x^0 \pm x^3, \quad \vec{x}_\perp = (x^1, x^2)$

Transverse density on the light front

[Miller:2010nz, Freese:2022fat]

$$\rho_T(\vec{r}_\perp) = \int \frac{d^2 q_\perp}{(2\pi)^2 2P^+} e^{-i\vec{q}_\perp\cdot\vec{r}_\perp} \langle P+\tfrac{1}{2}q|J^+(0)|P-\tfrac{1}{2}q\rangle\Big|_{q^+=0}$$
$$= \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp\cdot\vec{r}_\perp} F_1(\vec{q}_\perp^{\,2})$$

- Frame independent: boost invariance in light-front dynamics
- Local probabilistic interpretation: $J^+ \sim \sum_i \bar{q}_i\gamma^+ q_i \sim \sum_i e_i N_i$, vacuum suppressed
- Intrinsically relativistic and related to the forward generalized parton density $q(x, \vec{b}_\perp)$, i.e. what the probes "see" in high-energy collision experiments [Burkardt:2000za]

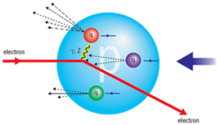
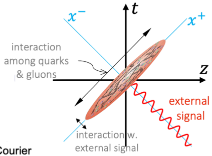
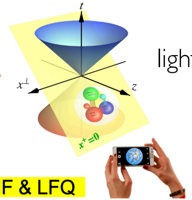


Image credit: CERN Courier

Experiment: DIS



$\lambda_\gamma \sim r_N \gg 1/P^+$



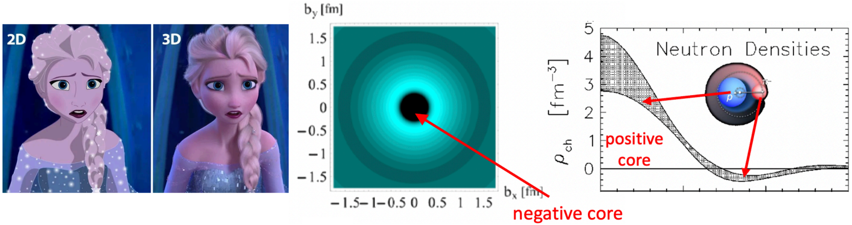
Theory: IMF & LFQ

light-cone coordinates:

$$x^\pm = x^0 \pm x^3,$$
$$\vec{x}_\perp = (x^1, x^2)$$

What does the proton look like in 3D?

- Light-front densities are 2D $\stackrel{?}{\to}$ 3D [Panteleeva:2021iip]
- Light-front densities can be understood as equal-time densities in the infinite momentum frame which could be counter-intuitive [Lorce:2020onh]
- Physical densities associated with "bad" components, e.g. J^-, are not well explained
- Amplitude vs. quantum expectation value: what is truly probed by a semiclassical electromagnetic field is the quantum expectation value $j^\alpha(x) = \langle\Psi|J^\alpha(x)|\Psi\rangle$ where $|\Psi\rangle$ is a generic hadronic state

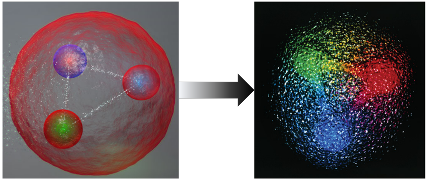


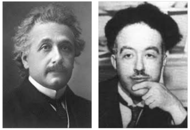
What does the proton look like in 3D?

resolving a non-relativistic particle: $r_{\text{hadron}} \gg \lambda_\gamma \gg \lambda_{\text{hadron}} \geq \lambda_{\text{C}}$

resolving a relativistic hadron: $\lambda_{\text{hadron}} \gtrsim r_{\text{hadron}} \sim \lambda_{\text{C}} \gg \lambda_\gamma$

where $\lambda_{\text{C}} = M^{-1}$ is the Compton wavelength, $\lambda_\gamma = Q^{-1}$ is the photon wavelength. λ_{hadron} is the de Broglie wavelength. r_{hadron} is the hadron radius.





- $\lambda_\gamma \ll \lambda_{\text{hadron}}$: the photon ``sees'' a de Broglie wave (medium)! [de Broglie, 1924]
- The relevant theory is Minkowski-Einstein-Laub's *relativistic theory of macroscopic electromagnetism*
- Mass & spin decomposition as multi-fluid description [Lorce:2017xzd]

Relativistic theory of macroscopic electromagnetism

$$j^\beta = j_f^\beta + \partial_\alpha M^{\alpha\beta} \;\Rightarrow\; \partial_\alpha H^{\alpha\beta} = j_f^\beta, \qquad (H^{\alpha\beta} \equiv F^{\alpha\beta} - M^{\alpha\beta})$$

where, $j_f^\alpha = \varrho_f u^\alpha + j^\alpha_{\text{Ohm}}$ is the free current, consisting of the convective current and the Ohmic current. and $M^{\alpha\beta} = u^\alpha \mathcal{P}^\beta - u^\beta \mathcal{P}^\alpha + \varepsilon^{\alpha\beta\rho\sigma} u_\rho \mathcal{M}_\sigma$ is the medium polarization tensor.

- Our ``medium'' is non-dissipative, unmagnetized, unpolarized ($\alpha_s \gg \alpha_{\text{QED}}$)
- Densities (frame dependent): charge $\rho = j^0$, bound charge density $\rho_b = \nabla \cdot \vec{P}$, effective magnetic charge density $\rho_m = \nabla \cdot M$, magnetization current $\vec{j}_m = \nabla \times \vec{M}$
- Co-moving densities -- densities measured in local rest frame, are Lorentz invariants:
  - Free charge density: $\varrho_f = u_\alpha j_f^\alpha$
  - Bound charge density: $\varrho_b = -\partial_\alpha \mathcal{P}^\alpha$
  - Total charge density: $\varrho = \varrho_f + \varrho_b$
  - Effective magnetic charge density: $\varrho_m = -\partial_\alpha \mathcal{M}^\alpha$

[Jackson, Classical Electrodynamics (John Wiley & Sons, 1999)]
[W. Pauli. Theory of Relativity (Oxford Univ. Press, 1958)]
[S.R. de Groot & L.G. Suttorp, Foundations of electrodynamics (North-Holland, 1972)]

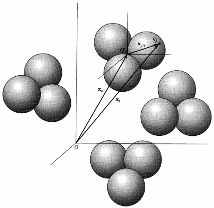
Factorization

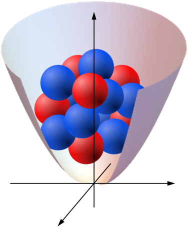
Consider media comprising of composite particles, e.g. atoms, molecules. The total current density factorizes into a convolution of an internal density and a convective current density:

$$j^\alpha(x) = \int \mathrm{d}^3 r \, \mathscr{J}^\alpha(\vec{r}) \rho_f(\vec{x} - \vec{r}, t)$$

Here, $\mathscr{J}^\alpha(\vec{x})$ is known as the atomic/molecular density.
Similar factorization formula for nuclear densities,

$$\rho(\vec{r}) = \int \mathrm{d}^3 R \, \rho_{\text{cm}}(\vec{R}) \rho_{\text{int}}(\vec{r} - \vec{R})$$





Relativistic quantum current

[Li:2022ldb, Freese:2022fat]

$$j^\mu(x) \equiv \langle\Psi|J^\mu(x)|\Psi\rangle$$
$$= \frac{1}{M}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}\int \frac{\mathrm{d}^3p'}{(2\pi)^3 2p'^0}\widetilde{\Psi}^*_{s'}(p')\widetilde{\Psi}_s(p)e^{iq\cdot x}$$
$$\times\, \overline{u}_{s'}(p')\Big[P^\mu G_E(q^2) + \frac{i}{2M}\varepsilon^{\mu\nu\rho\sigma}q_\nu P_\rho \gamma_\sigma\gamma_5 G_M(q^2)\Big]u_s(p)$$

where, $P = (p'+p)/2$ and $q = p'-p$. $\Psi_s(\vec{p}) = \langle p, s|\Psi\rangle$ is the momentum-space wavepacket, which is arbitrary, e.g. Gaussian $\widetilde{\Psi}_s(\vec{p}) \propto \exp\left\{\frac{(\vec{p}-\vec{p}_0)^2}{4\sigma^2}\right\}$ and stationary planewave $\widetilde{\Psi}_s(\vec{p}) \propto \delta^3(\vec{p})$ [Ernst:1960zza]

- Introduce the ``coordinate space wave function'' as the F.T. of $\widetilde{\Psi}_s(\vec{p})$,

$$\Psi(x) = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}\widetilde{\Psi}_s(\vec{p})u_s(p)e^{ip\cdot x}.$$

  where $\Psi(x)$ satisfies the Dirac equation

- Conserved number current: $(f\overleftrightarrow{\partial}g \equiv f\partial g - \partial f g)$

$$n^\mu(x) \equiv \frac{1}{2M}\overline{\Psi}(x)i\overleftrightarrow{\partial}^\mu\Psi(x) \quad\Rightarrow\quad \partial_\mu n^\mu = 0$$

Quantum wave kinematics

[Li:2022ldb]

$$n^\alpha = n u^\alpha, \quad (u_\alpha u^\alpha = 1).$$

$n = n_\alpha u^\alpha$ is the proper number density

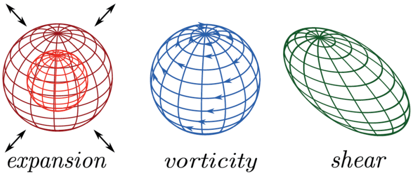
- The medium (wave) velocity u:

$$u^\alpha(x) \equiv \overline{\Psi}(x)\mathcal{U}^\alpha\Psi(x) = \frac{1}{\sqrt{4M^2+\partial^2}}[\overline{\Psi} i \overleftrightarrow{\partial}^\alpha \Psi] = n^\alpha - \frac{1}{8M^2}\partial^2 n^\alpha + \frac{3}{16M^4}\partial^4 n^\alpha - \cdots$$

- Consider the gradient of the velocity vector:

$$\partial_\alpha u_\beta = u_\alpha a_\beta + \Omega_{\alpha\beta} + \Sigma_{\alpha\beta} + \frac{1}{3}\theta\Delta_{\alpha\beta},$$

a is the 4-acceleration, $a_\alpha u^\alpha = 0$; $\Omega_{\alpha\beta}$ is the vorticity tensor; $\Sigma_{\alpha\beta}$ is the shear tensor; θ is the expansion scalar, $\theta = 0$ for hadrons, and $\Delta^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ is the spatial metric tensor (spatial projector)



[Rezzolla, Relativistic Hydrodynamics, 2013]

Electromagnetic structure: spin-0

Recall, in Minkowski-Einstein-Laub's theory,

$$j^\alpha = \rho_f u^\alpha + \partial_\beta M^{\alpha\beta}$$

- Free current of spin-0 particles:

$$j_f^\alpha = \mathcal{Q} n^\alpha = \mathcal{Q}\Phi^* i\overleftrightarrow{\partial}^\alpha \Phi \quad \Rightarrow \quad \rho_f = \mathcal{Q} n = \mathcal{Q}\Phi^* i\overleftrightarrow{\partial}_\alpha \mathcal{U}^\alpha \Phi$$

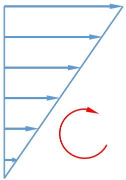
  here, $\mathcal{Q} = F(0)$ is the charge number

- The polarization tensor can be constructed as,

$$M^{\alpha\beta}(x) = \Big\langle \frac{1}{M} \int \frac{d^3 q}{(2\pi)^3} e^{iq\cdot z} q^{[\alpha} i\overleftrightarrow{\partial}^{\beta]} \frac{F(q^2) - F(0)}{iq^2} \Big\rangle$$

$$\Rightarrow \rho_b(x) = \Big\langle \int \frac{d^3 q}{(2\pi)^3} e^{iq\cdot z} \sqrt{1 - \frac{q^2}{4M^2}} [F(q^2) - F(0)] \Big\rangle,$$

$$\rho_m(x) = 0.$$



  where, the quantum average is defined as,

$$\langle \mathcal{O}(x) \rangle \equiv 2M \int d^3 z \, \Phi^*(x - z) \mathcal{O}(z) \Phi(x - z)$$

Electromagnetic structure: spin-0

- Full electric density in the local rest frame (LRF):

$$\rho = \rho_f + \rho_b = \langle \varrho(z) \rangle$$
$$= 2M \int \mathrm{d}^3 z\, \Phi^*(x-z) \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{iq\cdot z} \sqrt{1 - \frac{q^2}{4M^2}} F(q^2) \Phi(x-z)$$

- Full density is a convolution of the wave density $\propto \Phi^*\Phi$, i.e. the c.m. density and the hadronic density $\varrho(z)$, i.e. the intrinsic distribution
- The full current can also be written as,

$$j^\alpha = \langle \varrho \mathcal{U}^\alpha \rangle$$

corresponds to a new current decomposition in relativistic hydrodynamics (for spin-0):

$$j^\alpha = \rho u^\alpha$$

Electromagnetic structure: spin-1/2

Generalizing the current decomposition to polarized relativistic matter:

- Electromagnetic decomposition:

$$j^\alpha = \rho u^\alpha + \partial_\beta M^{\alpha\beta}$$

  - Special case: Minkowski-Einstein-Laub decomposition:

$$j^\alpha = \rho_f u^\alpha + \partial_\beta M^{\alpha\beta}$$

- Bemfica-Disconzi-Noronha-Kovtun (BDNK) formalism: [Kovtun:2016lfw, Hernandez:2017mch, Bemfica:2019knx]

$$j^\alpha = \mathcal{N} u^\alpha + \mathcal{J}^\alpha \qquad (\mathcal{J}^\alpha \equiv \Delta^\alpha_\beta j^\beta)$$

  where the relation between the e.m. and the 3+1 decompositions is,

$$\mathcal{N} = \rho + \partial_\alpha \mathcal{P}^\alpha + \mathcal{P}_\alpha a^\alpha + \mathcal{M}_\alpha \Omega^\alpha, \quad \mathcal{J}^\alpha = u_\beta \dot{\mathcal{P}}^{[\beta} u^{\alpha]} + \varepsilon^{\alpha\beta\rho\sigma} u_\rho \partial_\beta \mathcal{M}_\sigma$$

  where, $\Omega^\alpha = \frac{1}{2}\varepsilon^{\alpha\beta\rho\sigma} u_\beta \Omega_{\rho\sigma}$ is the vorticity vector, and $\dot{a} = u_\alpha \partial^\alpha a$

No dissipation (no entropy production), no gradient expansion, no local thermal equilibrium assumption

Electromagnetic decomposition

Hadron matrix elements expressed using Dirac-Pauli form factors: $(F_{1+2} = F_1 + F_2)$

$$\langle p', s'|J^\mu(0)|p, s\rangle = \overline{u}_{s'}(p')\Big[\frac{P^\mu}{M}F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M}F_{1+2}(q^2)\Big]u_s(p),$$

$$\Rightarrow \quad j^\alpha = \Big\langle \varrho\mathcal{U}^\alpha + \frac{1}{2M}\sigma^{\alpha\beta}\partial_\beta\mathcal{F}_{1+2}\Big\rangle$$

where, the quantum average for spin-1/2 particles is,

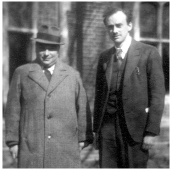
$$\langle\mathcal{O}(x)\rangle = \int \mathrm{d}^3z\, \overline{\Psi}(x-z)\mathcal{O}(z)\Psi(x-z)$$

- One can identify the F.T. of the Dirac form factor as the hadronic electric charge density,

$$\varrho(z) = \int \frac{\mathrm{d}^3q}{(2\pi)^3} e^{iq\cdot z}\sqrt{1-\frac{q^2}{4M^2}}F_1(q^2)$$

- Identify the F.T. of the F_{1+2} as the polarization (magnetic dipole) density,

$$\mathcal{M}^{\alpha\beta} = \frac{1}{2M}\sigma^{\alpha\beta}\mathcal{F}_{1+2}, \qquad \mathcal{F}_{1+2}(z) = \int \frac{\mathrm{d}^3q}{(2\pi)^3}e^{iq\cdot z}F_{1+2}(q^2)$$



3+1 decomposition

Hadron matrix elements expressed using Sachs form factors:

$$\langle p', s' | J^\mu(0) | p, s\rangle = \overline{u}_{s'}(p')\Big[\frac{P^\mu}{M} G_E(q^2) + \frac{1}{4M^2} i\varepsilon^{\mu\nu\rho\sigma} q_\nu P_\rho \gamma_\sigma\gamma_5 G_M(q^2)\Big] u_s(p),$$

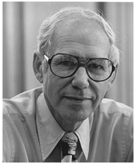
$$\Rightarrow \quad j^\alpha = \Big\langle \mathcal{N}\mathcal{U}^\alpha + \frac{1}{2M}\varepsilon^{\alpha\beta\rho\sigma}\mathcal{U}_\rho\gamma_\sigma\gamma_5\partial_\beta\mathcal{G}_M\Big\rangle$$

- One can identify the F.T. of the Sachs form factor G_E as the hadronic temporal charge density,

$$\mathcal{N}(z) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{iq\cdot z}\sqrt{1 - \frac{q^2}{4M^2}} G_E(q^2)$$

- Identify the F.T. of the Sachs form factor G_M as the magnetization density,

$$\mathcal{M}^\alpha = \frac{\gamma^\alpha\gamma_5}{2M}\int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{iq\cdot z}\sqrt{1 - \frac{q^2}{4M^2}} G_M(q^2) \propto \mathcal{J}_A^\alpha$$



Factorization of the hadronic density

[Li:2022ldb; Freese:2022fat]

$$\rho(x) = \int \mathrm{d}^3 z\, \overline{\Psi}(x-z)\left\{\int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{iq\cdot z}\sqrt{1-\frac{q^2}{4M^2}}F_{\text{ch}}(q^2)\right\}\Psi(x-z)\Big|_{z^0=0}$$

The hadronic part is *not factorizable* due to the dependence of $\vec{P} = (-i/2)\overleftrightarrow{\nabla}_x$ in $q^2 = (q^0)^2 - \vec{q}^2$, where $q^0 = \sqrt{(\vec{P}+\frac{1}{2}\vec{q})^2 + M^2} - \sqrt{(\vec{P}-\frac{1}{2}\vec{q})^2 + M^2}$

- Taylor expansion around $\vec{P} = 0$: multipole series,

$$\rho(\vec{x}) = \sum_{n=0}^{\infty} \frac{(-i)^n}{2^n n!} \rho_n^{i_1 i_2 \cdots i_n}(\vec{x}) \overleftrightarrow{\nabla}^{i_1} \overleftrightarrow{\nabla}^{i_2} \ldots \overleftrightarrow{\nabla}^{i_n}$$

- Monopole density gives the Breit-frame distribution (Sachs distribution)

$$\rho(r) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\sqrt{1+\frac{\vec{q}^2}{4M^2}}F_{\text{ch}}(-\vec{q}^2)e^{-i\vec{q}\cdot\vec{r}} + \mathrm{O}\big(\max\{r_{\text{hadron}}, \lambda_{\text{C}}\}/\lambda_{\text{hadron}}\big)$$

- High-multipole moments exist due to Lorentz distortion
- No special frame or non-relativistic approximation is taken
- Convergence of the multipole series: $\lambda_{\text{hadron}} \gg \max\{r_{\text{hadron}}, \lambda_{\text{C}}\}$ -- sufficiently *delocalized* wavepacket

Light-front distribution from infinite momentum limit

[Li:2022ldb]

Is the multipole expansion unique? No! $\to$ Alternative: Taylor (Laurent) expansion around $1/|\vec{P}| = 0$

- Sufficient to take $P_z \to \infty \;\Rightarrow\; |\vec{P}| = \sqrt{\vec{P}_\perp^2 + P_z^2} \to \infty$
- Monopole density gives the 2D light-front distribution

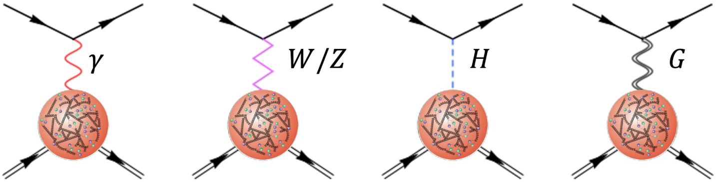
$$\rho_{\text{Mon}}(r) = \delta(r_\parallel) \int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^3} \sqrt{1 + \frac{\vec{q}_\perp^2}{4M^2}} F_{\text{ch}}(-\vec{q}_\perp^2) e^{-i\vec{q}_\perp \cdot \vec{r}_\perp}$$

- No special frame, e.g. Drell-Yan $q^+ = 0$ frame, is chosen
- Relativistic, suitable also for massless hadrons (in contrast to the Sachs distribution)
- Convergence of the multipole series: $|\vec{P}| \gg \lambda^{-1}_{\text{hadron}} \gg \{M, r^{-1}_{\text{hadron}}\}$ -- sufficiently *localized* z-direction
- Consistent with analyses from the (2+2)D phase space approach and the light-front quantization approach. And we have retained full Lorentz covariance [Lorce:2020onh, Freese:2021mzg]
- In the infinite momentum frame (IMF), the current components form a hierarchy: $j^+ \gg \vec{j}_\perp \gg j^-$

Hadronic energy-momentum tensor

$$H = \int \mathrm{d}^3x\, T^{00}(x) \quad \Rightarrow \quad t^{\alpha\beta}(x) = \langle \Psi | T^{\alpha\beta}(x) | \Psi \rangle$$

Hadronic energy-momentum tensor encodes the energy-stress densities inside hadrons



Hadronic matrix elements and gravitational form factors (GFFs): [Kobzarev:1962wt, Pagels:1966zza]

$$\langle p', s' | T_i^{\mu\nu}(0) | p, s \rangle = \frac{1}{M} \bar{u}_{s'}(p') \Big[P^\mu P^\nu A_i(q^2) + \frac{1}{2} i P^{\{\mu} \sigma^{\nu\}\rho} q_\rho J_i(q^2) + \frac{1}{4}(q^\mu q^\nu - g^{\mu\nu} q^2) D_i(q^2) + g^{\mu\nu} \bar{c}_i(q^2) \Big] u_s(p)$$

Light front energy and momentum densities [Xu:2024cfa, cf. Freese:2021czn, Freese:2021mzg]

$$\mathcal{T}^{\alpha\beta}_{ss'}(\vec{r}_\perp;P) = \int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2 2P^+} e^{-i\vec{q}_\perp\cdot\vec{r}_\perp} \langle P+\tfrac{1}{2}q, s'|T^{\alpha\beta}(0)|P-\tfrac{1}{2}q, s\rangle\Big|_{q^+=0}$$

$$\mathcal{P}^+(r_\perp) \equiv \mathcal{T}^{++}(r_\perp;P) = P^+\mathcal{A}(r_\perp),$$

$$\int \mathrm{d}^3x\, T^{+\mu}(x) = P^\mu \quad \Rightarrow \quad \mathcal{P}^i_\perp(r_\perp) \equiv \mathcal{T}^{+i}_{ss}(r_\perp;P) = P^i_\perp \mathcal{A}(r_\perp) + (\nabla\times\vec{\mathcal{S}})^i, \qquad (i=1,2),$$

$$\mathcal{P}^-(r_\perp) \equiv \mathcal{T}^{+-}_{ss}(r_\perp;P) = \frac{P^2_\perp \mathcal{A}(r_\perp) + \vec{P}_\perp\cdot(\nabla\times\vec{\mathcal{S}}) + \mathcal{M}^2(r_\perp)}{P^+}$$

- $\mathcal{A}(r_\perp)$ can be interpreted as the number density (cf. mass density, tensor density, long. momentum density)
- $\mathcal{M}^2(r_\perp)$ can be interpreted as the invariant mass squared density
- $\vec{\mathcal{S}}(r_\perp)$ can be interpreted as the spin current density

$$P^- = \frac{P^2_\perp + M^2}{P^+}$$

$$\mathcal{A}(r_\perp) = \int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp\cdot\vec{r}_\perp} A(q^2_\perp),$$

$$\mathcal{M}^2(r_\perp) = \int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp\cdot\vec{r}_\perp} \Big[(M^2 + \frac{1}{4}q^2_\perp)A(q^2_\perp) + \frac{1}{2}q^2_\perp D(q_\perp)\Big],$$

$$\vec{\mathcal{S}}(r_\perp) = 2\vec{s}\int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp\cdot\vec{r}_\perp} J(q^2_\perp)$$

Frame independent!

Relativistic quantum many-body rep'n on the light front

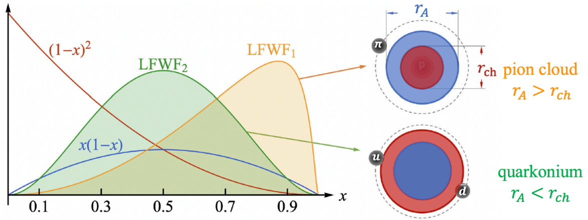
- Drell-Yan-West formula: transverse charge density as exact OBD: [Drell:1969km, West:1970av]

$$\rho_{\text{ch}}(r_\perp) = \sum_n \int \left[\mathrm{d}x_i \mathrm{d}^2 r_{i\perp}\right]_n \left|\widetilde{\psi}_n(\{x_i, \vec{r}_{i\perp}\})\right|^2 \sum_j e_j \delta^2(r_\perp - r_{j\perp}) \equiv \Big\langle \sum_j e_j \delta^2(r_\perp - r_{j\perp}) \Big\rangle$$

- Brodsky-Hwang-Ma-Schmidt formula: number density $\mathcal{A}(r_\perp)$ as exact OBD : [Brodsky:2000ii]

$$\mathcal{A}(r_\perp) = \Big\langle \sum_j x_j \delta^2(r_\perp - r_{j\perp}) \Big\rangle$$

Number density $\mathcal{A}(r_\perp)$ mainly samples the valence partons $x_j \sim O(1)$; wee parton $x_j \ll 1$ contributions suppressed



Wave function representation of D-term

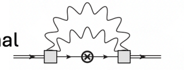
International Journal of Modern Physics A | Vol. 33, No. 26, 1830025 (2018)
| Reviews

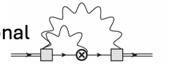
Forces inside hadrons: Pressure, surface tension, mechanical radius, and all that

Maxim V. Polyakov and Peter Schweitzer

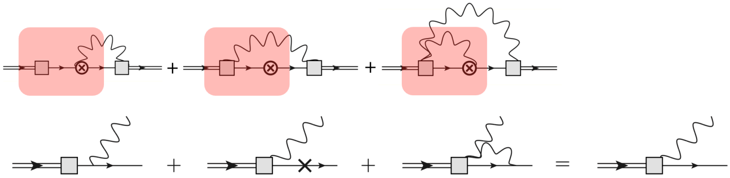
https://doi.org/10.1142/S0217751X18300259 | **Cited by:** 212 **(Source: Crossref)**

$\hat{T}_{++}$ of the EMT. Being related to the stress tensor $\hat{T}_{ij}$ the form factor $D(t)$ naturally "mixes" good and bad light-front components and is described in terms of transitions between different Fock state components in overlap representation. As a quantity intrinsically nondiagonal in a Fock space, it is difficult to study the D-term in approaches based on light-front wave functions. This is due to the rela-

diagonal 

non-diagonal 

- In an explicit model calculation (scalar model in 3+1D), we showed that all non-diagonal contributions add up to a diagonal contribution [Cao:2023ohj]



LFWF representation of energy and stress

[Cao:2023ohj, Cao:2024rul]

$$t^{12} = \frac{1}{2}\Big\langle \sum_j e^{-i\vec{q}_\perp\cdot\vec{r}_{j\perp}} \frac{i\overleftrightarrow{\nabla}^1_{j\perp} i\overleftrightarrow{\nabla}^2_{j\perp} - q^1_\perp q^2_\perp}{x_j} \Big\rangle,$$

$$t^{+-} = 2\Big\langle \underbrace{\sum_j e^{i\vec{r}_{j\perp}\cdot\vec{q}_\perp} \frac{-\frac{1}{4}\overleftrightarrow{\nabla}^2_{j\perp} + m_j^2 - \frac{1}{4}q^2_\perp}{x_j}}_{\text{kinetic part}} + \underbrace{V e^{i\vec{r}_{N\perp}\cdot\vec{q}_\perp}}_{\text{potential part}} \Big\rangle$$

where, $V = M^2 - \sum_j \frac{-\nabla^2_{j\perp}+m_j^2}{x_j}$, and the quantum average is defined as,

$$\langle O\rangle \equiv \sum_n \int \left[\mathrm{d}x_i \mathrm{d}^2 r_{i\perp}\right]_n \widetilde{\psi}^*_n(\{x_i, \vec{r}_{i\perp}\}) O_n \widetilde{\psi}_n(\{x_i, \vec{r}_{i\perp}\})$$

- The off-shell factors $e^{i\vec{r}_{j\perp}\cdot\vec{q}_\perp} \xrightarrow{\text{F.T.}} \delta^2(r_\perp - r_{j\perp})$ indicate the location of the graviton coupling
- Large contributions from the wee parton -- condensates?
- Use scalar model as an example

EMT of a relativistic spin medium

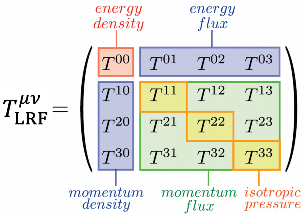
[Rezzolla, Relativistic Hydrodynamics, 2013]

$$t^{\alpha\beta} = eu^\alpha u^\beta - p\Delta^{\alpha\beta} + \tfrac{1}{2}\partial_\sigma(u^{\{\alpha}s^{\beta\}\sigma}) + \pi^{\alpha\beta} - g^{\alpha\beta}\Lambda + \text{dissipative terms}$$

where, u^α is the medium velocity with $u_\alpha u^\alpha = 1$, $\Delta^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ is the spatial metric tensor. $a^{\{\mu}b^{\nu\}} = a^\mu b^\nu + a^\nu b^\mu$.

- $e(x)$ -- proper energy density, i.e. energy density measured in local rest frame (LRF)
- $c^{\alpha\beta} = \pi^{\alpha\beta} - p\Delta^{\alpha\beta}$ -- Cauchy stress tensor, consisting of a traceless shear tensor and a normal pressure $p(x)$.
- $\pi^{\alpha\beta}(x)$ -- shear tensor, dissipative in fluids but non-dissipative in solids
- $s^{\alpha\beta}(x)$ -- spin tensor, recently proposed by Fukushima et. al. in relativistic spin hydrodynamics
- Λ -- cosmological constant term, non-conserving, representing an external pressure [Teryaev:2013qba, Teryaev:2016edw, Liu:2023cse]
- Coupled multifluids vs Interacting multifluids

[Fukushima:2020ucl, cf. Li:2020eon]



Hadronic energy-momentum tensor

[Li:2024vgv]

- It can be shown that the quantum expectation value of the EMT tensor can be written as,

$$\langle\Psi|T^{\alpha\beta}(x)|\Psi\rangle = \langle \mathcal{E}\mathcal{U}^\alpha\mathcal{U}^\beta - \mathcal{P}\Delta^{\alpha\beta} + \tfrac{1}{2}\partial_\rho(\mathcal{U}^{\{\alpha}\mathcal{S}^{\beta\}\rho}) + \mathit{\Pi}^{\alpha\beta} - g^{\alpha\beta}\Lambda\rangle_\Psi$$

where,

$$\langle\mathcal{O}(x)\rangle_\Psi = \int \mathrm{d}^3 z\, \overline{\Psi}(z)\mathcal{O}(x-z)\Psi(z)\Big|_{x^0=z^0},$$

is a convolution with the wavepacket $\Psi(x)$.

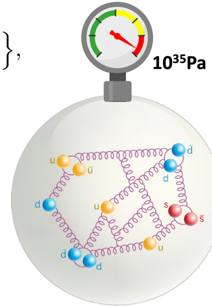
$$\mathcal{E}(x) = M\int\frac{\mathrm{d}^3q}{(2\pi)^3}e^{iq\cdot x}\Big\{\Big(1-\frac{q^2}{4M^2}\Big)A(q^2) + \frac{q^2}{4M^2}\Big[2J(q^2)-D(q^2)\Big]\Big\},$$

$$\mathcal{P}(x) = \frac{1}{6M}\int\frac{\mathrm{d}^3q}{(2\pi)^3}e^{iq\cdot x}q^2D(q^2),$$

$$\mathcal{S}^{\alpha\beta}(x) = \int\frac{\mathrm{d}^3q}{(2\pi)^3}e^{iq\cdot x}\Big\{i\sigma^{\alpha\beta}\sqrt{1-\frac{q^2}{4M^2}} - \frac{U^{[\alpha}q^{\beta]}}{2M}\Big\}J(q^2),$$

$$\mathit{\Pi}^{\alpha\beta}(x) = \frac{1}{4M}\int\frac{\mathrm{d}^3q}{(2\pi)^3}e^{iq\cdot x}\Big(q^\alpha q^\beta - \frac{q^2}{3}\Delta^{\alpha\beta}\Big)D(q^2),$$

$$\Lambda = -M^2\int\frac{\mathrm{d}^3q}{(2\pi)^3}e^{iq\cdot x}\bar{c}(q^2)$$



Factorization of the hadronic density: stationary limit [Li:2022ldb, cf. Freese:2022fat]

$$e(x) = \int \mathrm{d}^3 z\, \overline{\Psi}(x-z)\Big\{ M \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{iq\cdot x} \Big\{ \Big(1 - \frac{q^2}{4M^2}\Big) A(q^2) \; + \frac{q^2}{4M^2}\Big[2J(q^2) - D(q^2)\Big] \Big\} \Big\} \Psi(x-z)\Big|_{z^0=0}$$

The hadronic part is *not factorizable* due to the dependence of $\vec{P} = (-i/2)\overleftrightarrow{\nabla}_x$ in $q^2 = (q^0)^2 - \vec{q}^2$, where $q^0 = \sqrt{(\vec{P}+\frac{1}{2}\vec{q})^2 + M^2} - \sqrt{(\vec{P}-\frac{1}{2}\vec{q})^2 + M^2}$

- Taylor expansion around $\vec{P} = 0$: multipole series,

$$\mathcal{E}(\vec{r}) = \sum_{n=0}^{\infty} \frac{(-i)^n}{2^n n!} \mathcal{E}_n^{i_1 i_2 \cdots i_n}(\vec{r}) \overleftrightarrow{\nabla}^{i_1} \overleftrightarrow{\nabla}^{i_2} \ldots \overleftrightarrow{\nabla}^{i_n}$$

- Monopole density gives the Breit-frame distribution (Sachs distribution)

$$\mathcal{E}_0(\vec{r}) = M \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \Big\{ \Big(1 + \frac{\vec{q}^2}{4M^2}\Big) A(-\vec{q}^2) - \frac{\vec{q}^2}{4M^2}\Big[2J(-\vec{q}^2) - D(-\vec{q}^2)\Big] \Big\}$$

- High-multipole moments exist due to Lorentz distortion
- Note that for spin-0, the monopole density $\mathcal{E}_0$ differs from the Breit-frame distribution given by Polyakov & Schweitzer

Infinite momentum limit

[Li:2022ldb]

Is the multipole expansion unique? No! $\to$ Alternative: Taylor (Laurent) expansion around $1/|\vec{P}| = 0$

- Sufficient to take $P_z \to \infty \;\Rightarrow\; |\vec{P}| = \sqrt{\vec{P}_\perp^2 + P_z^2} \to \infty$
- Monopole density gives the 2D light-front distribution

$$\mathcal{E}_0(x) = \delta(x_\parallel) M \int \frac{\mathrm{d}^2 q_\perp}{(2\pi)^2} e^{-i\vec{q}_\perp\cdot\vec{x}_\perp} \Big[\Big(1 + \frac{\vec{q}_\perp^{\,2}}{4M^2}\Big) A(-\vec{q}_\perp^{\,2}) - \frac{\vec{q}_\perp^{\,2}}{4M^2}\Big(2J(-\vec{q}_\perp^{\,2}) - D(-\vec{q}_\perp^{\,2})\Big)\Big].$$

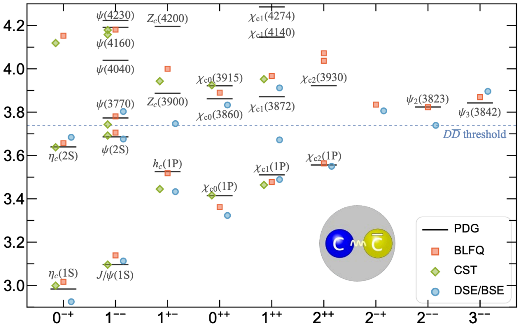
- No special frame (e.g. Drell-Yan $q^+ = 0$ frame) is chosen
- Relativistic, suitable also for massless hadrons (in contrast to the Sachs distribution)
- Convergence of the multipole series: $|\vec{P}| \gg \lambda_{\text{hadron}}^{-1} \gg \{M, r_{\text{hadron}}^{-1}\}$ -- sufficiently *localized* z-direction
- In the infinite momentum frame (IMF), components of the EMT form a hierarchy:

$$\underbrace{\mathcal{T}^{++} \sim P_z^2}_{\text{best}}, \quad \underbrace{\mathcal{T}^{+i} \sim P_z^1}_{\text{good}}, \quad \underbrace{\mathcal{T}^{+-} \sim \mathcal{T}^{ij} \sim P_z^0}_{\text{bad}}, \quad \underbrace{\mathcal{T}^{-i} \sim P_z^{-1}}_{\text{worse}}, \quad \underbrace{\mathcal{T}^{--} \sim P_z^{-2}}_{\text{worst}}$$

Charmonium: "hydrogen atom" of QCD

[Li:2015zda, Li:2017mlw]

$$H = \sum_i \frac{p_{i\perp}^2 + m_i^2}{x_i} + U_i^{(0)} + \sum_{ij} \alpha_s U_{ij}^{(1)} + \sum_{ij} V_{ij}^{\text{QCD}} - U_{ij}$$

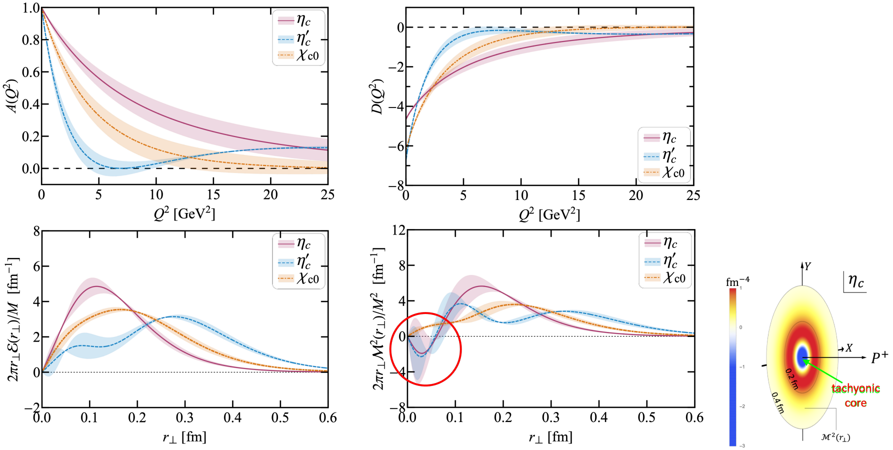


- Basis light-front quantization: two free parameters (m_c, κ), rms deviation: 30 MeV
- Good agreement with the PDG data for both the masses and decay constants and radiative widths

Charmonium gravitational form factors

[Xu:2024cfa; Hu:2024edc]

- Obtained GFF D from both T^{+-} (via localizing P^-) and T^{12}, results consistent with each other
- Energy density $\mathcal{E}(r_\perp)$ is positive. However, $\mathcal{M}^2(r_\perp)$ is negative at small $r_\perp$: tachyonic core within hadrons?

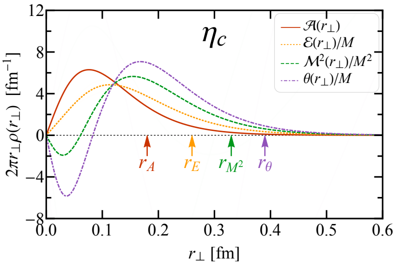


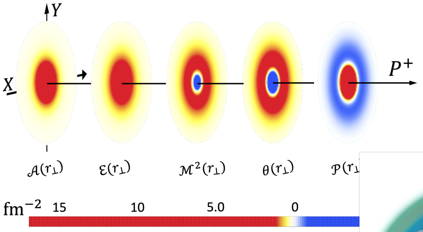
Physical densities

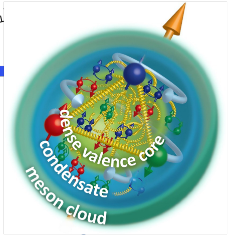
[Xu:2024cfa]

- Number density $\mathcal{A}(r_\perp)$, energy density $\mathcal{E}(r_\perp)$, invariant mass squared density $\mathcal{M}^2(r_\perp)$ and trace scalar density $\theta(r_\perp)$
- Onion like structure -- true for all hadrons with negative D

$$r_A < r_E < r_{M^2} < r_\theta < r_D$$





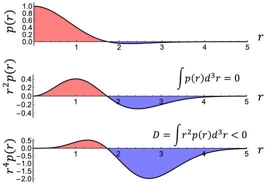


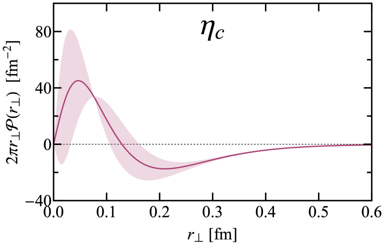
Mechanical stability conjecture $D < 0$

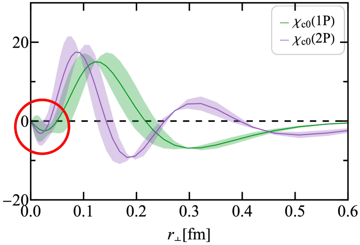
[Hu:2024edc]

$$D = \int \mathrm{d}^3 r \, r^2 \mathcal{P}(r) \overset{???}{<} 0$$

- Speculation: a mechanically stable system must have a repulsive core and an attractive edge
- We find that while η_c has a repulsive core, χ_{c0} has an attractive cores, and both have negative D!







Summary

- Hadrons are unique relativistic systems $r \sim \lambda_C$ which enables us to view them as de Broglie waves. Minkowski-Einstein-Laub's relativistic theory of macroscopic electromagnetism provides the relevant concepts & tools for describing their electromagnetic structure in 3D
- We showed the multipole structures of relativistic currents (first proposed by Born) -- Sachs & light-front distributions can be understood as monopole densities
- I also discussed the relativistic quantum many-body description of form factors as 2D Fourier transform (F.T.) of one-body densities (OBDs) on the light front
- I extended the concepts to the hadronic energy-momentum tensor and discussed the mechanical properties of hadrons
- So far, I focused on the interpretations. Are there any physical consequences of the interpretations? (Gravity, soft-photon, energy conditions, ...)

Thank you!

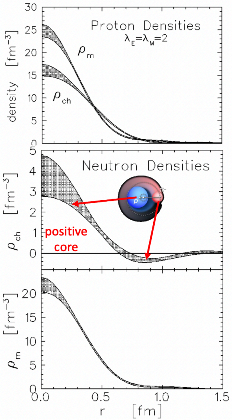

I think we're gonna need
Backup

Transverse charge densities of the nucleons

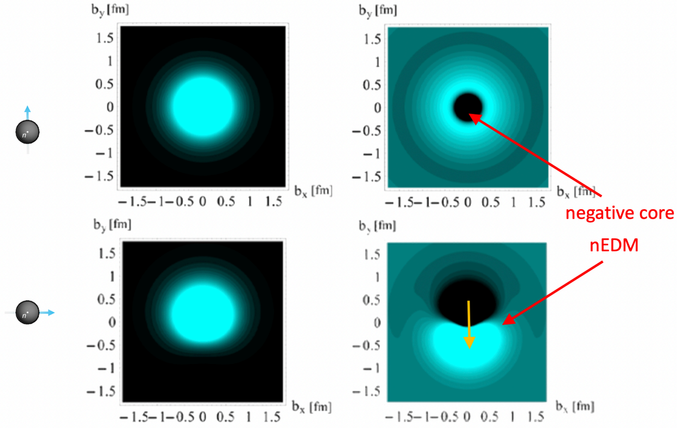
[Miller:2010nz]

$$\rho_{\text{ch}}^{\text{Sachs}}(r) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{\langle +\frac{1}{2}\vec{q}_\perp, s|J^0(0)| -\frac{1}{2}\vec{q}_\perp, s\rangle}{2E}$$

$$\rho_{\text{ch}}^{\text{LF}}(\vec{b}_\perp; \vec{S}) = \int \frac{d^2q_\perp}{(2\pi)^2} e^{i\vec{q}_\perp\cdot\vec{b}_\perp} \frac{\langle P^+, +\frac{1}{2}\vec{q}_\perp, \vec{S}|J^+(0)|P^+, -\frac{1}{2}\vec{q}_\perp, \vec{S}\rangle}{2P^+}$$



proton **neutron**



Phase space approach

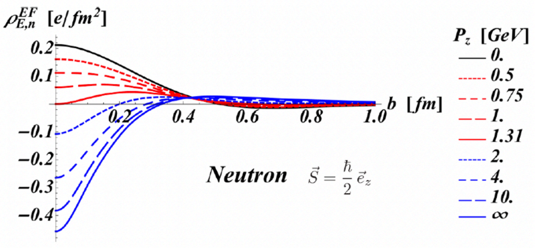
[Lorce:2020onh, Chen:2022smg]

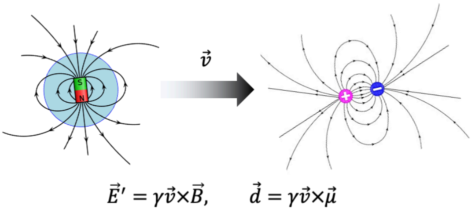
Lorcé interpreted the F.T. as the Weyl function of the current operator,

$$\rho_W(\vec{r}, \vec{P}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} \frac{\langle \vec{P} + \frac{1}{2}\vec{q} | J^0 | \vec{P} - \frac{1}{2}\vec{q} \rangle}{2P^0}$$

- Weyl quantization $O \leftrightarrow O_W(\vec{r}, \vec{p})$, eps. the Wigner function $\varrho \leftrightarrow W(\vec{r}, \vec{p})$
- Sachs distribution: $\vec{P} = 0$; Light-front distribution: $\vec{P} \to \infty$ (IMF)
- Quasi-probabilistic
- Still need a special frame, $q^0 = 0$ (elastic frame)

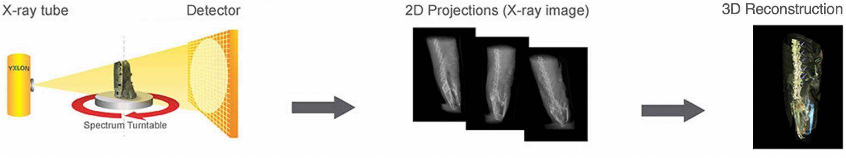
$$\overline{O}_\Psi = \int \mathrm{d}^3 r \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \times O_W(\vec{r}, \vec{p}) W(\vec{r}, \vec{p})$$





From 2D to 3D: Abel-Radon transformation

[Panteleeva:2021iip]



- An Abel-Radon signal at angle $\hat{n}$ and distance s is obtained by the line integral along the path of the X ray.

$$\mathcal{R}f(s,\hat{n}) = \int \mathrm{d}^3 r\, f(\vec{r})\delta(s - \hat{n}\cdot\vec{r})$$

- Inversion problem
- Panteleeva & Polyakov showed with some reasonable assumptions the 3D Sachs and the 2D light-front distributions can be related by the Abel-Radon transformation [Panteleeva:2021iip]

Example: Gaussian wavepacket

- Normalization of the momentum space wave function:

$$\langle\Psi|\Psi\rangle = 1,\ \langle p'|p\rangle = 2p^0(2\pi)^3\delta^3(p-p') \quad \Rightarrow \quad \int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}\widetilde{\Psi}^*(\vec{p})\widetilde{\Psi}(\vec{p}) = 1$$

where, $p^0 = \sqrt{\vec{p}^2 + M^2}$ is the on-shell energy.

- Gaussian wavepacket:

$$\widetilde{\Psi}(\vec{p}) = N_\sigma e^{-\frac{\vec{p}^2}{2\sigma^2}}$$

where, σ is the width of the Gaussian, and $N_\sigma = 4\pi^{\frac{3}{4}}/[\sigma U^{-\frac{1}{2}}(\frac{1}{2}, 0, M^2/\sigma^2)]$. $U(a,b,c)$ is the confluent hypergeometric function of the second kind. Plane wave limit: $\sigma \to 0$ and localization limit $\sigma \to \infty$

- Normalization of the coordinate-space wave function,

$$2M\int \mathrm{d}^3x\,\big|\Psi(x)\big|^2 = \frac{M}{\sigma}\frac{U(1,\frac{1}{2},\frac{M^2}{\sigma^2})}{U(\frac{1}{2},0,\frac{M^2}{\sigma^2})} = \begin{cases} 1 - \frac{3\sigma^2}{4M^2} + O(\frac{\sigma^4}{M^4}) & (\sigma\to 0), \\ \frac{\sqrt{\pi}M}{\sigma} - \frac{\pi M^2}{\sigma^2} + O(\frac{M^3}{\sigma^3}) & (\sigma\to\infty) \end{cases}$$

The wave function cannot be normalized in the localization limit or the massless limit.

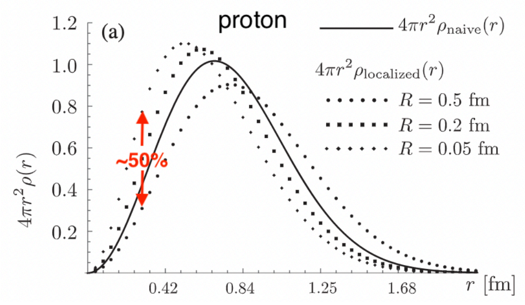
Dependence on wavepacket

[Jaffe:2020ebz; Lorce:2020onh; Belitsky:2003nz; Ji:2004gf; Belitsky:2005qn]

- Consider a Gaussian wavepacket:

$$r_\Psi^2 \equiv \int \mathrm{d}^3x\, \vec{x}^2 \,\langle\Psi|J^0(x)|\Psi\rangle = 6F'(0) + 3R_\Psi^2$$

  where, R_Ψ is the size of the wavepacket.
- Sachs distribution is valid (i.e. $r_\Psi^2 \approx 6F'(0)$) if, $r_\text{hadron} \gg Q_\text{max}^{-1} \gg R_\Psi \gg M^{-1}$
- However, for hadrons $r_\text{hadron} M \sim 1$
- Factorization approach works for $R_\Psi \gg r_\text{hadron}$



	r	M	rM
pion	0.67 fm	0.14 GeV	0.5
charmonium	0.25--0.4 fm	3.0 GeV	3.8--6
proton	0.87 fm	0.94 GeV	4
nuclei	1.3 $A^{\frac{1}{3}}$ fm	0.94A GeV	6$A^{\frac{4}{3}}$

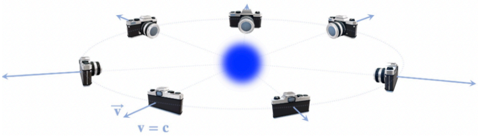
Sharply localized wavepacket

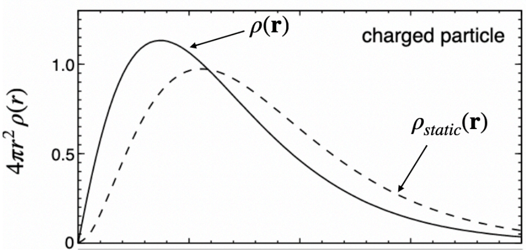
[Epelbaum:2022fjc, Panteleeva:2022khw]

Epelbaum et al. proposed to use sharply localized wavepacket $R_\Psi \to 0$ with spherical symmetry,

$$\rho(\vec{r}) = \int \frac{\mathrm{d}^3}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \int_{-1}^{+1} \mathrm{d}\alpha \frac{1}{2} F_{\text{ch}}\Big[-(1-\alpha^2)\vec{q}^2\Big]$$

- Angle-averaged light-front density
- Particle localization problem in QFTs
- $R_\Psi \to 0$ limit is not well defined, e.g. $R_x = 2R_y \to 0$ gives different results





Factorization formula

[Li:2022ldb; cf. Freese:2022fat]

$$j^\alpha(x) = \int \frac{\mathrm{d}^3 P}{(2\pi)^3} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \widetilde{\Phi}^*(\vec{P} + \tfrac{1}{2}\vec{q})\widetilde{\Phi}(\vec{P} - \tfrac{1}{2}\vec{q}) \frac{P^\alpha}{2p^0 p'^0} F_{\mathrm{ch}}(q^2) e^{iq\cdot x},$$

$$= \int \mathrm{d}^3 x_1 \Phi^*(\vec{x}_1, t) \int \mathrm{d}^3 x_2 i\partial^\alpha \Phi(\vec{x}_2, t) \int \frac{\mathrm{d}^3 P}{(2\pi)^3} e^{i\vec{P}\cdot(\vec{x}_1 - \vec{x}_2)} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} F_{\mathrm{ch}}(q^2) e^{-i\vec{q}\cdot(\vec{x} - \frac{\vec{x}_1 + \vec{x}_2}{2})}$$

$$- \int \mathrm{d}^3 x_1 i\partial^\alpha \Phi^*(\vec{x}_1, t) \int \mathrm{d}^3 x_2 \Phi(\vec{x}_2, t) \int \frac{\mathrm{d}^3 P}{(2\pi)^3} e^{i\vec{P}\cdot(\vec{x}_1 - \vec{x}_2)} \int \frac{\mathrm{d}^3 q}{(2\pi)^3} F_{\mathrm{ch}}(q^2) e^{-i\vec{q}\cdot(\vec{x} - \frac{\vec{x}_1 + \vec{x}_2}{2})}$$

The above integral is not factorizable because q^0 is not independent of $\vec{P}$:

$$q^0 = \sqrt{(\vec{P} + \tfrac{1}{2}\vec{q})^2 + M^2} - \sqrt{(\vec{P} - \tfrac{1}{2}\vec{q})^2 + M^2}$$

However, we can evaluate the $\mathrm{d}^3 P$ integration, provided all $\vec{P} \to (-i/2)\overleftrightarrow{\nabla}$ inside q^0:

$$\boxed{j^\alpha(x) = \int \mathrm{d}^3 r \, \Phi(\vec{r}, t) i\overleftrightarrow{\partial}^\alpha_r D(\vec{x} - \vec{r}; \tfrac{-i}{2}\overleftrightarrow{\nabla}_r)\Phi(\vec{r}, t)}$$

where, the intrinsic density is idensitified as,

$$D(\vec{x}; \vec{P}) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3} F_{\mathrm{ch}}(q^2) e^{-i\vec{q}\cdot\vec{x}}$$

Covariant analysis of EM current on the light cone [Zhang:2025wli]

In Drell-Yan frame ($\omega \cdot q = 0$):

$$\langle p' | J^\mu(0) | p \rangle = 2P^\alpha F(-q^2) + \frac{M^2 \omega^\mu}{\omega \cdot P} S(-q^2),$$

where, $P = (p' + p)/2$, $q = p' - p$. $\omega^\mu = (\omega^+, \omega^-, \vec{\omega}_\perp) = (0, 2, 0)$ is a null vector indicating the orientation of the quantization surface.

- Emergence of spurious form factors S due to the violation of dynamical Lorentz symmetries in practical calculations, which usually contain uncanceled divergences
- EM current is automatically conserved in the Drell-Yan frame $q^+ = 0$
- Identify J^+, $\vec{J}_\perp$ as the good currents that are free of spurious form factors or divergence

Covariant analysis of EMT on the light cone

[Cao:2024rul]

In Drell-Yan frame ($\omega \cdot q = 0$):

$$\langle p'|T_i^{\alpha\beta}(0)|p\rangle = 2P^\alpha P^\beta A_i(-q^2) + \frac{1}{2}(q^\alpha q^\beta - q^2 g^{\alpha\beta})D_i(-q^2) + 2M^2 g^{\alpha\beta}\bar{c}_i(-q^2) + \frac{M^4\omega^\alpha\omega^\beta}{(\omega\cdot P)^2}S_{1i}(-q^2) + (V^\alpha V^\beta + q^\alpha q^\beta)S_{2i}(-q^2),$$

where, $P = (p' + p)/2$, $q = p' - p$. $\omega^\mu = (\omega^+, \omega^-, \vec{\omega}_\perp) = (0, 2, 0)$ is a null vector indicating the orientation of the quantization surface. Vector V^α is defined as $V^\alpha = \varepsilon^{\alpha\beta\rho\sigma}P_\beta q_\rho \omega_\sigma/(\omega\cdot P)$.

- Emergence of spurious form factors $S_{1,2}$ due to the violation of dynamical Lorentz symmetries in practical calculations, which usually contain uncanceled divergences
- Identify T^{++}, T^{+i}, T^{12}, T^{+-} as the good currents that are free of spurious form factors or divergence

$$t_i^{++} = 2(P^+)^2 A_i(q_\perp^2),$$

$$t_i^{12} = \frac{1}{2}q_\perp^1 q_\perp^2 D_i(q_\perp^2),$$

$$t_i^{+-} = 2(M^2 + \frac{1}{4}q_\perp^2)A_i(q_\perp^2) + q_\perp^2 D_i(q_\perp^2) + 4M^2\bar{c}_i(q_\perp^2)$$

$$t_i^{--} = 2\Big(\frac{M^2 + \frac{1}{4}q_\perp^2}{P^+}\Big)^2 A_i(q_\perp^2) + \frac{4M^4}{(P^+)^2}S_{1i}(q_\perp^2).$$

$$t_i^{11} + t_i^{22} = -\frac{1}{2}q_\perp^2 D_i(q_\perp^2) - 4M^2\bar{c}_i(q_\perp^2) + 2q_\perp^2 S_{2i}(q_\perp^2).$$