Graphics&Geometric Computing Lab @USTC

Differential Geometry & Discrete Operators

Ligang Liu
Graphics&Geometric Computing Lab
USTC
http://staff.ustc.edu.cn/~lgliu

Curves

- Tangent vector to curve $C(t)=(x(t),y(t))$ is

$$T = C'(t) = \frac{dC(t)}{dt} = \left[x'(t), y'(t)\right]$$

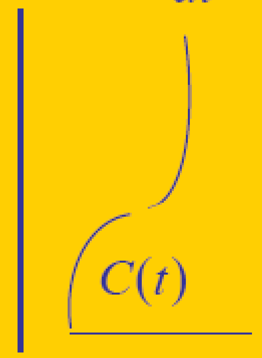
- Unit length tangent vector

$$\vec{T} = \vec{C}(t) = \frac{\left[x'(t), y'(t)\right]}{\sqrt{x'(t)^2 + y'(t)^2}}$$

- Curvature

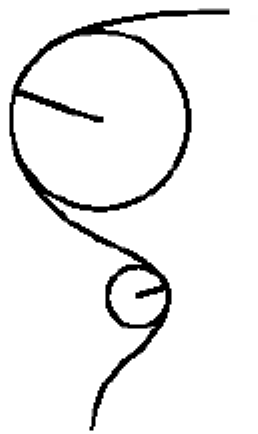
$$k(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}$$

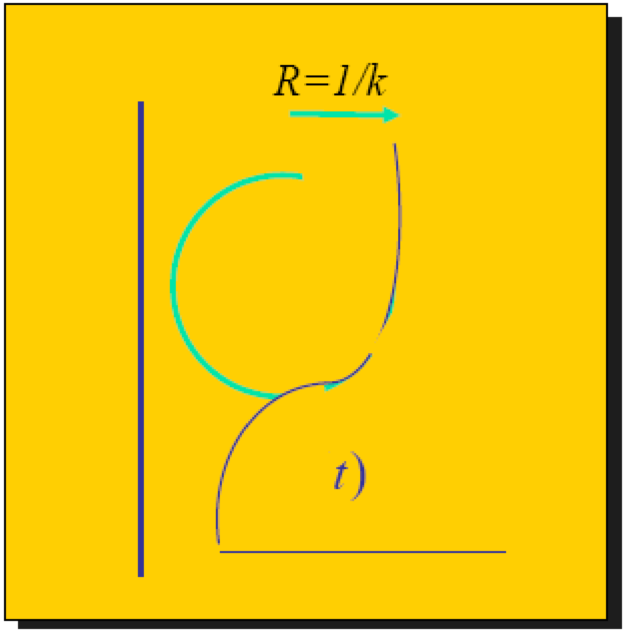
$$T = \frac{dC(t)}{dt}$$



Curve Curvature

- Curvature is **independent** of parameterization
  - $C(t)$, $C(t+5)$, $C(2t)$ have same curvature (at corresponding locations)
- Corresponds to radius of osculating circle $R=1/k$
- Measure curve bending





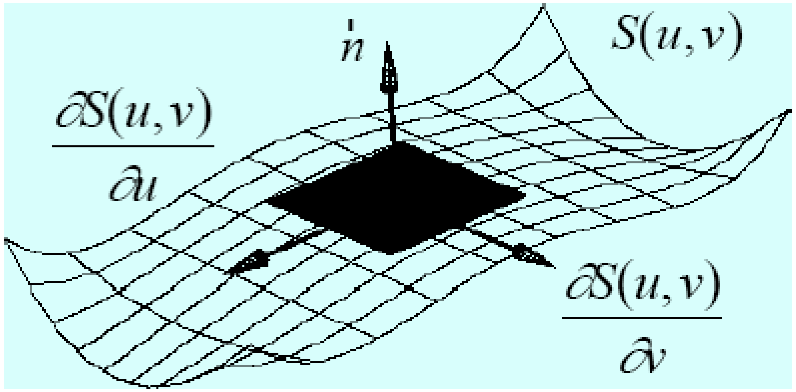
Surfaces

- Tangent plane to surface $S(u,v)$ is spanned by two partials of S:

$$\frac{\partial S(u,v)}{\partial u} \qquad \frac{\partial S(u,v)}{\partial v}$$

- **Normal** to surface

$$\vec{n} = \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}$$



  - perpendicular to tangent plane
- Any vector in tangent plane is tangential to $S(u,v)$

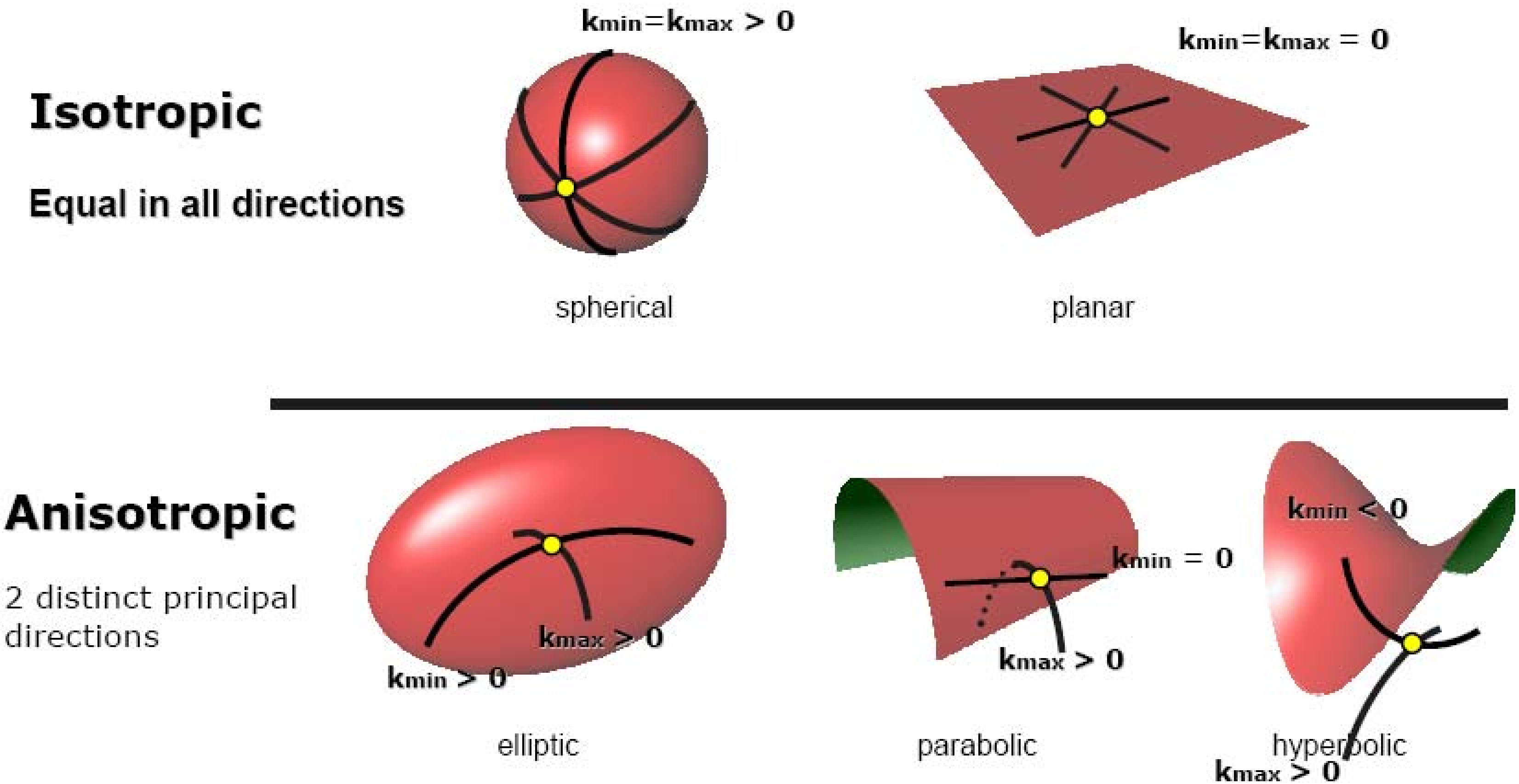
Surface Curvature

- ***Normal curvature*** of surface is defined for each tangential direction

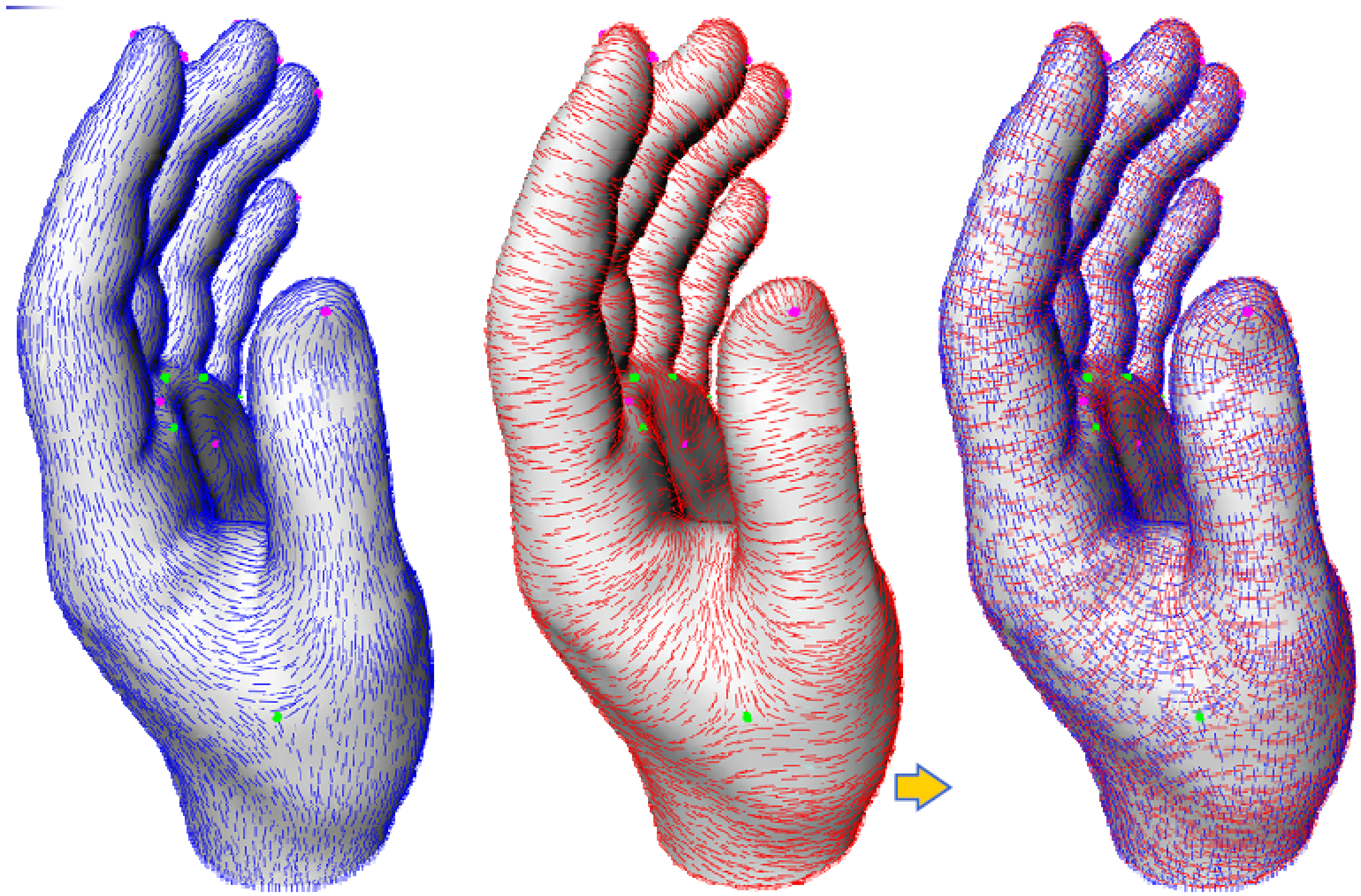
$$\kappa^N(\theta) = \kappa_1 cos^2(\theta) + \kappa_2 sin^2(\theta)$$

- ***Principal curvatures*** $Kmin$ & $Kmax$: maximum and minimum of normal curvature
  - Correspond to two **orthogonal** tangent directions
    - Principal directions
  - Not necessarily partial derivative directions
  - Independent of parameterization

Surface Curvature



Principal Directions



Min Curvature Max Curvature

Surface Curvatures

- Typical measures:
  - ***Gaussian*** curvature

$$K = k_{\min} k_{\max}$$

  - ***Mean*** curvature

$$H = \frac{k_{\min} + k_{\max}}{2}$$

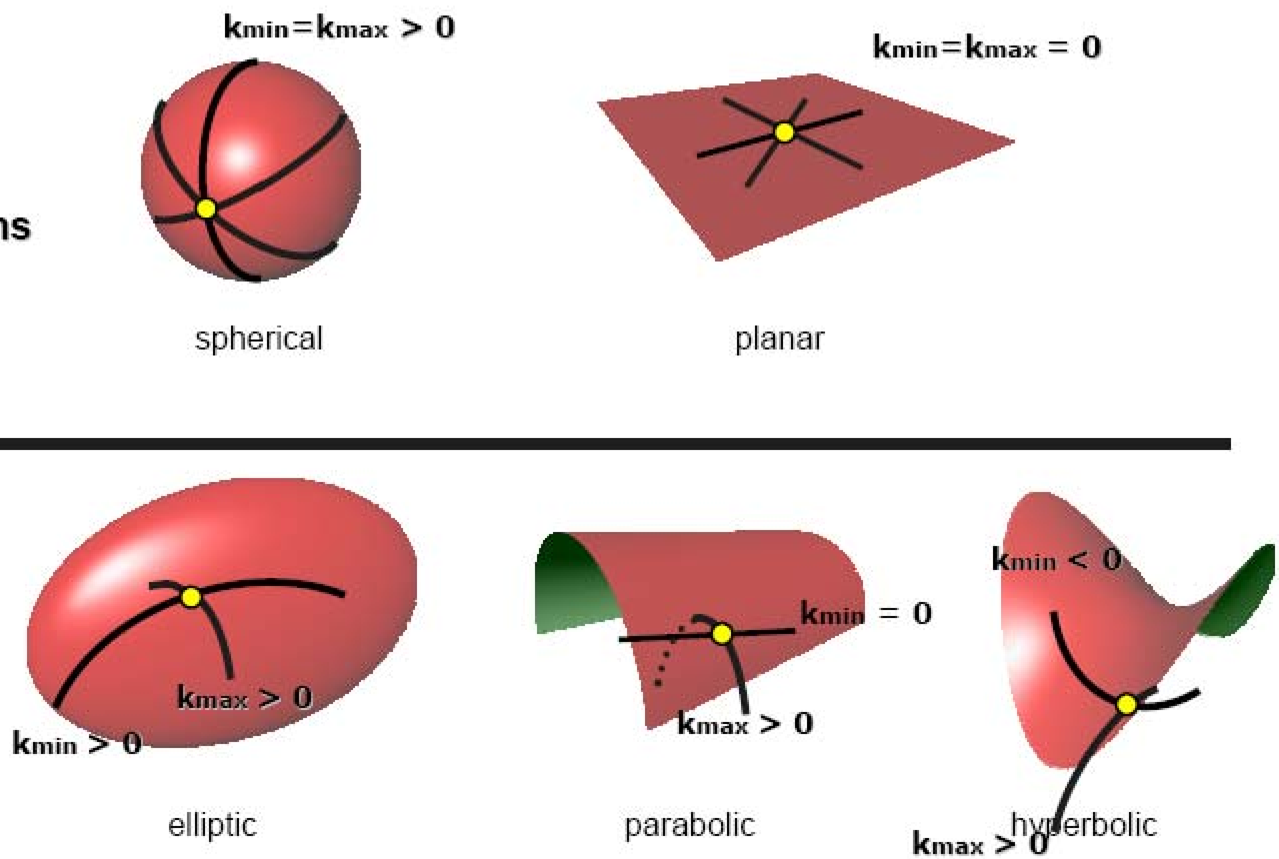
Surface Curvature

Isotropic

Equal in all directions

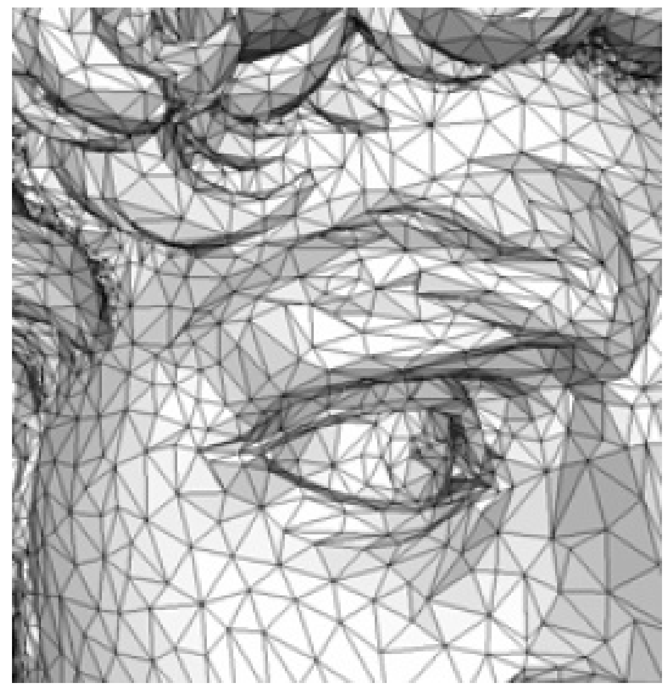
Anisotropic

2 distinct principal directions



However, meshes are only C^0

- Meshes are piecewise linear surfaces
  - Infinitely continuous on triangles
  - C^0 at edges and vertices



Discrete Differential Geometry

- How to apply the traditional differential geometry on discrete mesh surfaces?
  - Normal estimation
  - Curvature estimation
  - Principal curvature directions
  - …

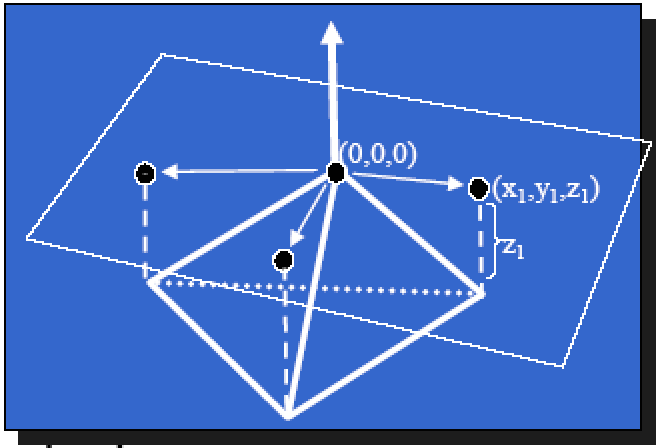
Estimation of Differential Measures

- Approximate the (unknown) underlying surface
  - Continuous approximation
    - Approximate the surface & compute continuous differential measures (normal, curvature)
  - Discrete approximation
    - Approximate differential measures for mesh

Continuous Approximation

Quadratic Approximation

- Approximate surface by quadric
- At each mesh vertex (use surrounding triangles)
  - Compute normal at vertex
    - Typically average face normals
  - Compute tangent plane & local coordinate system
    - (node = (0,0,0))



  - For each neighbor vertex compute location in local system
    - relative to node and tangent plane

Quadratic Approximation (2)

- Find quadric function approximating vertices

$$F(x,y,z) = ax^2 + bxy + cy^2 - z = 0$$

- To find coefficients use least squares fit

$$\min \sum_i (ax_i^2 + bx_iy_i + cy_i^2 - z_i)$$

Quadratic Approximation (3)

Finding the quadric function approximating points
$F(x,y,z) = ax^2 + bxy + cy^2 - z = 0$

To find coefficients use least square fit to find minimum: $min \sum_i \left(ax_i^2 + bx_i y_i + cy_i^2 - z_i\right)$

$$\begin{pmatrix} x_1^2 & x_1 y_1 & y_1^2 \\ \dots & \dots & \dots \\ x_n^2 & x_n y_n & y_n^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix} \quad A = \begin{pmatrix} x_1^2 & x_1 y_1 & y_1^2 \\ \dots & \dots & \dots \\ x_n^2 & x_n y_n & y_n^2 \end{pmatrix}, \; X = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \; b = \begin{pmatrix} z_1 \\ \dots \\ z_n \end{pmatrix}$$

Approximation can be found by: $\tilde{X} = (A^T A)^{-1} A^T b$

Quadratic Approximation (4)

- Given surface F its principal curvatures k_{min} and k_{max} are real roots of:

$$k^2 - (a+c)k + ac - b^2 = 0$$

- *Mean curvature*: $H = (k_{min} + k_{max})/2$

- *Gaussian curvature:* $K = k_{min} k_{max}$

Other approximation

- Cubic approximation
  - J. Goldfeather and V. Interrante. A novel cubic-order algorithm for approximating principal direction vectors. ACM Transactions on Graphics 23, 1 (2004), 45–63.
- Implicit surface approximation
  - Yutaka Ohtake et al. Multi-level partition of unity implicits. Siggraph 2003.
- Many others…

Discrete Approximation

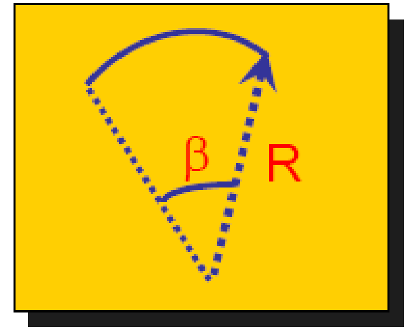
Normal Estimation

- Normal estimation on vertices
  - Defined for each face
  - Average face normals
    - Weighted: face areas, angles at vertex
- What happen at edges/creases?

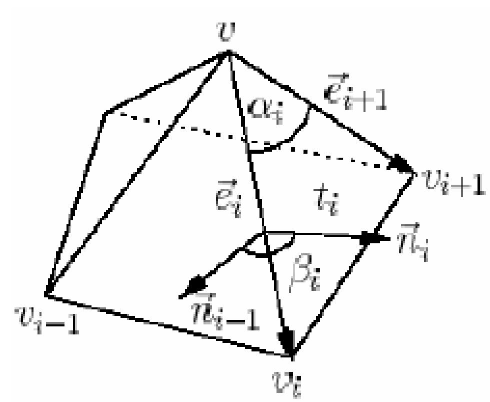
Mean Curvature

- Integral of curvature on circular arc
  - β - central angle

$$\int k = \frac{1}{R} arclenght = \frac{1}{R}\frac{\beta}{2\pi}2\pi R = \beta$$



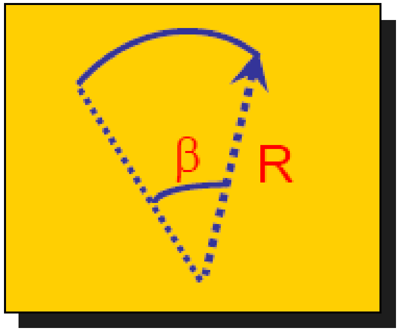
- On cylindrical parts $H = k_{max}/2$ ($k_{min}=0$)
- On planar faces $H=0$



Mean Curvature (2)

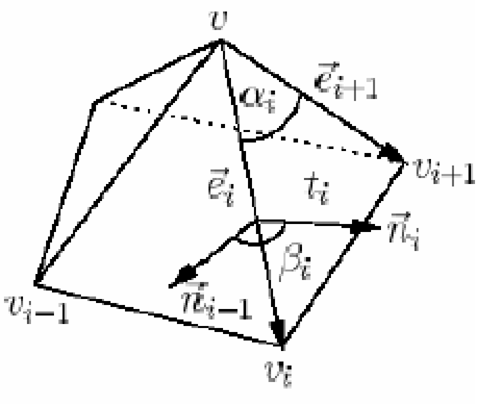
- For entire vertex region

$$\int H = \sum_i \beta_i / 2 \| e_i \| / 2 = \frac{1}{4} \sum_i \beta_i \| e_i \|$$



- Mean curvature at vertex (A_i triangle area)

$$H = \frac{3}{4 \sum A_i} \sum_i \beta_i \| e_i \|$$



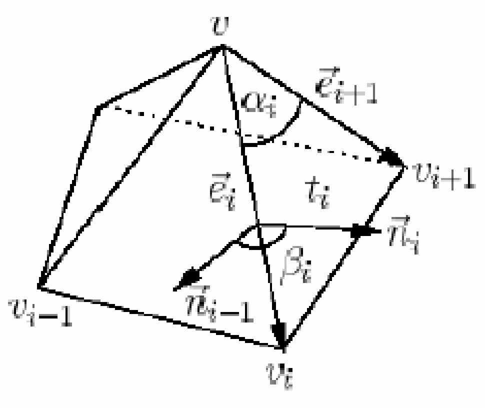
Gaussian Curvature

- Use Gauss-Bonnet Theorem

$$\int_T K = 2\pi - \sum_i \alpha_i - \int_{\partial T} k_{\partial T} = 2\pi - \sum_i \alpha_i$$

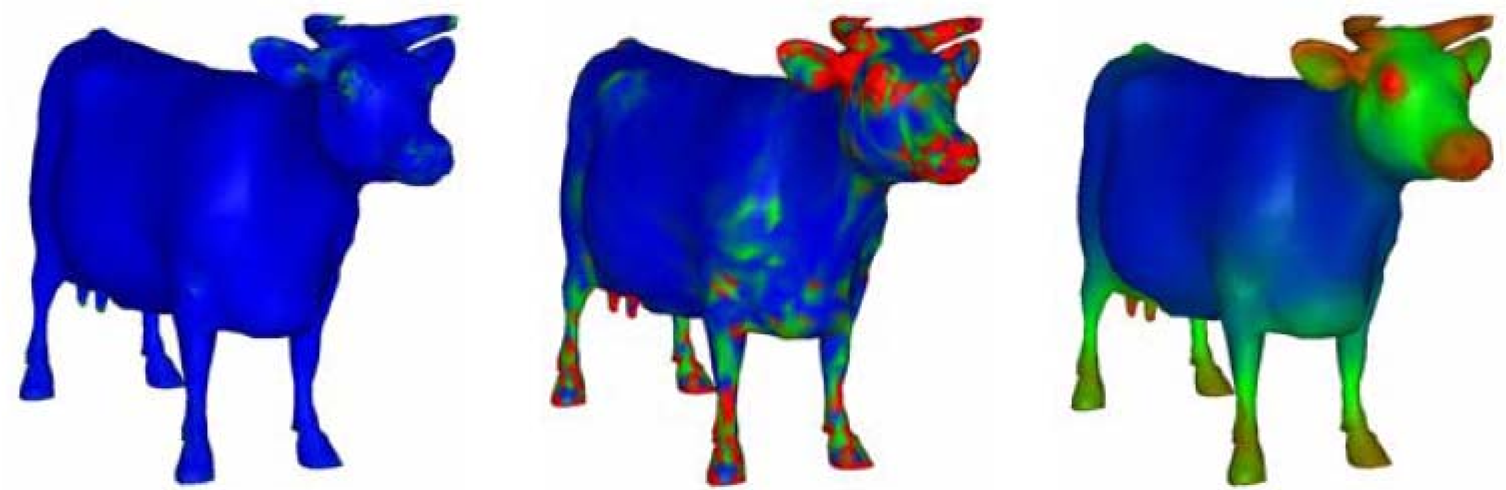
- Curvature at vertex

$$K = \frac{3(2\pi - \sum_i \alpha_i)}{\sum_i A}$$



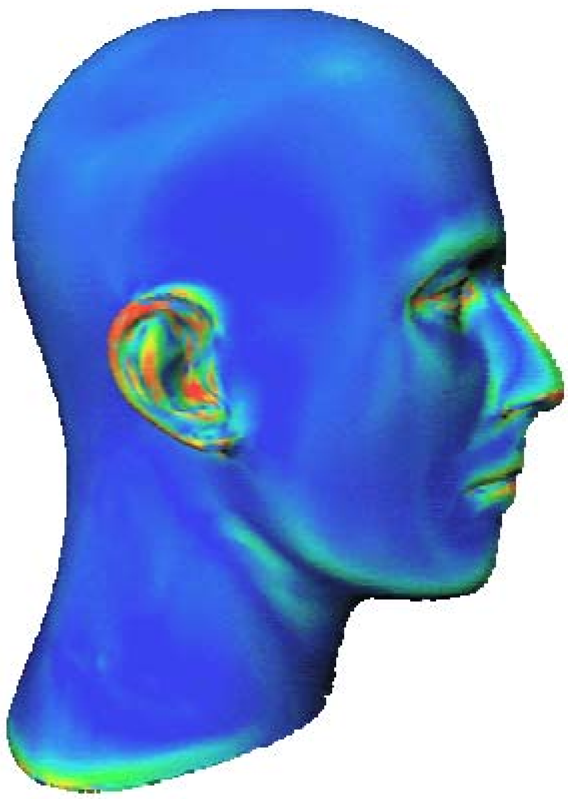
- Note (Gauss-Bonnet for closed surfaces) – Integral Gaussian curvature = genus

Gaussian Curvature Estimate – Example

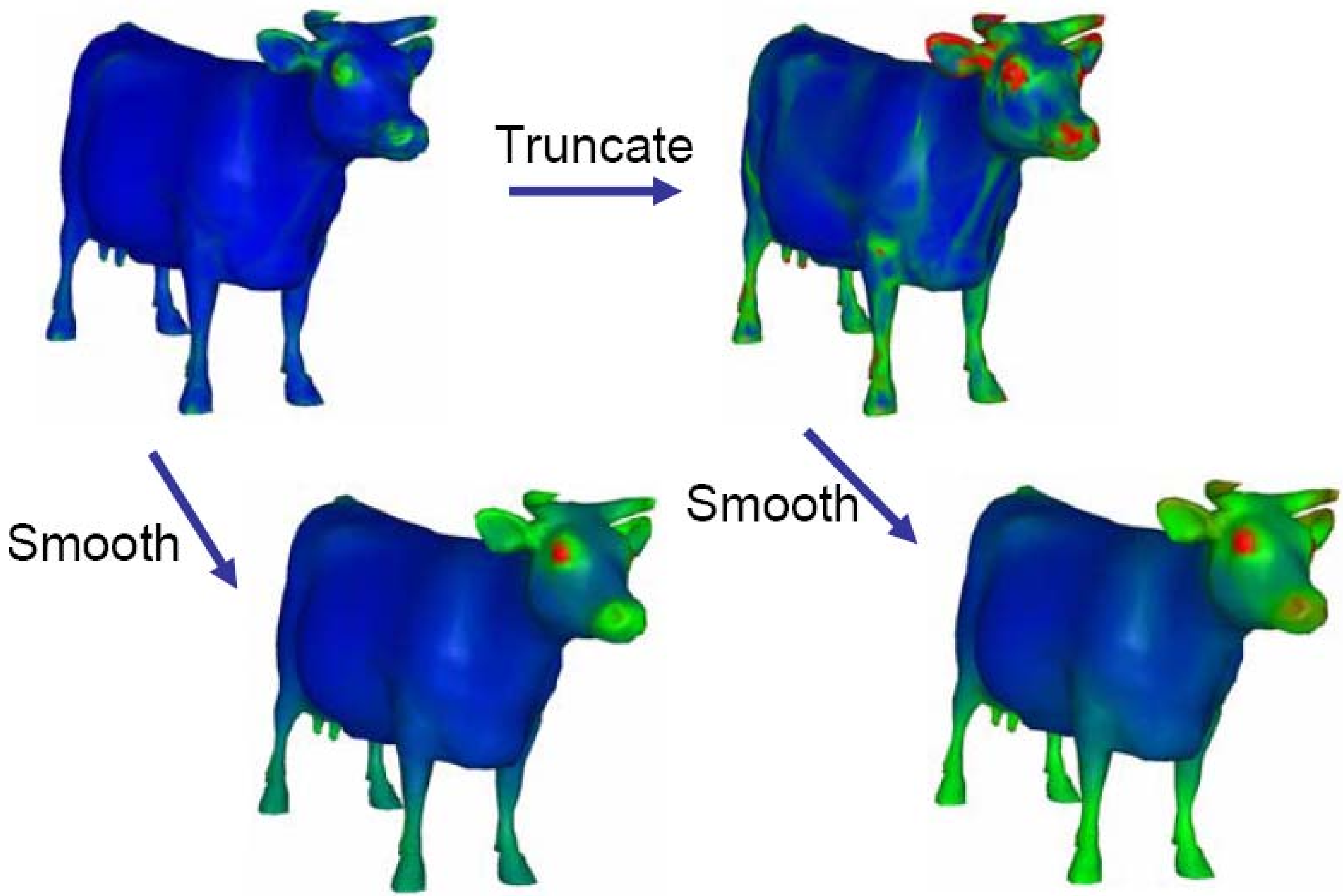


- Approximation always results in some noise
- Solution
  - Truncate extreme values
    - Can come for instance from division by very small area
  - Smooth
    - More later

Mean Curvature Estimate – Example



Mean Curvature



More…

- MEYER M., DESBRUN M., SCHRÖDER P., BARR A.: Discrete differential-geometry operators for triangulated 2-manifolds. In Visualization and Mathematics III, Hege H.-C., Polthier K., (Eds.). Springer, 2003, pp. 35–58. (PDF)

Applications

- Feature detection
- Shape recognition
- Mesh segmentation
- Any feature-aware applications
  - Preserving salient features in processing

- Challenges:
  - What are features on surfaces?

References

- TAUBIN G.: Estimating the tensor of curvature of a surface from a polyhedral approximation. In Proc. International Conference on Computer Vision (1995), pp. 902–907.
- MEYER M., DESBRUN M., SCHRÖDER P., BARR A.: Discrete differential-geometry operators for triangulated 2-manifolds. In Visualization and Mathematics III, Hege H.-C., Polthier K., (Eds.). Springer, 2003, pp. 35–58.
- CAZALS F., POUGET M.: Estimating differential quantities using polynomial fitting of osculating jets. In Eurographics Symposium on Geometry Processing (2003), pp. 177–187.
- COHEN-STEINER D., MORVAN J.: Restricted delaunay triangulations and normal cycle. In Proc. ACM Symposium on Computational Geometry (2003), pp. 312–321.
- GOLDFEATHER J., INTERRANTE V.: A novel cubic-order algorithm for approximating principal direction vectors. ACM Transactions on Graphics 23, 1 (2004), 45–63.
- MARTIN R. R.: Estimation of principal curvatures from range data. International Journal of Shape Modeling 4, 1 (1998), 99–109.
- OHTAKE Y., BELYAEV A., SEIDEL H.-P.: Ridge-valley lines on meshes via implicit surface fitting. ACM Transactions on Graphics 23, 3 (2004), 609–612. (Proc. SIGGRAPH'2004).
- PAGE D., SUN Y., KOSCHAN A., PAIK J., ABIDI M.: Normal vector voting: Crease detection and curvature extimation on large, noisy meshes. Graphical Models 64, 3-4 (2002), 199–229.

Q&A