中国科学技术大学
University of Science and Technology of China

GAMES102

GAMES 102在线课程

几何建模与处理基础

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GAMES 102在线课程：几何建模与处理基础

曲线光顺

光滑(Smooth)曲线

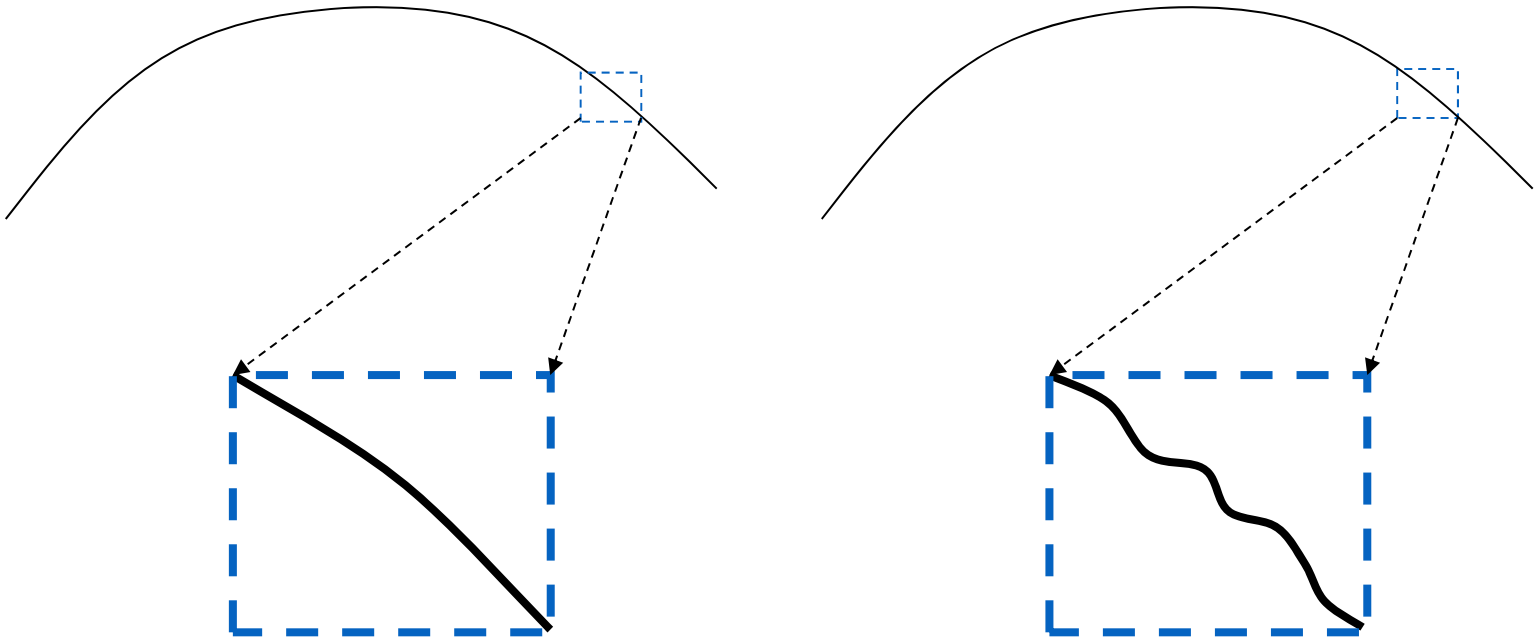
- 连续曲线(Continuous)：参数连续性
  - 给定 2 条曲线

    $\boldsymbol{x}_1(t)$ 定义在 $[t_0, t_1]$

    $\boldsymbol{x}_2(t)$ 定义在 $[t_1, t_2]$
  - 曲线 $\boldsymbol{x}_1$ 和 $\boldsymbol{x}_2$ 在 t_1 称为C^r连续的，如果它们的从 0th（0阶） 至 r^{th}（r阶）的导数向量在 t_1 处完全相同
  - 光滑：高阶连续
- 几何连续：
  - 与参数化无关，更刻画了曲线形状的本征光滑性
  - 更适合交互式曲线设计

什么是光顺曲线?

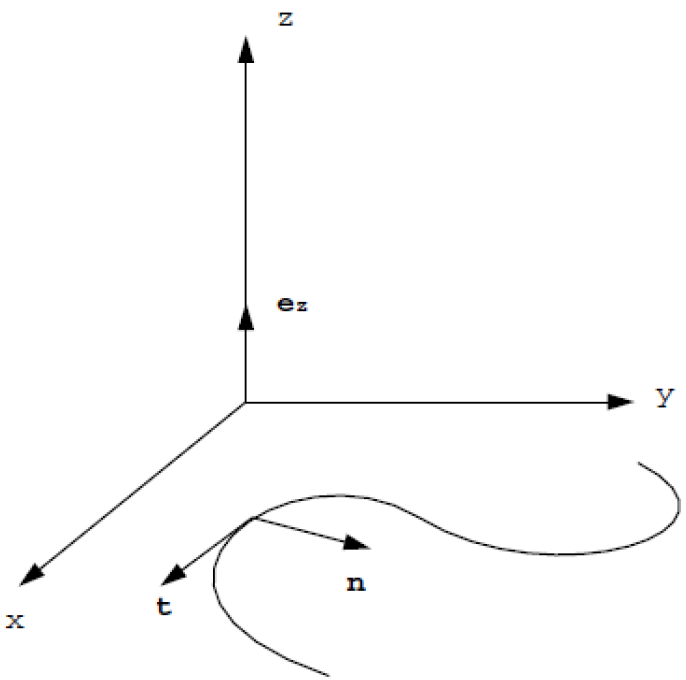
- Curve fairing

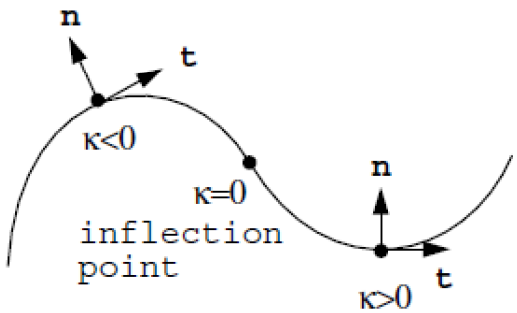


曲线的微分几何

单参数曲线的切线和法向

- 曲线：$\boldsymbol{r} = \boldsymbol{r}(t) = \big(x(t), y(t)\big)$，$t \in [0,1]$
- 切线：
  - $\boldsymbol{t} = \boldsymbol{r}'(t) = \big(x'(t), y'(t)\big)$
- 法线$\boldsymbol{n}$





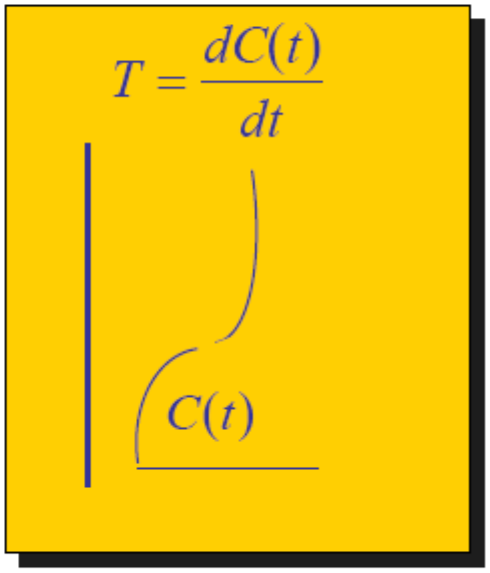
Curves

- Tangent vector to curve $C(t)=(x(t),y(t))$ is

$$T = C'(t) = \frac{dC(t)}{dt} = \left[x'(t), y'(t)\right]$$

- Unit length tangent vector

$$\vec{T} = \vec{C}(t) = \frac{\left[x'(t), y'(t)\right]}{\sqrt{x'(t)^2 + y'(t)^2}}$$



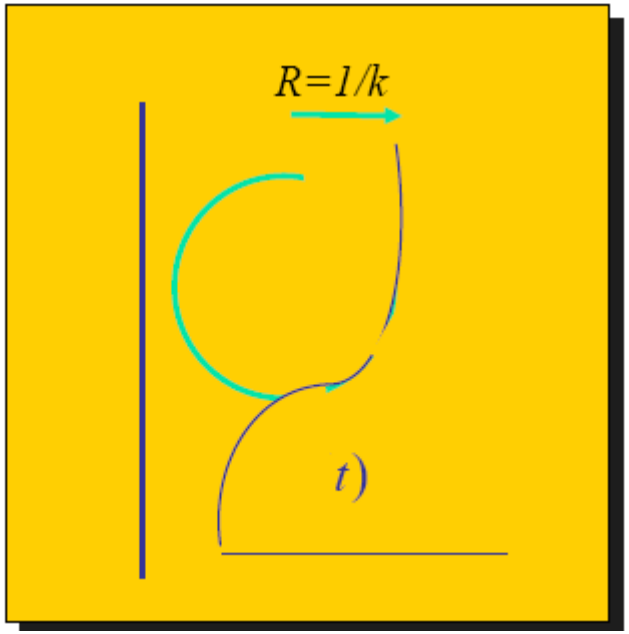
- Curvature

$$k(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{\left(x'(t)^2 + y'(t)^2\right)^{3/2}}$$

Curve Curvature

- Curvature is **independent** of parameterization
  - $C(t)$, $C(t+5)$, $C(2t)$ have same curvature (at corresponding locations)

- Corresponds to radius of osculating circle $R=1/k$



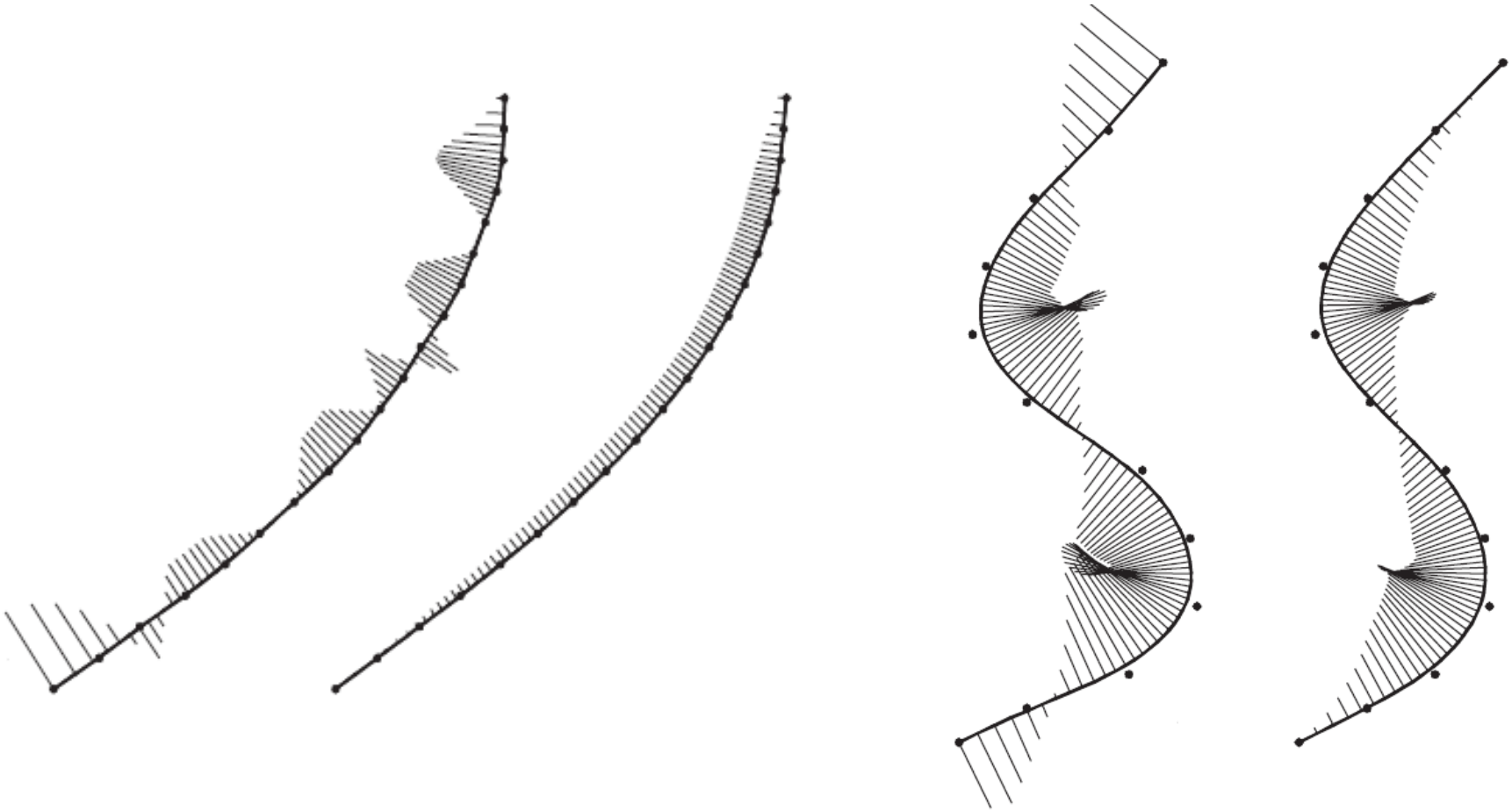
- Measure curve bending

曲线的光顺定义

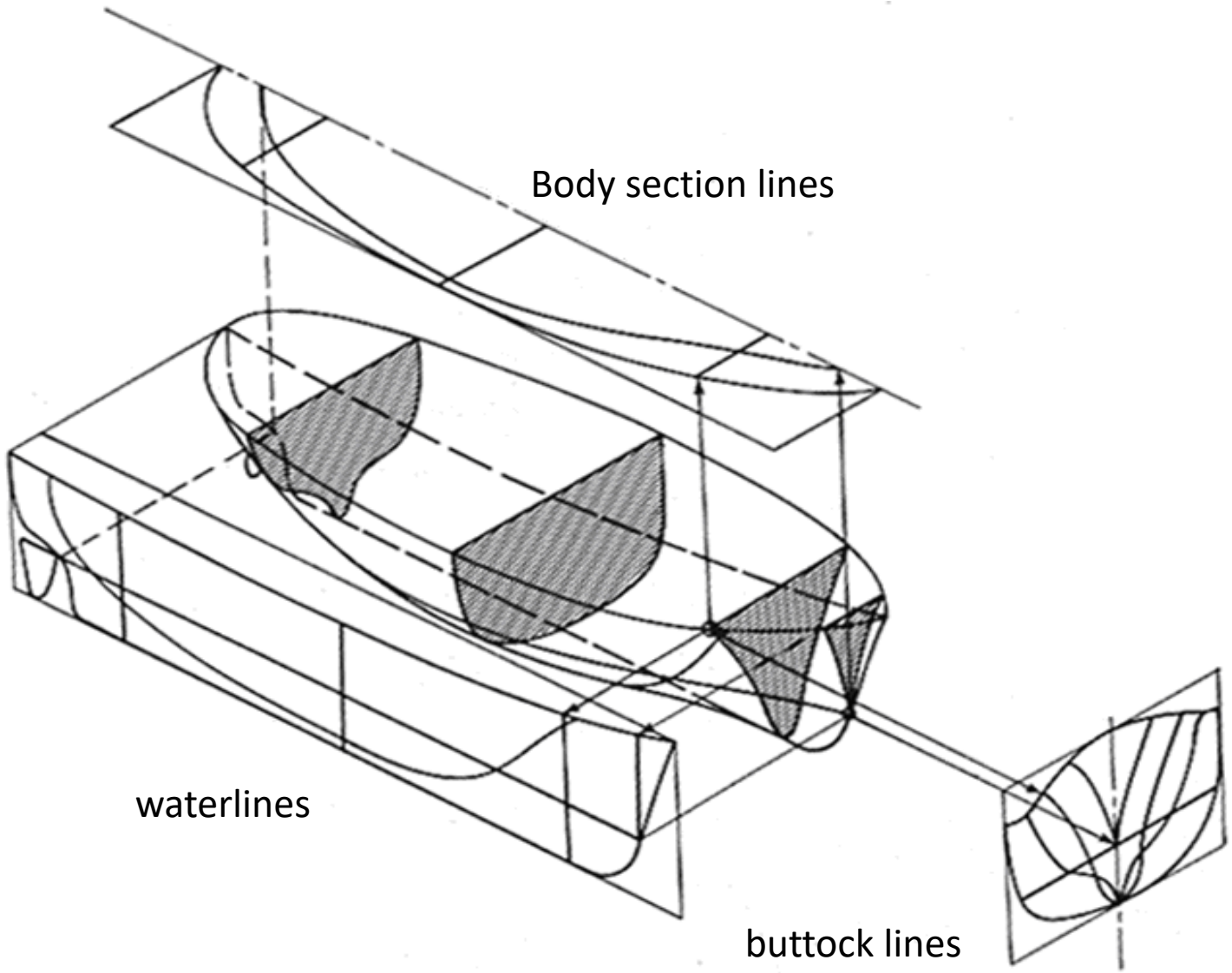
Geometric Design: Two Phases

- Shape design and modeling (Macro)
- Fairing design and modeling (Micro)
  - A post-processing after shape design
  - Less well studied
    - Difficult problem
    - Lack of solid theories
    - Far from solved

曲线的曲率图

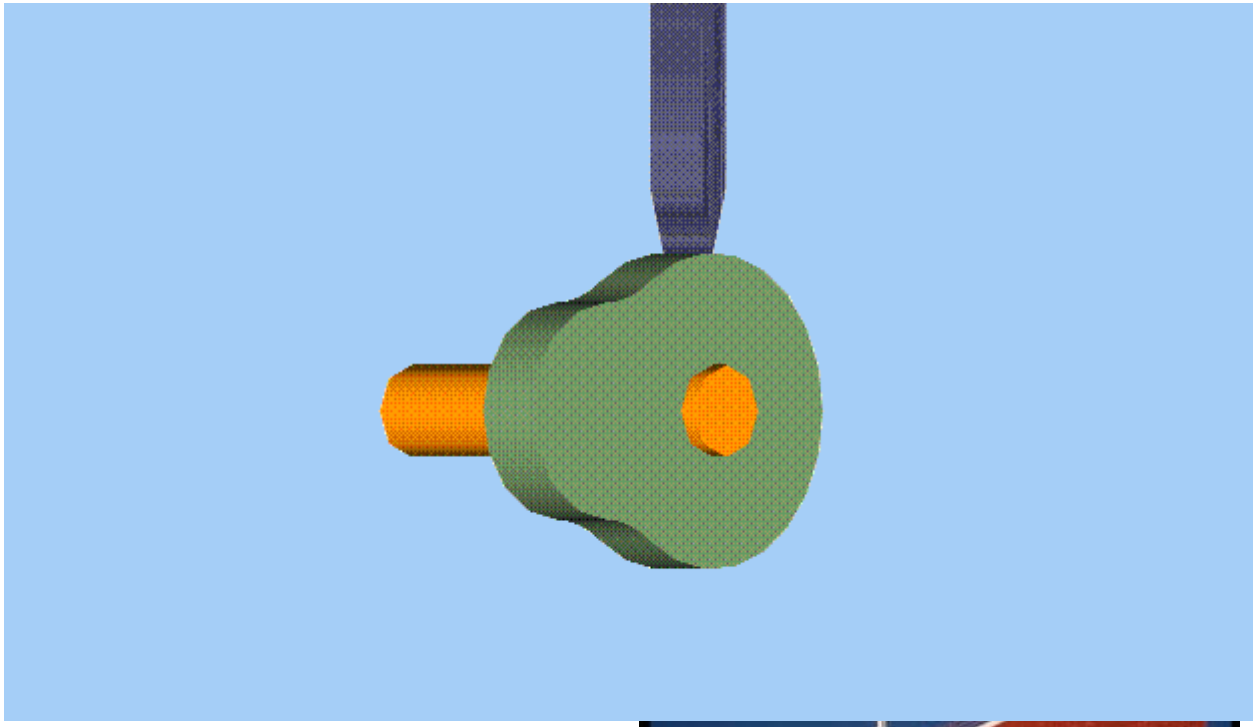


Why fairing curves?



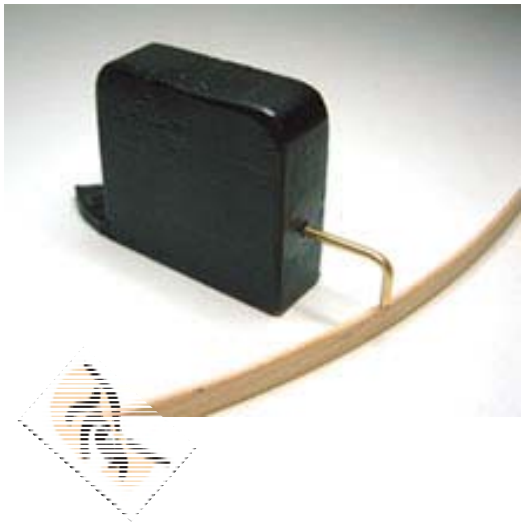
Fairing Design is Important!

- Shoe sole
- Cam profile
- Ship hull
- Car profile
- Plane profile
- …



Why Difficult?

- A subjective concept
  - The subtle bumps, wiggles, and inflection points of a curve
  - Related to human perception
  - Dependent on designer's experience
- A difficult task
  - Examining the curves by eye!
  - No objective measures
  - Cannot do it mathematically



Some ‘Definitions’ of Fairness

- [Su and Liu 1978]
  - A curve is fair if it is C^2 continuous and its curvature plot is free of any unnecessary variation, i.e., the distribution of curvature must be as uniform as possible.
- [Farin and Sapidis, 1989]
  - A curve is fair if its curvature plot consists of relatively few monotone pieces.
- [Farin 2002]
  - A curve is fair if its curvature plot is continuous and consists of only a few monotone pieces.
- [Roulier and Rando, 1994]
  - A curve is fair if it is C^2 continuous and minimizes the integral of the squared curvature with respect to arc length

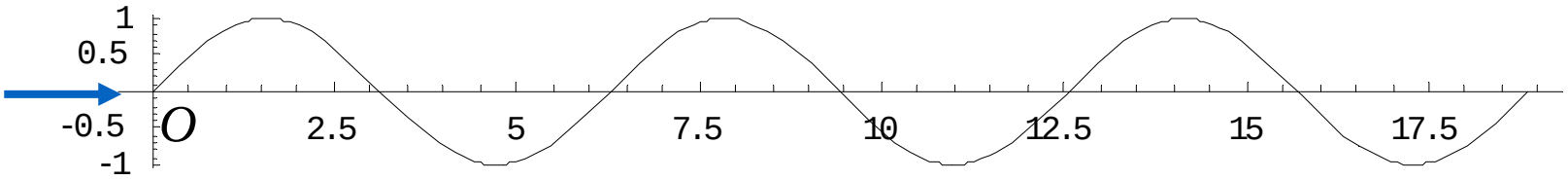
$$\int_C k^2 ds = \mathrm{MIN}$$

Observations of Fairness

- Neither a global problem nor a local problem, but a large local problem
  - Not an energy minimization problem
- Need not C^2 continuous
  - Circular spline
- Intimately related to uniform distribution of curvature
  - Curvature is a “magnifier” of the curve fairness

Example 1

$$y = \sin x, \qquad x \in [0, 6\pi]$$

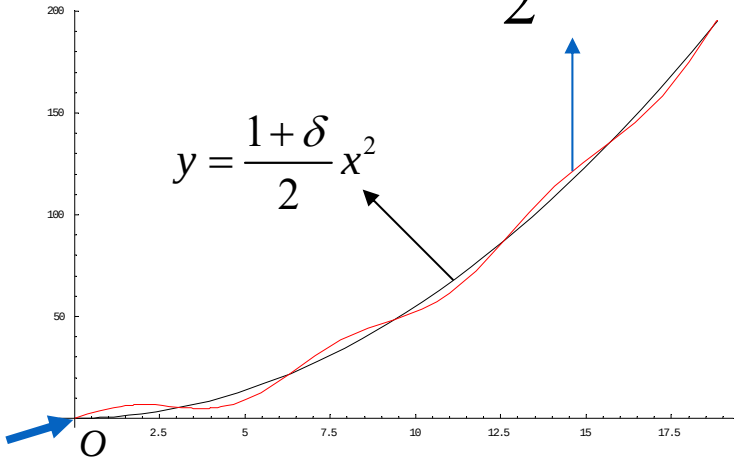


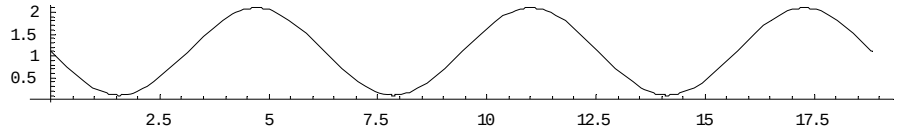
- The curve is C^{∞}
- The curve is not fair as the eye is very uncomfortable while viewing from point O
- Reason
  - It has too many inflections (vibration numbers)

(One vibration: from convex to concave or from concave to convex)

Example 2

$$y = \frac{1+\delta}{2}x^2 + \sin x, \qquad x \in [0, 6\pi], \qquad \delta > 0$$

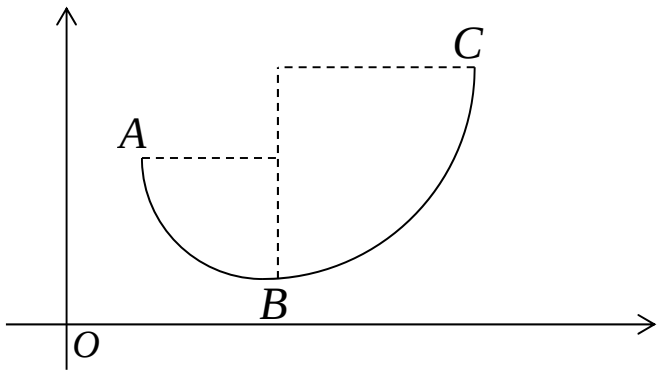


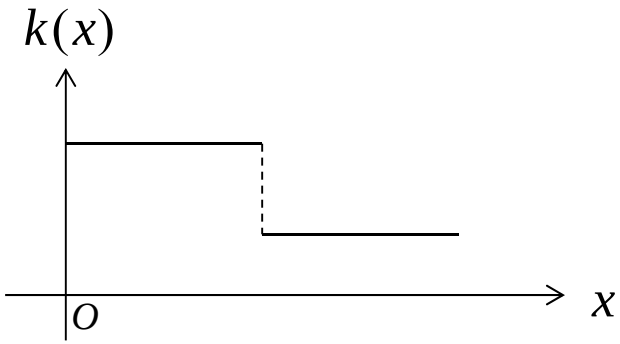


$$y'' = 1 + \delta - \sin x > 0$$

- The curve is C^∞ without any inflection point.
- The curve is not fair as the eye is very uncomfortable while viewing from point O (it winds along the black parabola curve)
- Reason
  - y”(x) has too many vibration numbers

Example 3





- The curvature function y”(x) is discontinuous.
- Vibration number of y”(x) can be defined if it is a bounded function.

$$y''(x) \text{ is bounded} \Rightarrow y'(x) \text{ has bounded variation} \Rightarrow y(x) \in C^{l+1}.$$

- The curve is not fair if k_1 and k_2 are much different.

- Reason
  - y”(x) has large amplitude at discontinuity point.

曲线的光顺的“新定义”

- 一条曲线是光顺的，如果

  (1) 它是 C^{1+l} $(l>0)$ 连续的;

  (2) 它的曲线本身拐点较少;

  (3) 它的曲率图的拐点较少;

  (4) 它的曲率图变化的振幅相对小.

说明 1. 条件(1)中的 C^{1+l} 是要求曲线为 C^1 连续而不必 C^2，但 C^1 的导数满足有界变差. 条件(4)则要求曲线在曲率非连续点处的跳跃要尽可能小.

说明 2. 满足(2)和(3)描述的曲线的它的曲率图含有的单调段都会相对少. 这与前面所述的判别准则 1-4 一致.

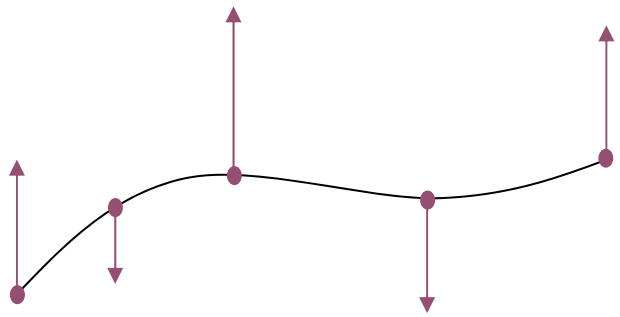
Remarks

- Vibration
  - Change from convex to concave or change from concave to convex
- First vibration number *R*
  - Vibration number of y(x)
- Second vibration number *S*
  - Vibration number of curvature function

曲线的光顺方法

函数型3次样条曲线

- 小扰度假设
  - 转角不大于60°



- $y'(x) \ll 1$
- $y''(x) \approx k(x)$

曲线的光顺方法

- C^1 continuous
- Decrease jump amplitude of curvature
- Decrease the first vibration number R
- Decrease the second vibration number S

Steps

- Coarse fairing
- Basic fairing
- Fine fairing

Step 1. 初光顺

- 定界法
  - Adjust the positions of control points
  - Decrease the jump amplitude of curvature
  - Remove some unwanted inflections
- Physical approach

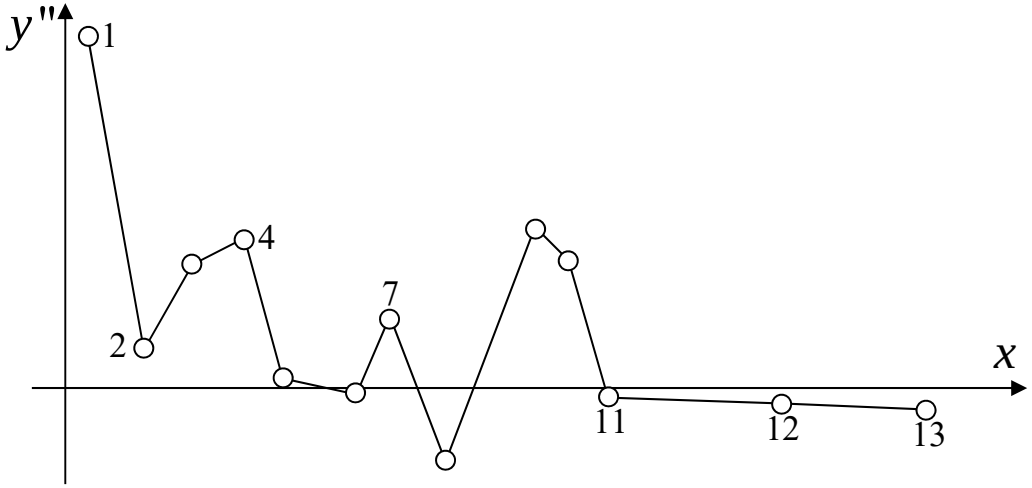
Step 2. 基本光顺

- 卡尺法
  - Adjust the positions of control points
  - Remove other redundant inflections
  - Decrease the first vibration number *R*
- Geometric approach

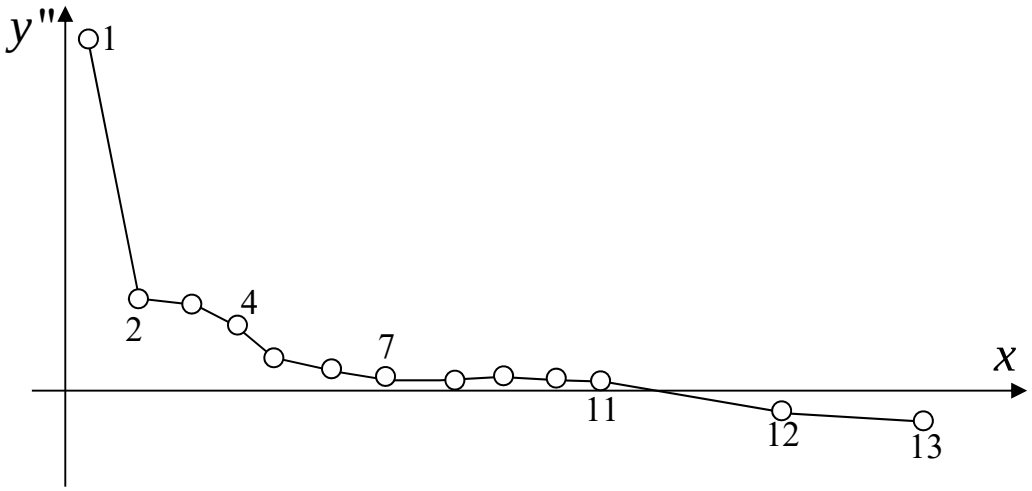
Step 3. 精光顺

- 回弹法
  - Check the signs of shear force at control points
  - Adjust the change numbers of shear force
  - Decrease the second vibration number *S*
- Physical approach

Example 1

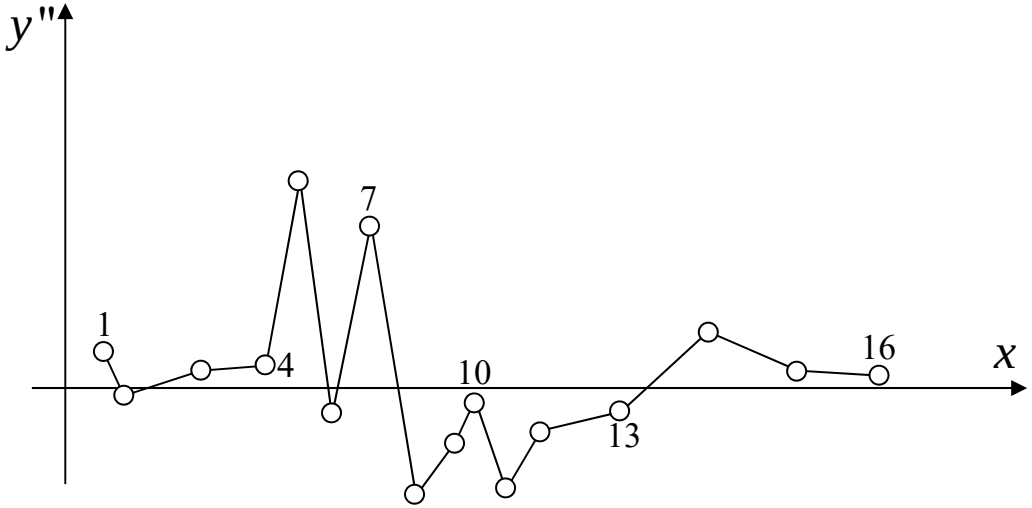


Before fairing

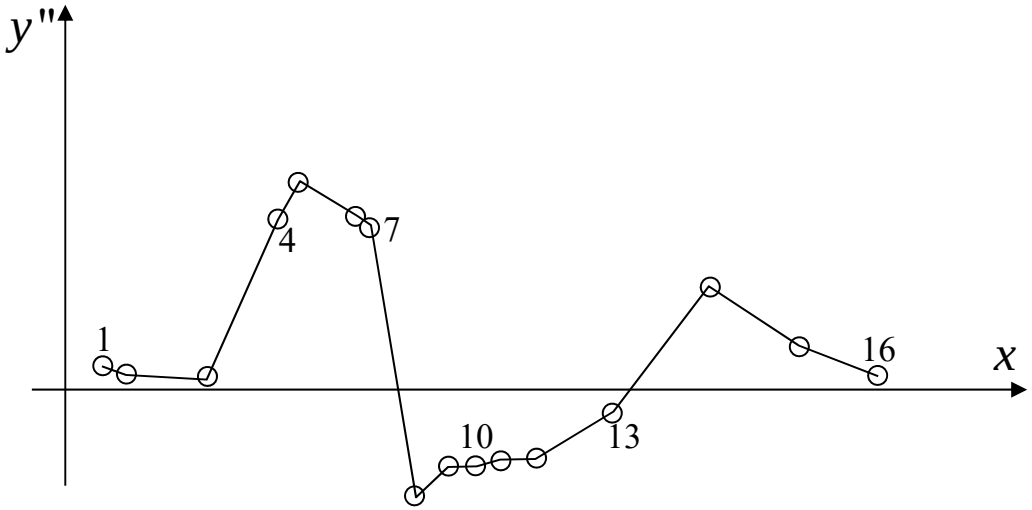


After fairing

Example 2

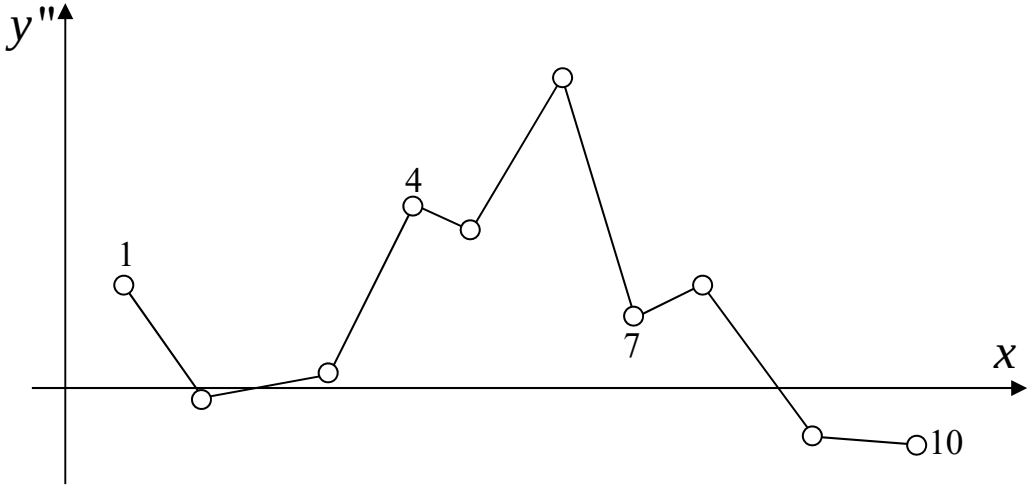


Before fairing

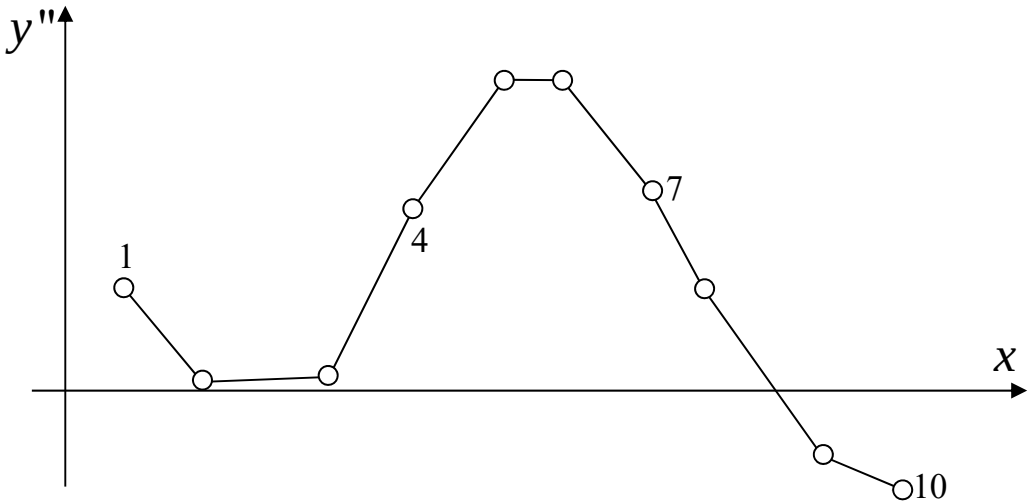


After fairing

Example 3

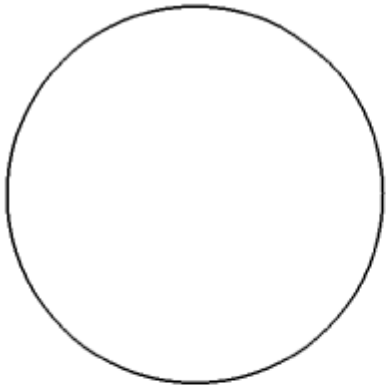


Before fairing

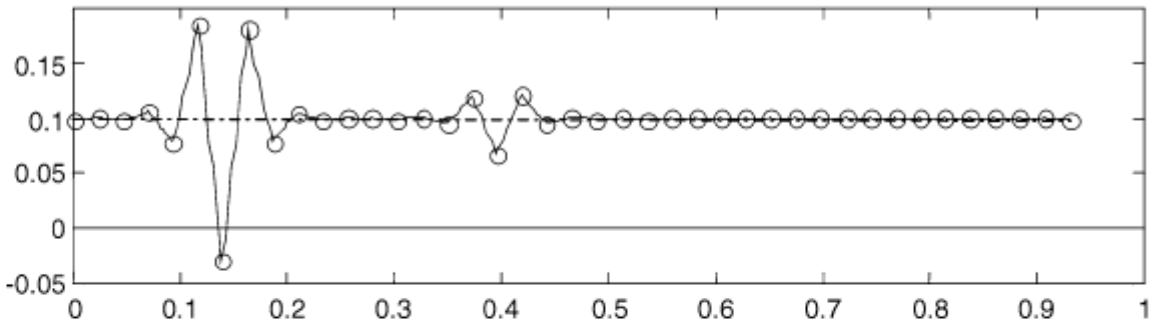


After fairing

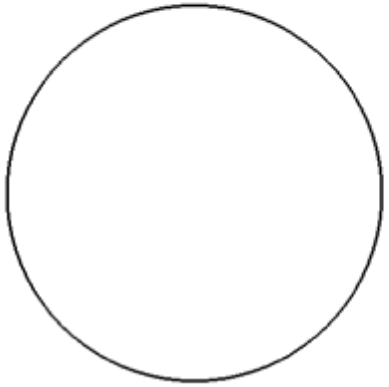
光顺结果



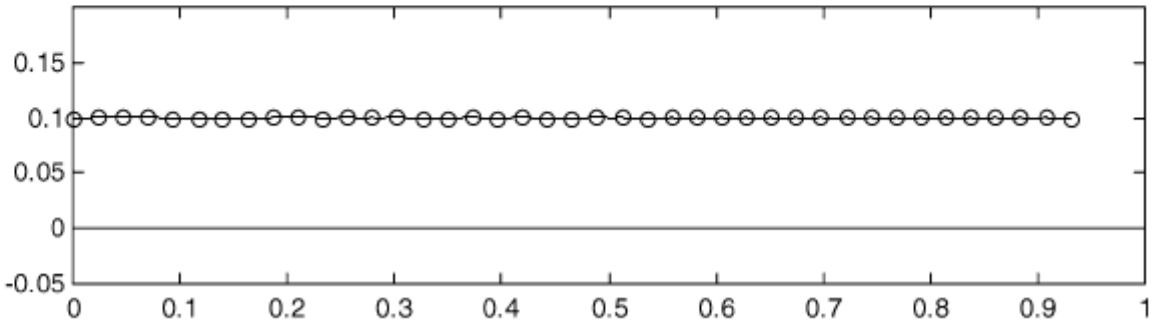
(a)


0.15
0.1
0.05
0
-0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

(b)

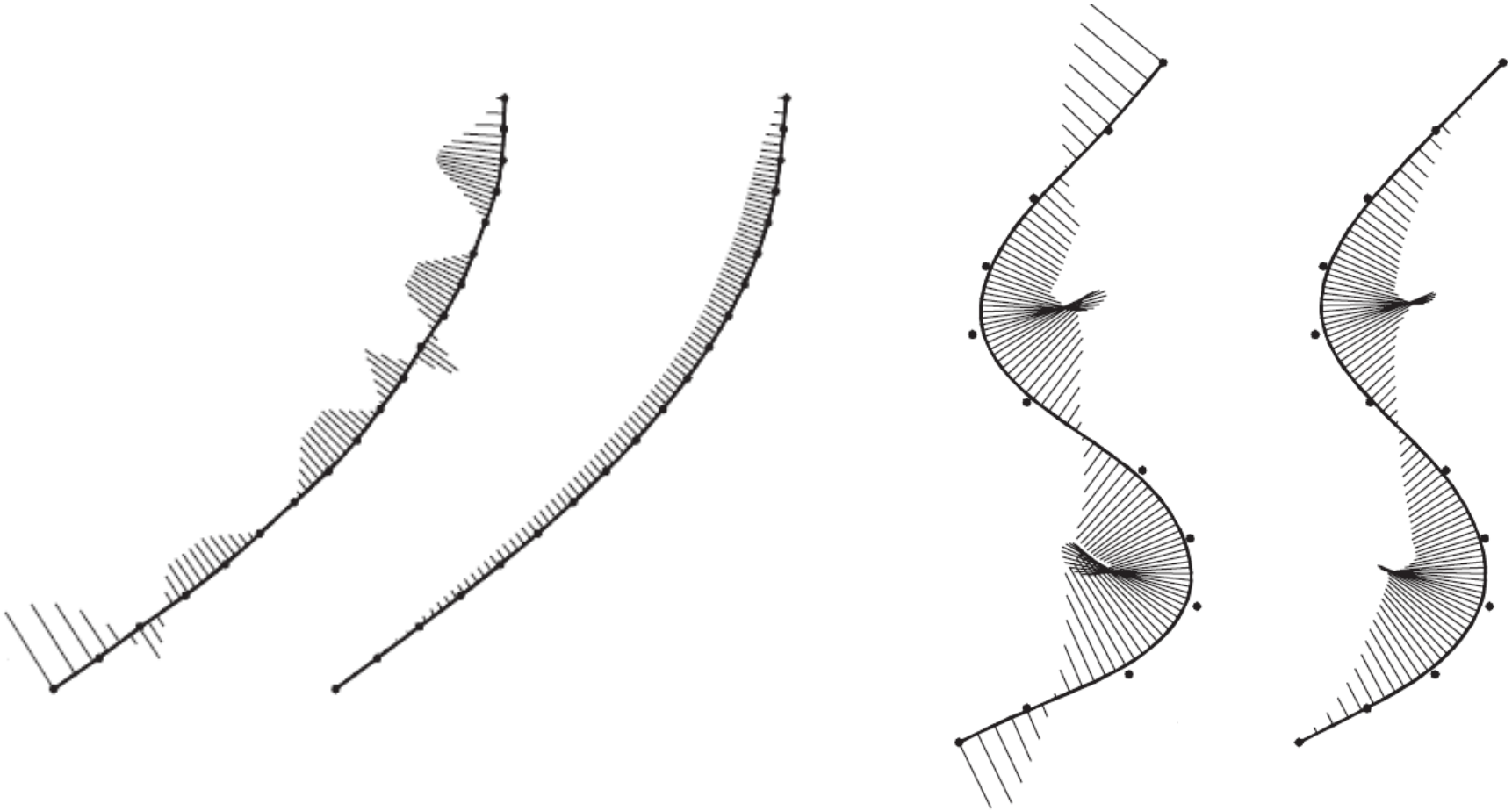


(c)


0.15
0.1
0.05
0
-0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

(d)

光顺结果



B样条曲线的光顺方法

- 基于稀疏优化的光顺优化方法

$$\min_{\tilde{d}} \| e(\tilde{d}) \|_1$$

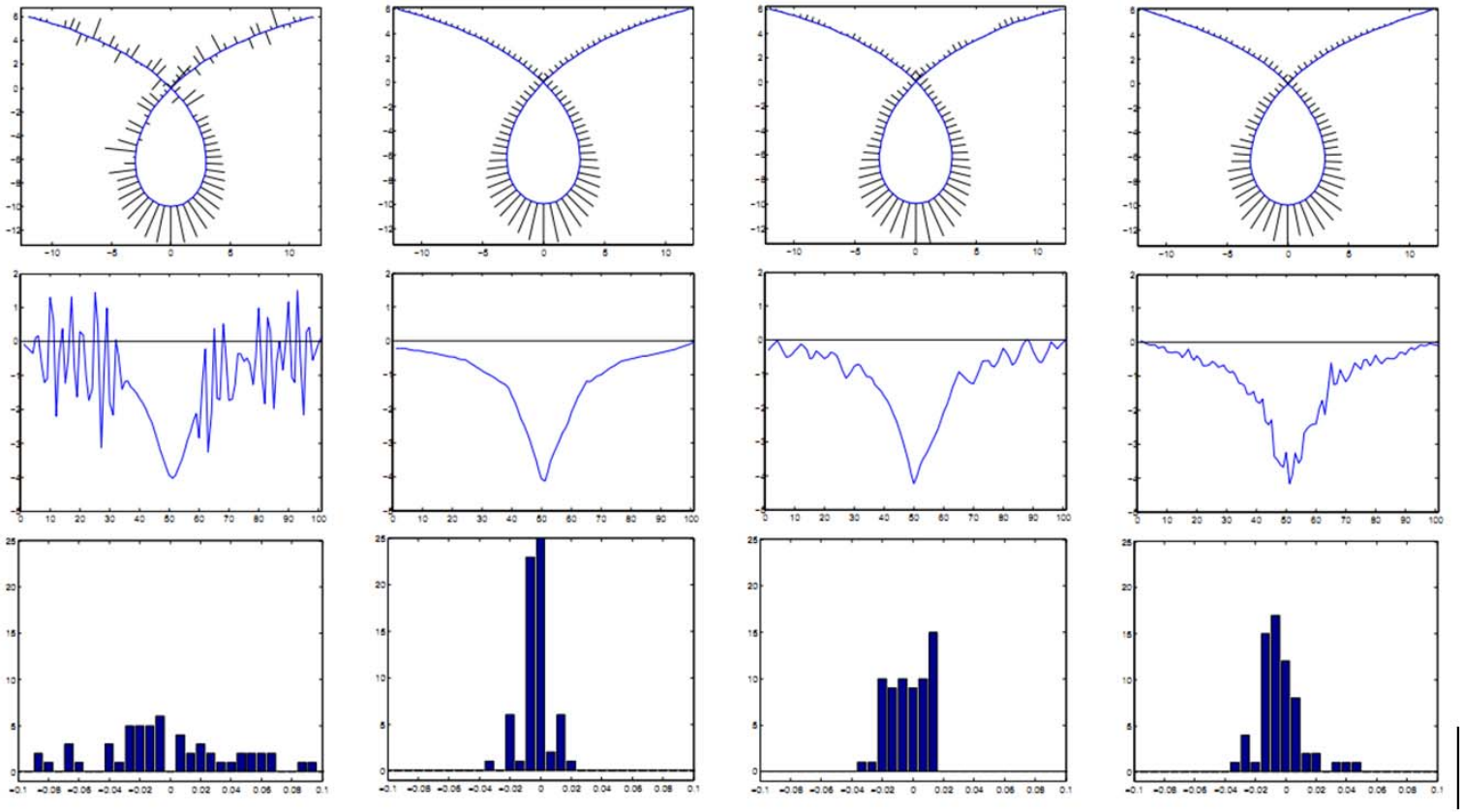
$$\text{s.t.} \quad \| \tilde{d} - d \|_\infty \le \varepsilon$$

曲率的二阶差分向量 e**.** 计算公式如下,

$$e_i = \frac{c_{i+1} - c_i}{t_{i+1} - t_i} - \frac{c_i - c_{i-1}}{t_i - t_{i-1}}, \quad i = 1, \cdots, n-3$$

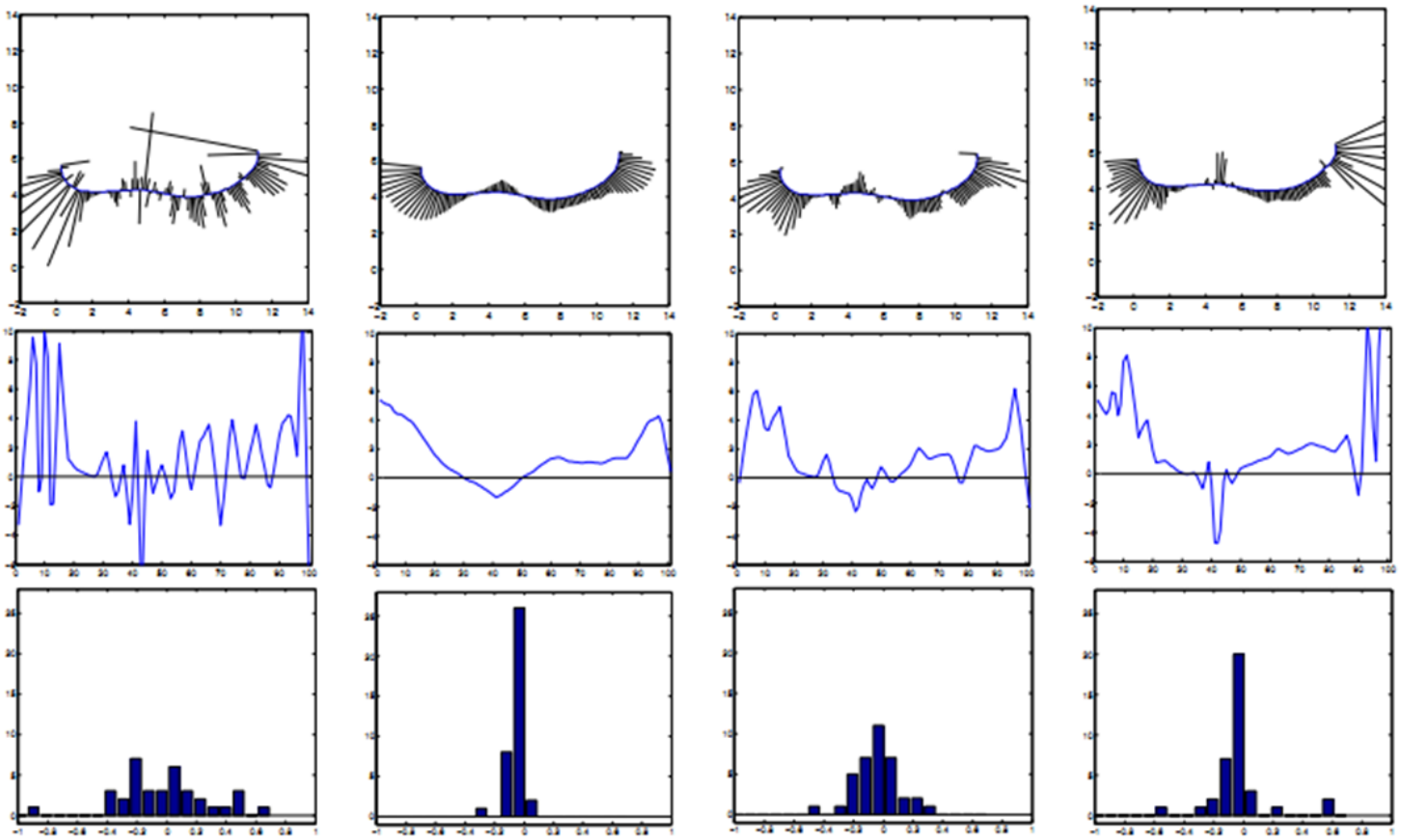
王士玮等，基于稀疏模型的曲线光顺算法，计算机辅助设计与图形学学报，2016.

光顺结果及比较



(a) 原始曲线 (b) 本文算法 (c) SF 算法 (d) MSSC 算法

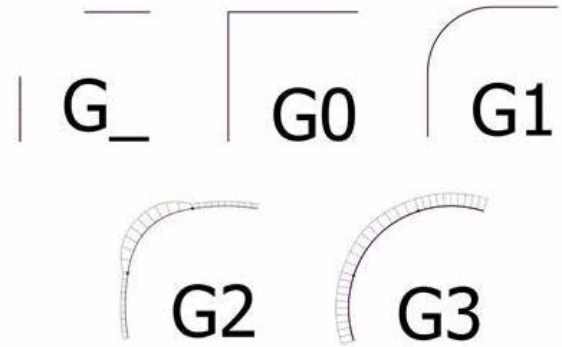
光顺结果及比较



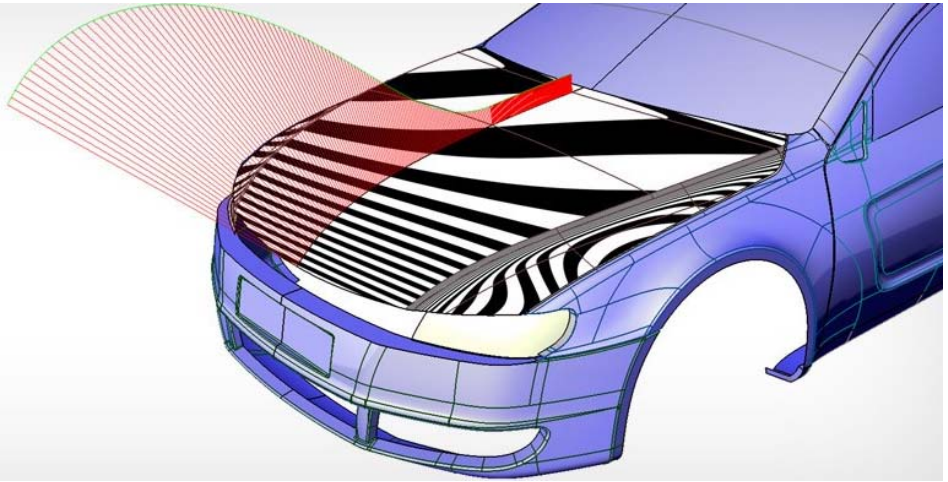
(a) Original (b) Ours (c) SF (d) MSSC

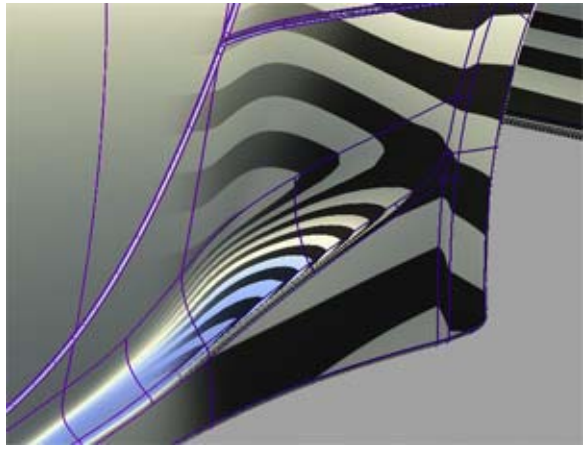
(a) 原始曲线 (b) 本文算法 (c) SF 算法 (d) MSSC 算法

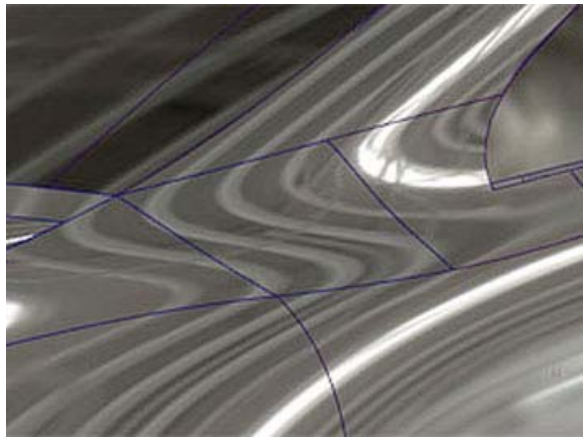
曲面的光顺



- 无严格定义
  - 工业界：Class A曲面（Dassault CATIA）







- 方法1：三向曲线光顺
- 方法2：能量法

光顺曲面

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谢　谢！