中国科学技术大学
University of Science and Technology of China

GAMES102

GAMES 102在线课程

几何建模与处理基础

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GAMES 102在线课程：几何建模与处理基础

采样与剖分

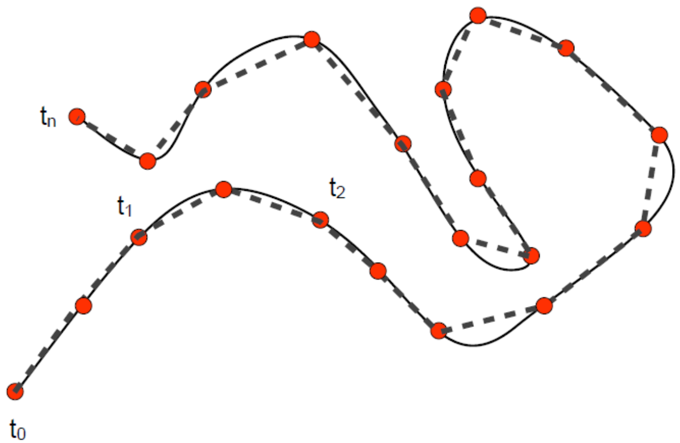
采样 (Sampling)

从连续到离散

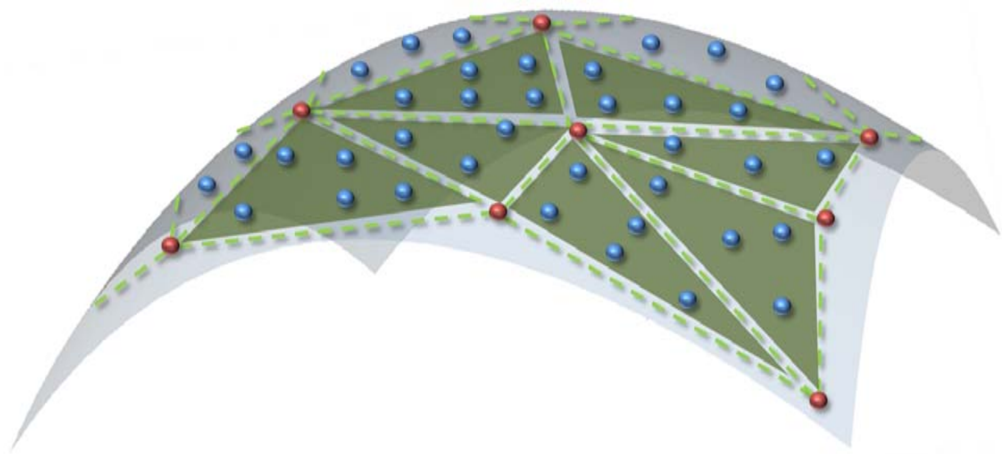
- 对象的表达
  - 在数学上，连续表达与计算
  - 在计算机中，离散表达与计算
- 数值方法：数值微分、数值积分、数值优化
  - 数值分析：离散计算对精确计算的近似程度
  - Fourier分析/变换：离散Fourier分析/变换
  - 卷积（滤波）
- 在计算机科学（计算机图形学）中，采样无处不在
  - 计算机只能表达离散的数值
  - 例子：int型的数据（量化）

曲线曲面的离散表达

- 曲线的绘制:
  - GDI/OpenGL 绘制基本单元：点、线段
  - 曲线须离散成多边形
- 曲面的绘制:
  - OpenGL 绘制基本单元：点、线、三角形
  - 曲面须离散成三角形网格



曲线的离散

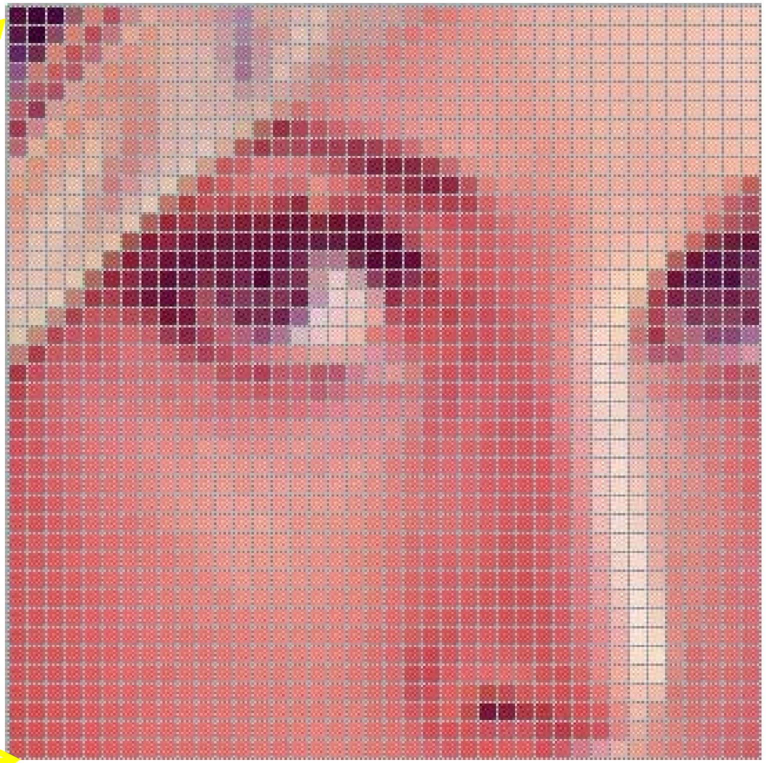


曲面的离散

离散的本质：采样 (Sampling)

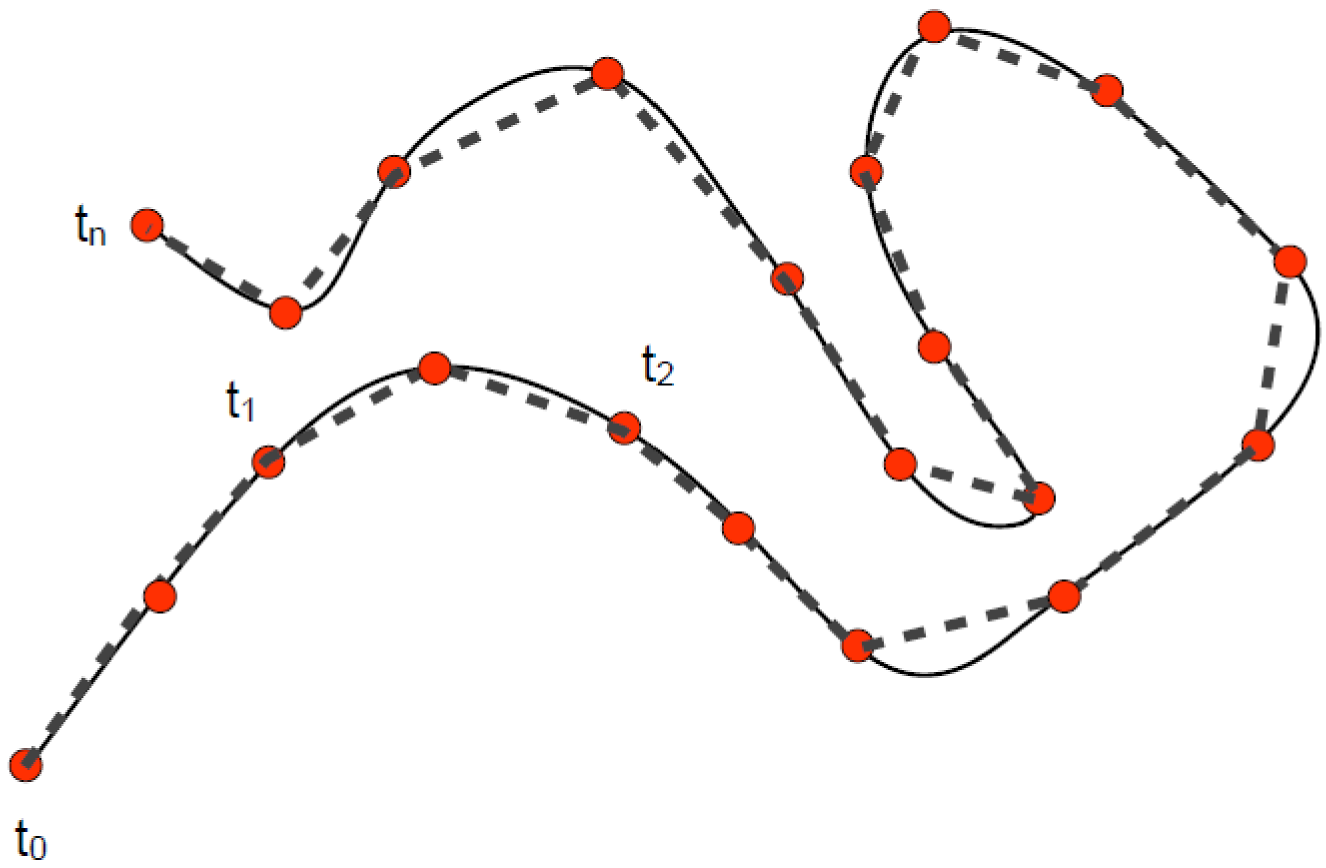
- 曲线曲面的采样
  - 在参数域上采样
  - 直接在原始曲线曲面采样
- NURBS曲线曲面的采样误差估计
  - 可进行理论上的误差分析
- 逆向工程：
  - 采样点的获取
    - 通过扫描硬件设备得到采样点
    - 通过（多视点几何）重建算法计算得到采样点
  - 重建问题：如何通过采样点重构原始曲线/曲面
    - 连续重建：用连续函数来拟合表达
    - 离散重建：直接得到离散基元表达

图像：区域的采样

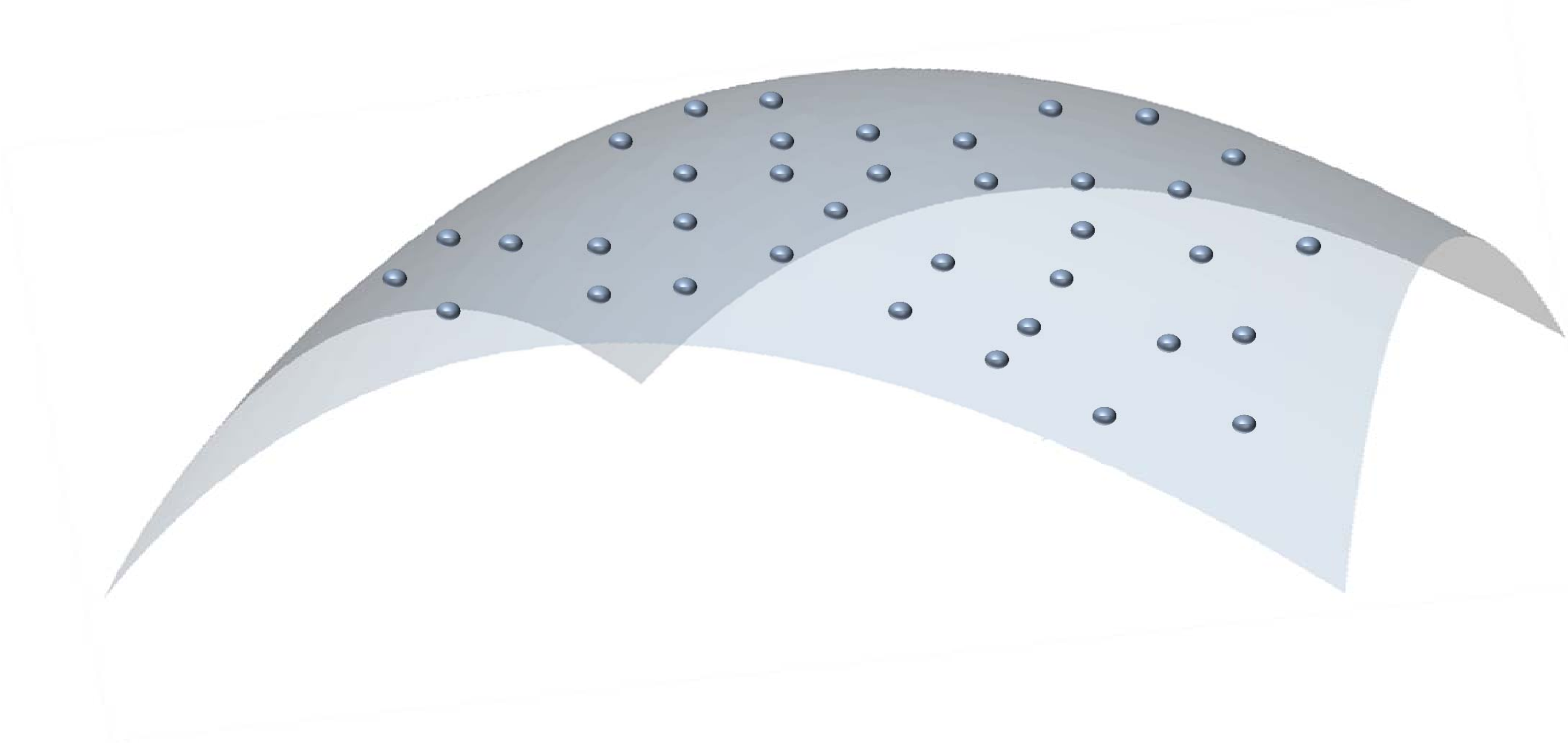


视频：时间的采样

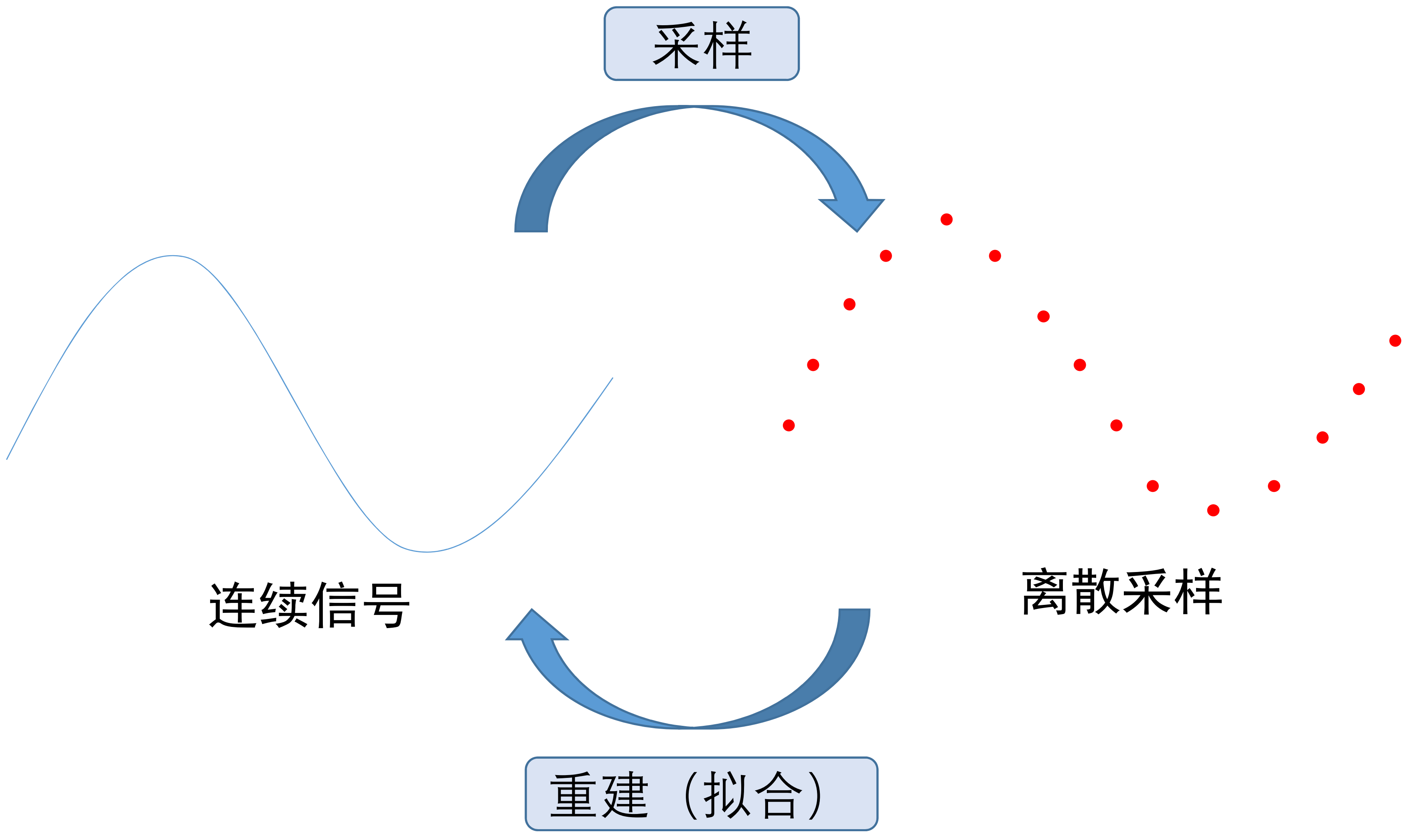
曲线的采样



曲面的采样



采样与重建



Sampling Theorem

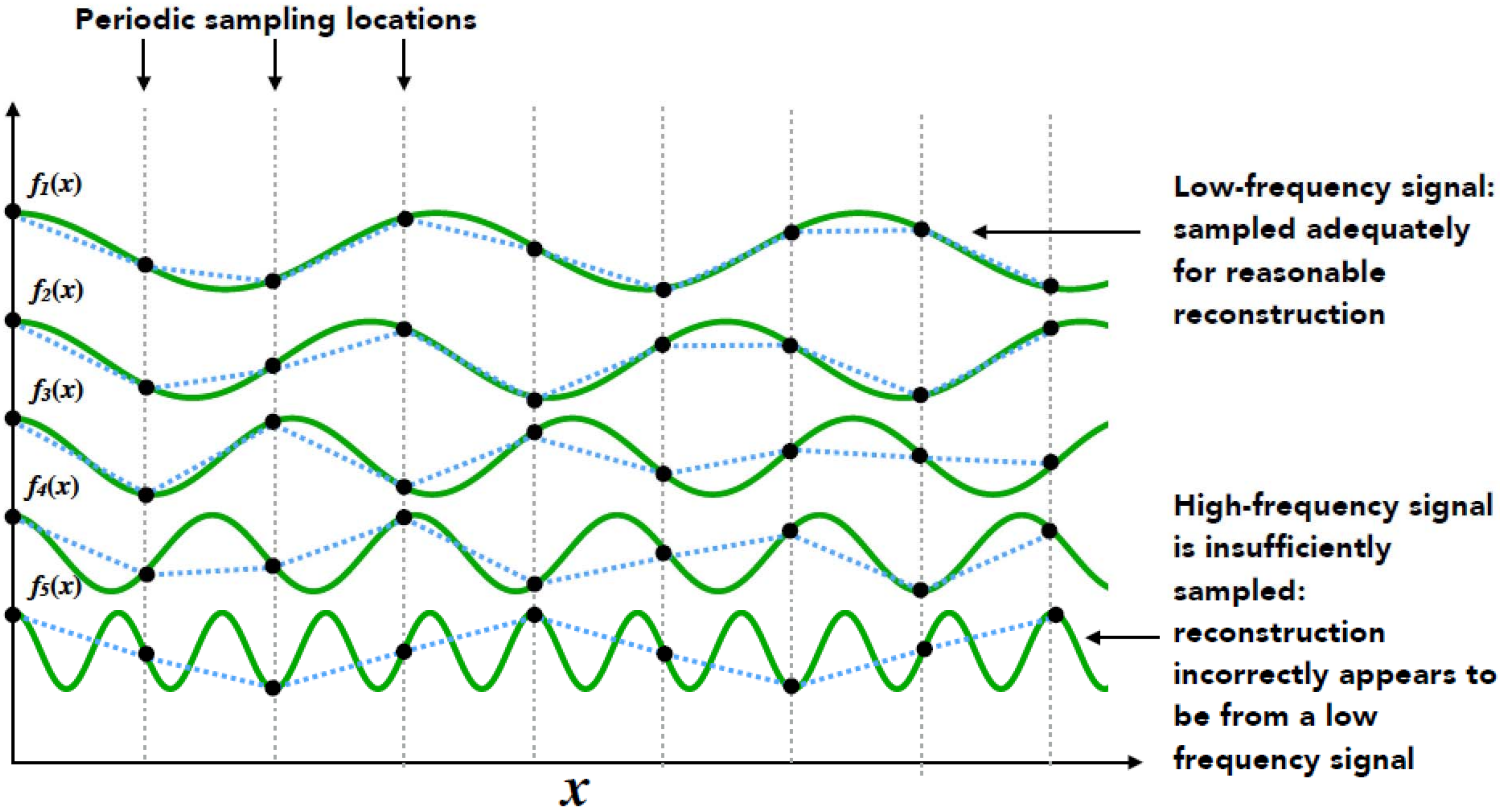
- Nyquist–Shannon sampling theorem

If a function x(t) contains no frequencies higher than B hertz, it is completely determined by giving its ordinates at a series of points spaced 1/(2B) seconds apart.

➡ Generally, a amount of samples are required to recover complex signal

采样与信号频率

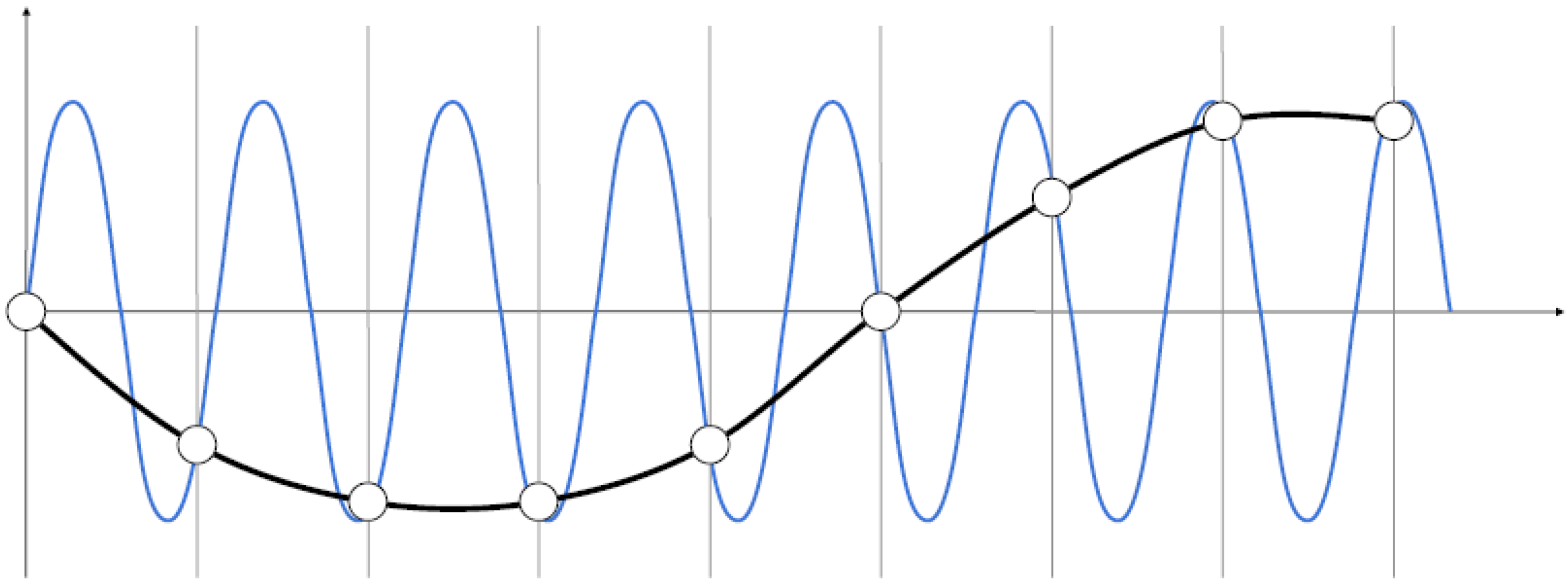
Fourier Analysis



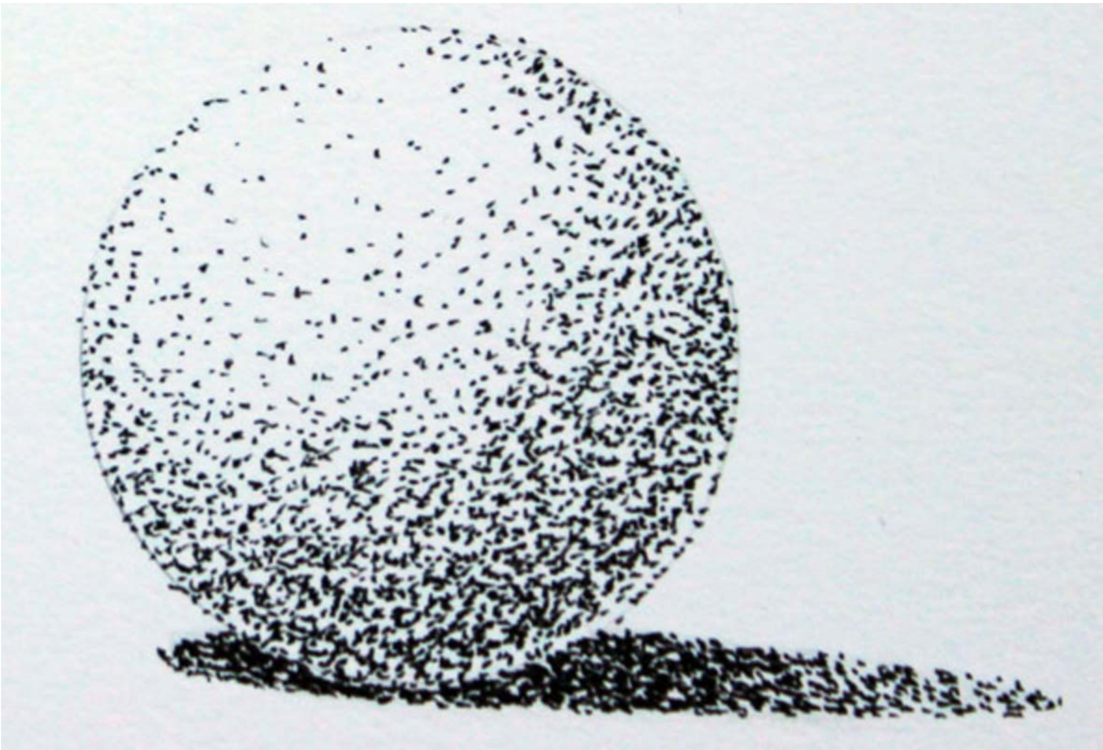
Slide courtesy of Prof. Ren Ng, UC Berkeley

欠采样产生频率的走样

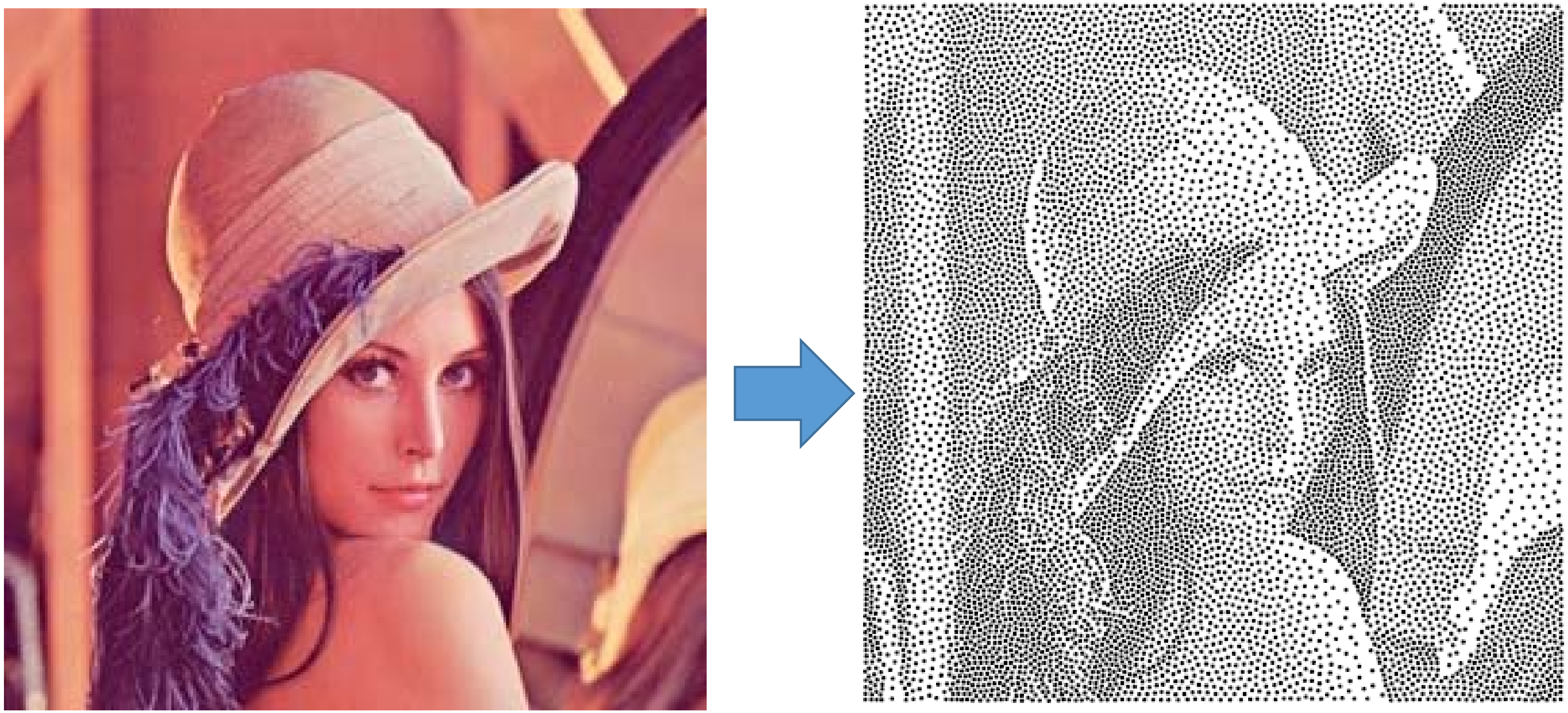
- 高频函数拟合低频信号：过拟合
- 低频函数拟合高频信号：欠拟合



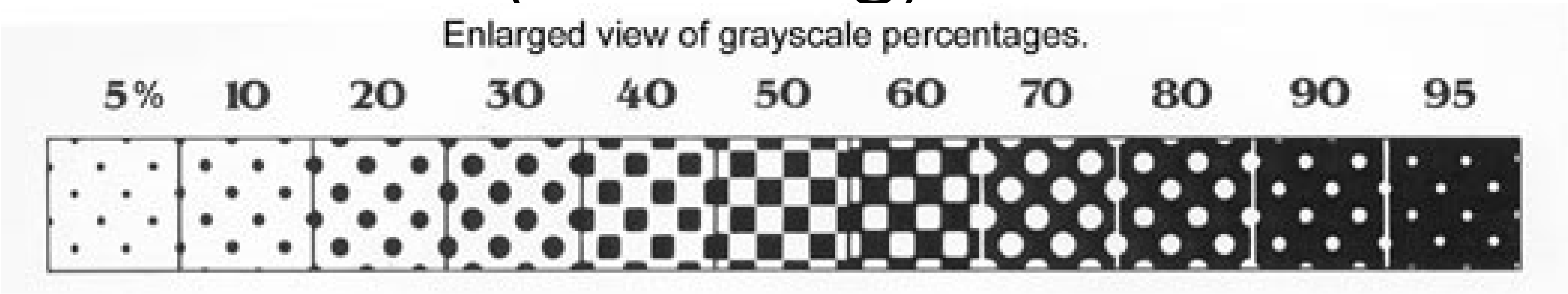
Stippling

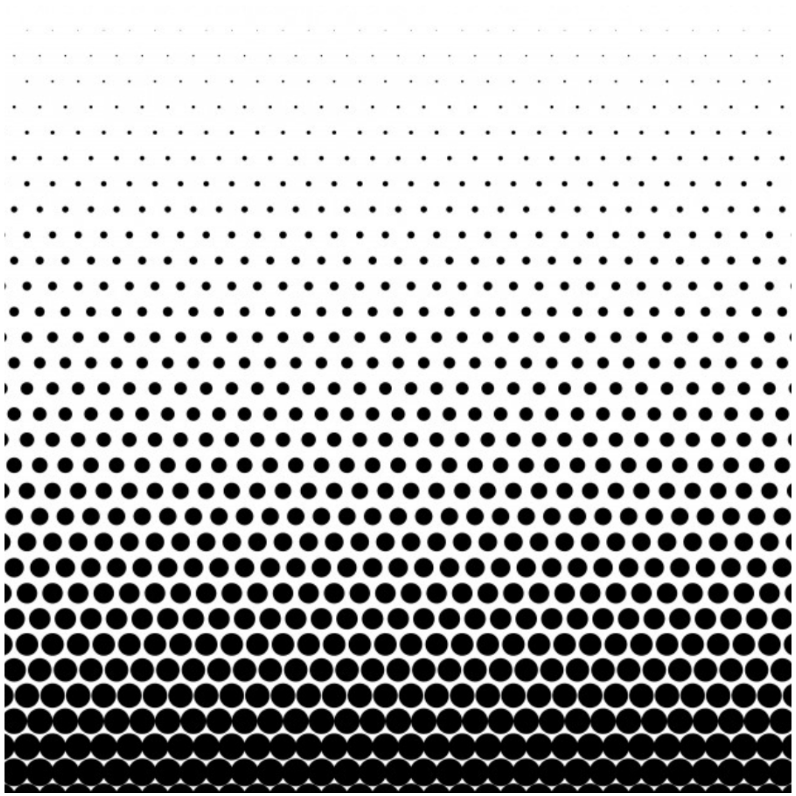


Stippling

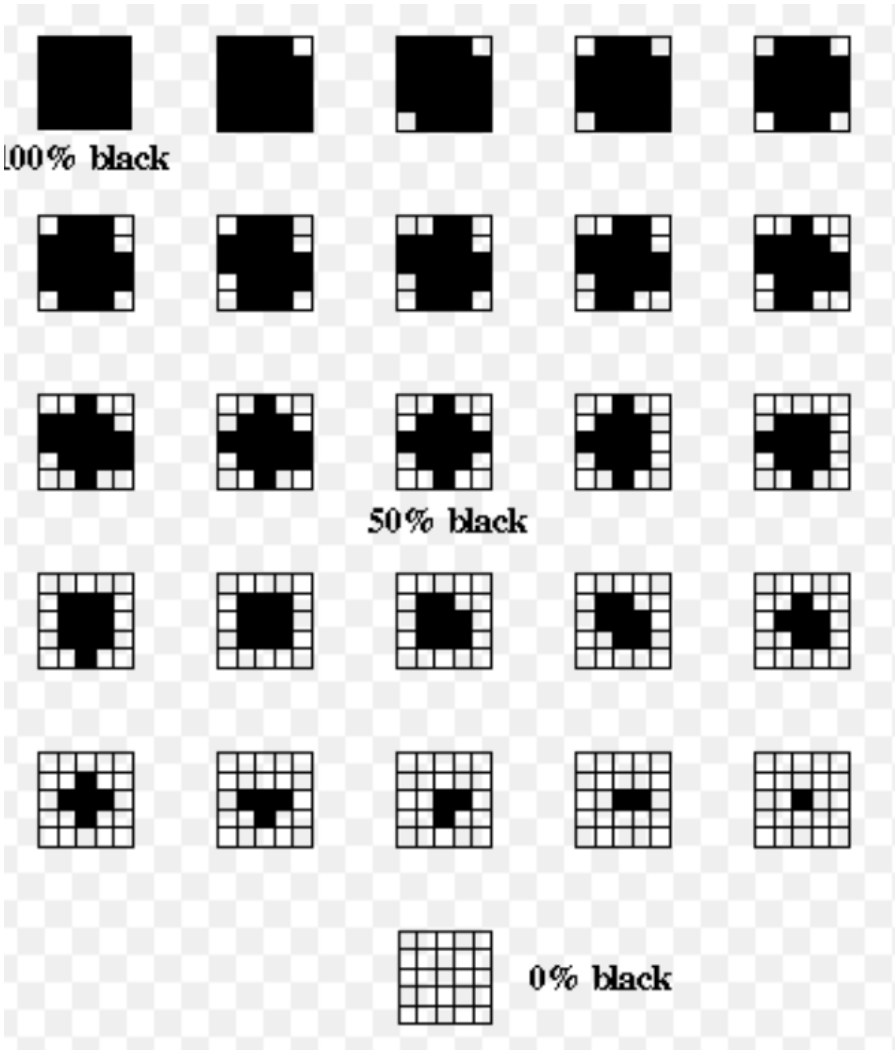


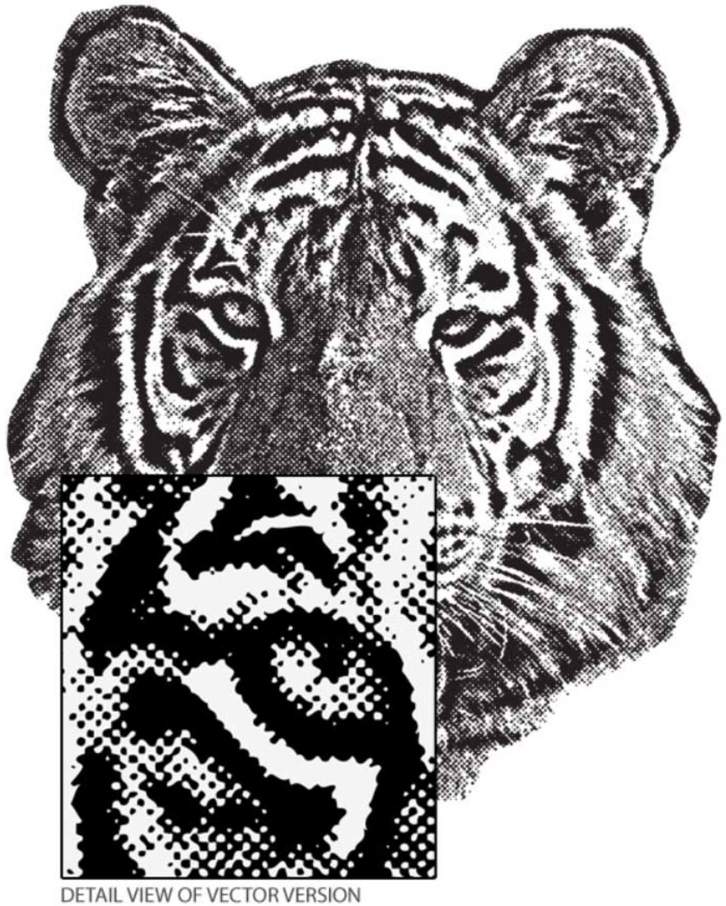
Half-tone (dithering)



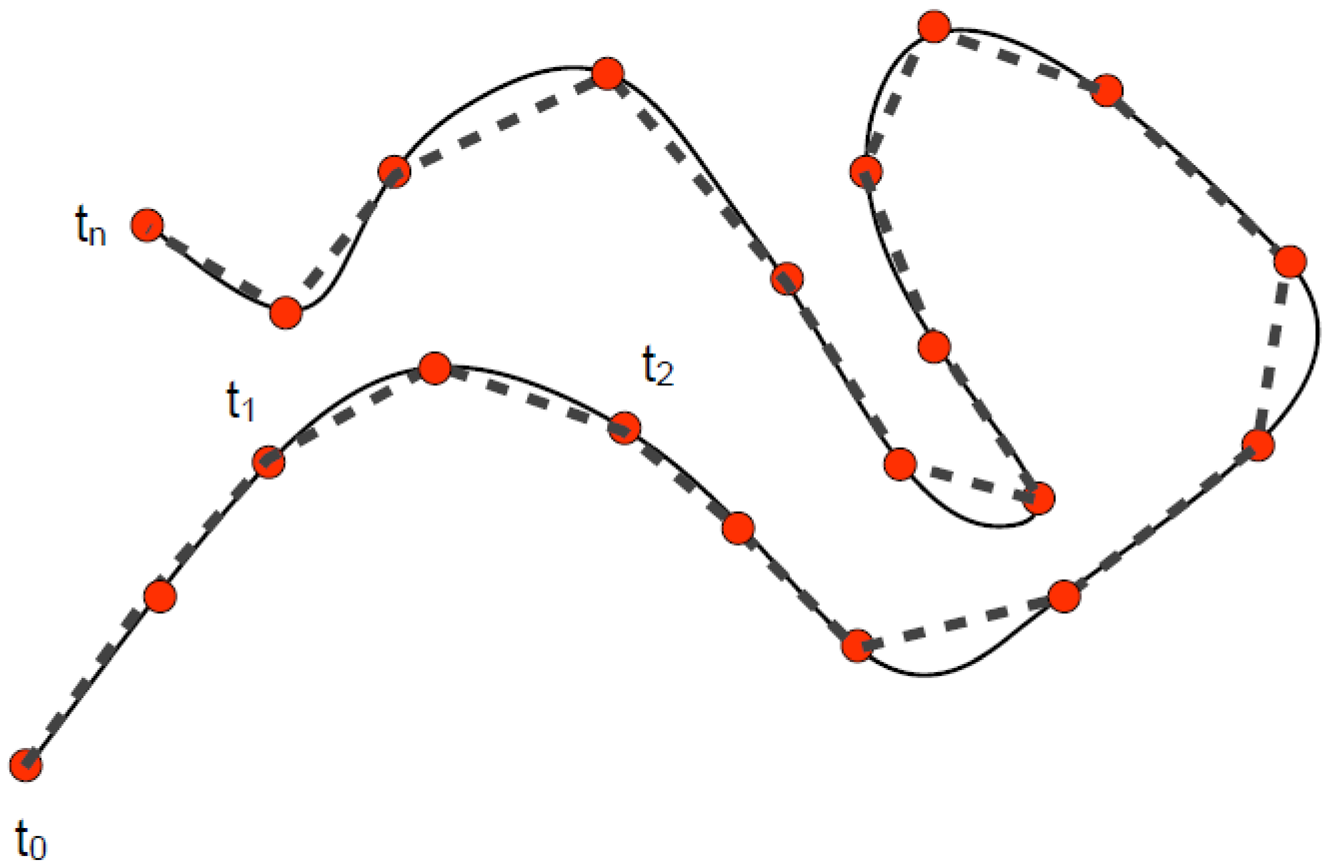


Halftoning Printing

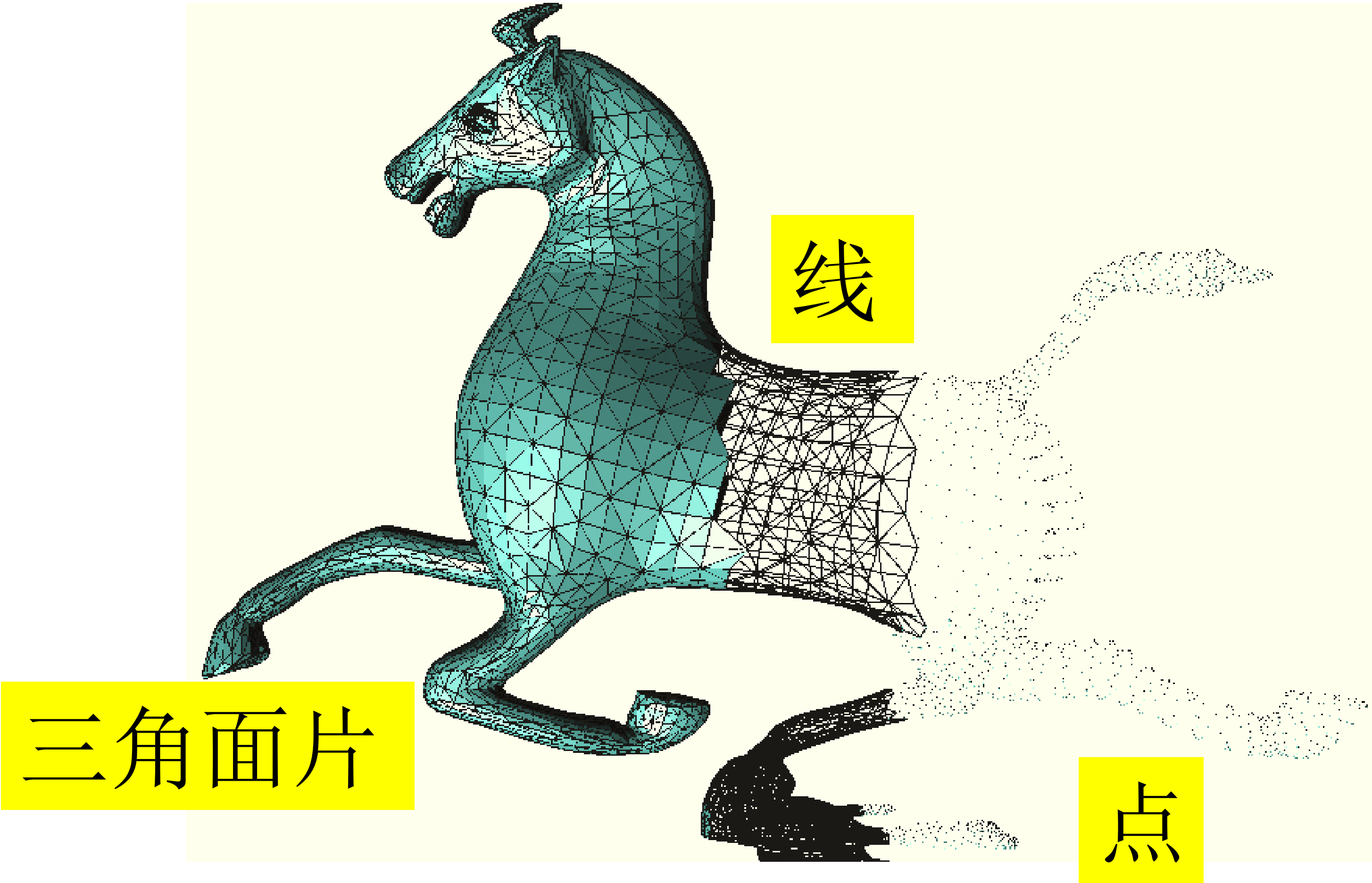




1D曲线的采样：分段线性逼近表达

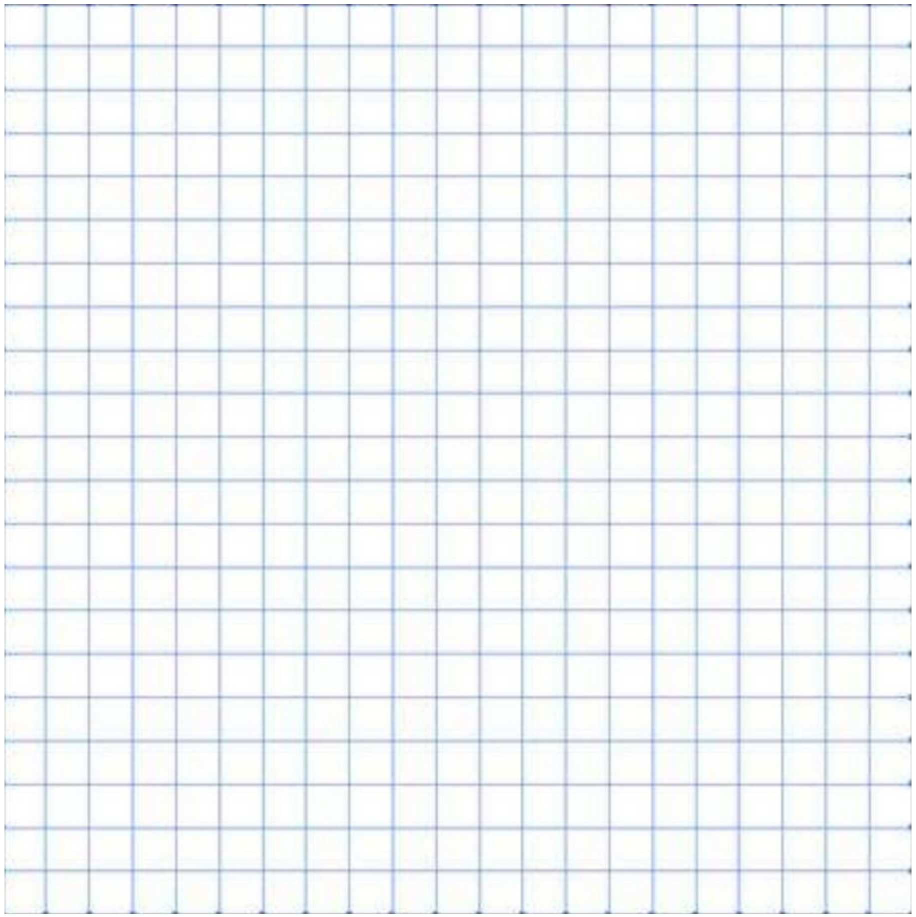


2D曲面的采样：分片线性逼近表达

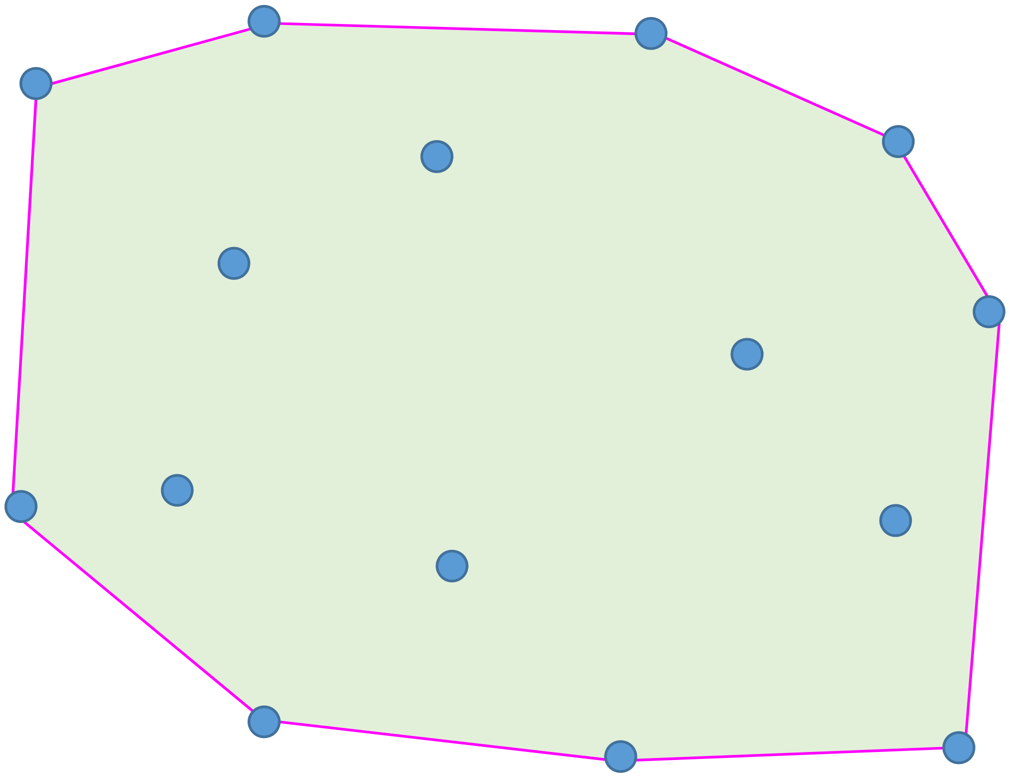


平面规则区域的采样

- 将一个区域分解为若干个小区域

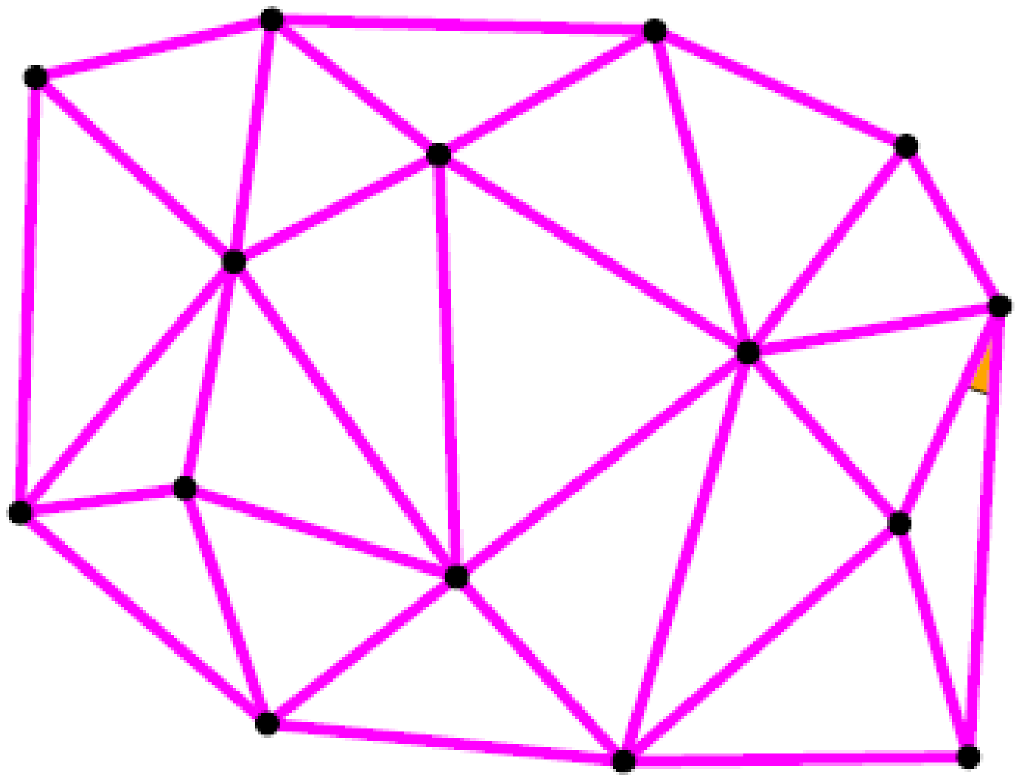


平面区域的不规则采样



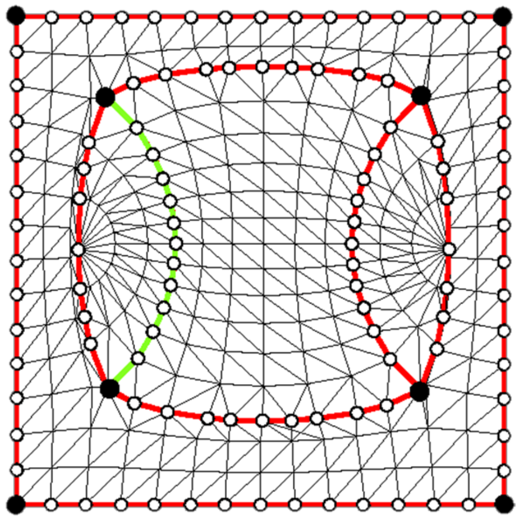
平面区域的不规则采样

- 将一个区域分解为若干个小区域

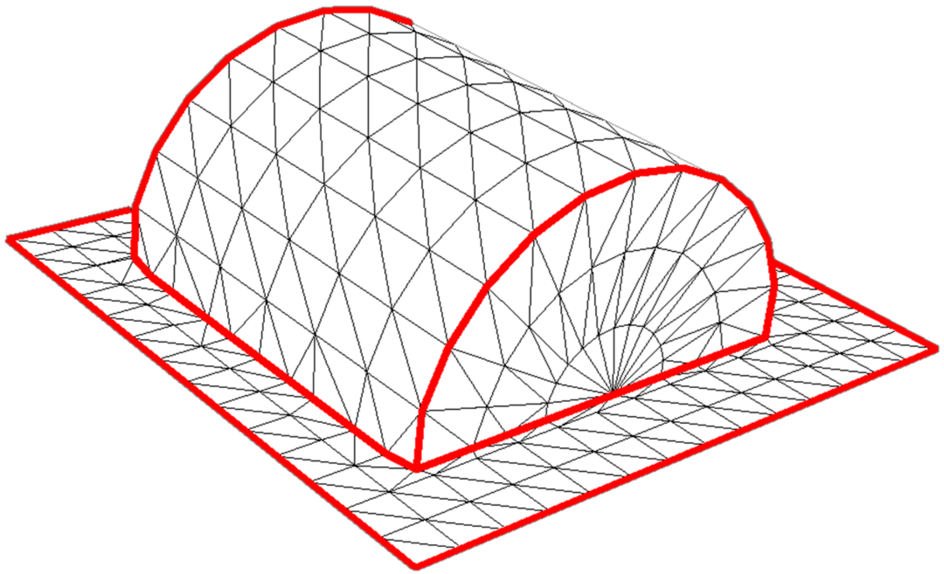


Triangulation

- 复杂函数的分片线性逼近 (piece-wise linear approximation)



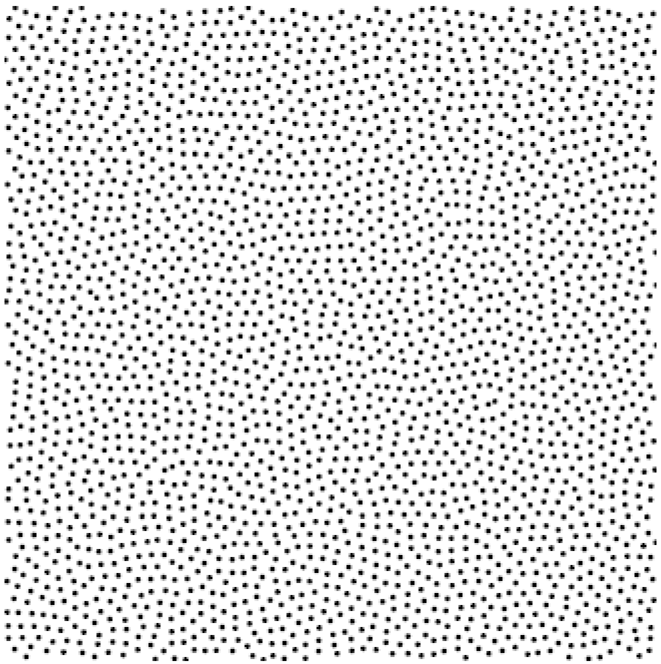
2D

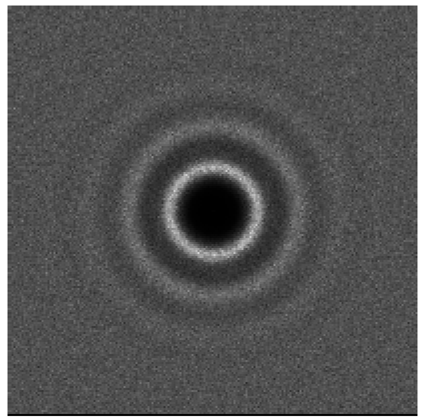


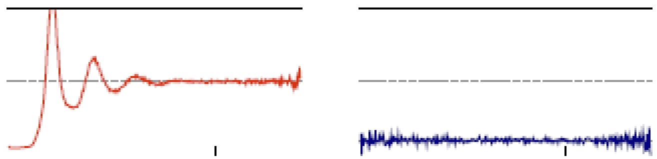
3D

Blue Noise Sampling

- (Left) A uniformly distributed yet randomly located point set
- (Right) The typical power spectrum, radially averaged power spectrum and anisotropy of blue noise distributions.

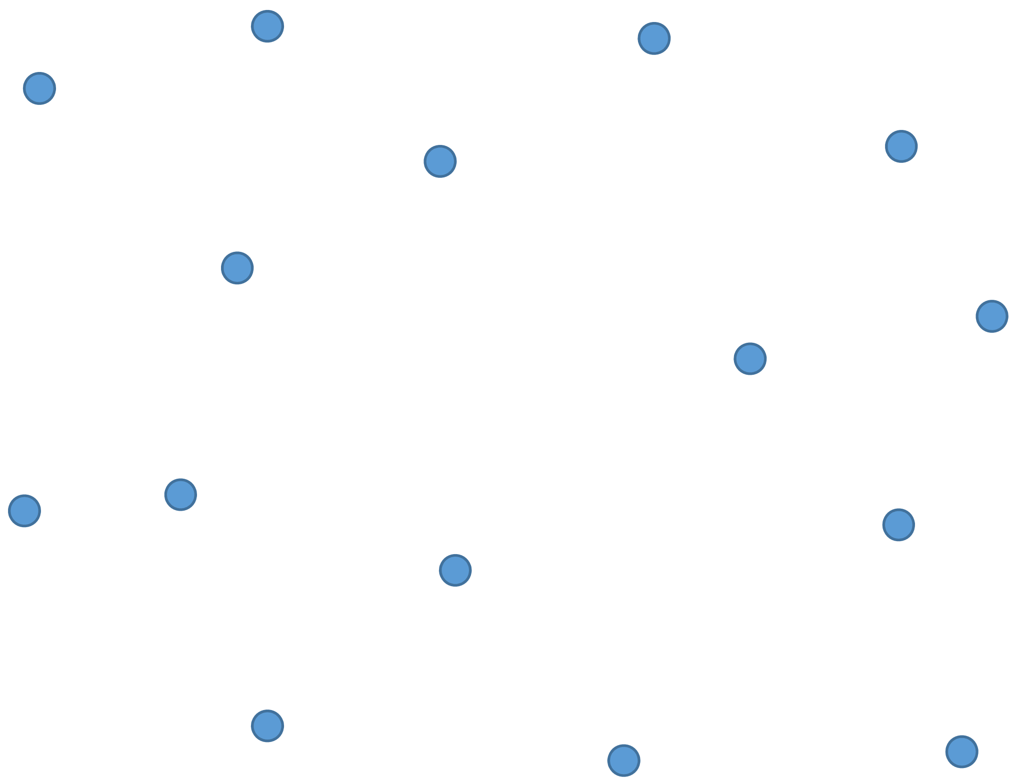




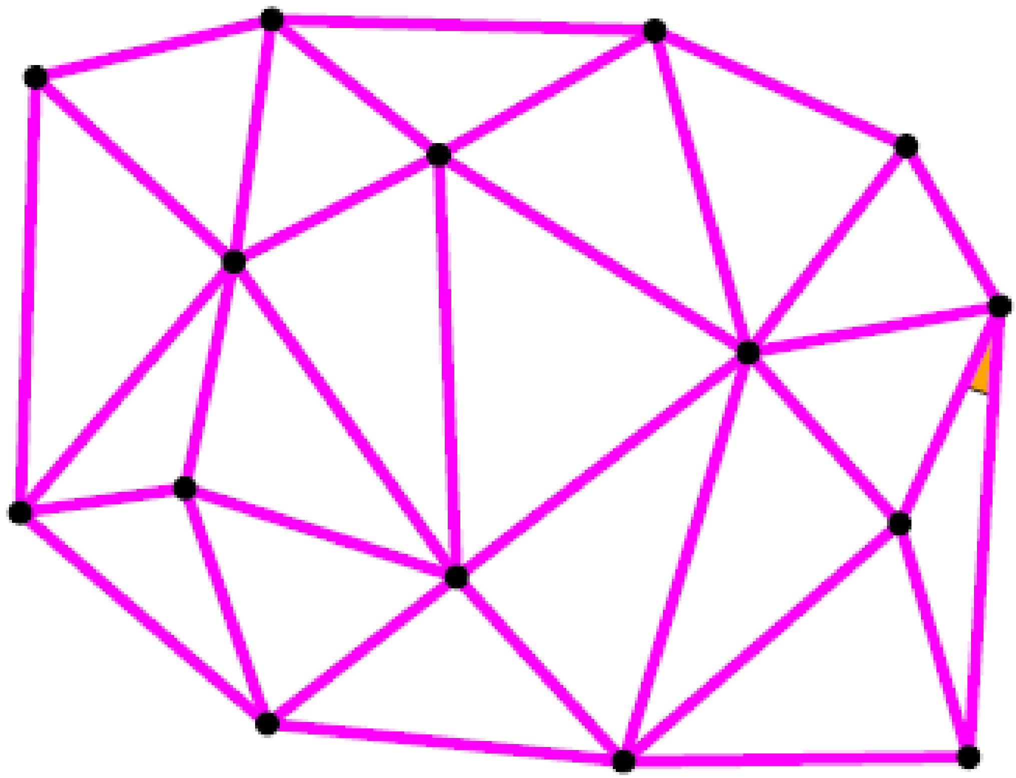


平面三角网格

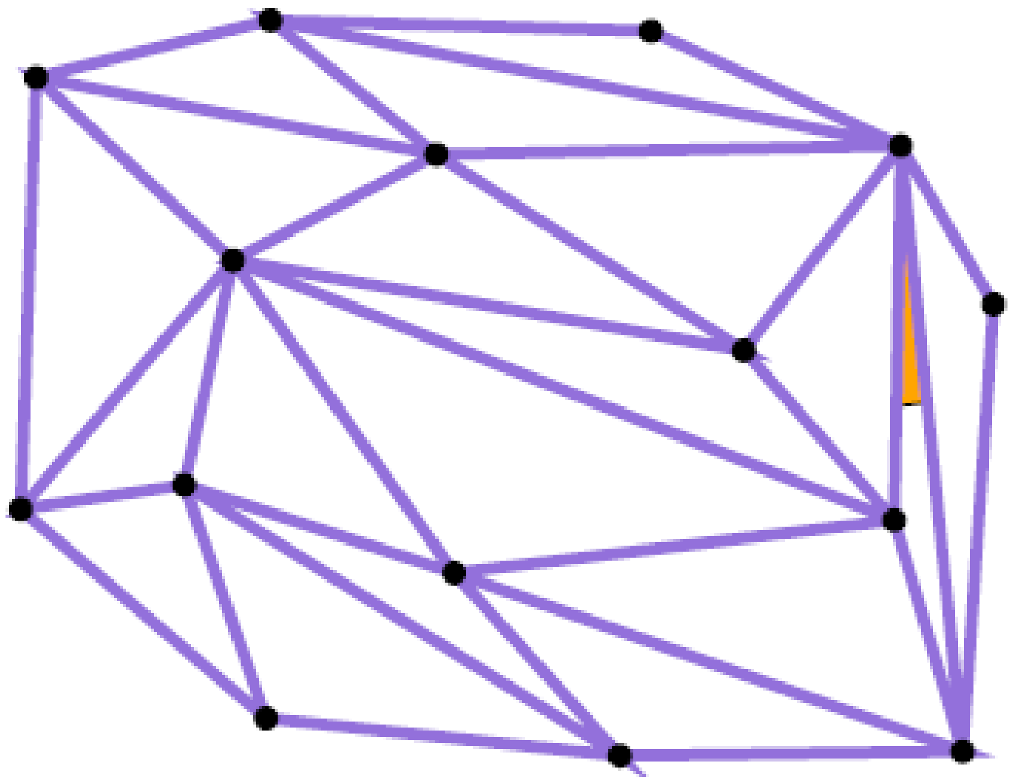
给定平面上一些点，
如何生成三角剖分？



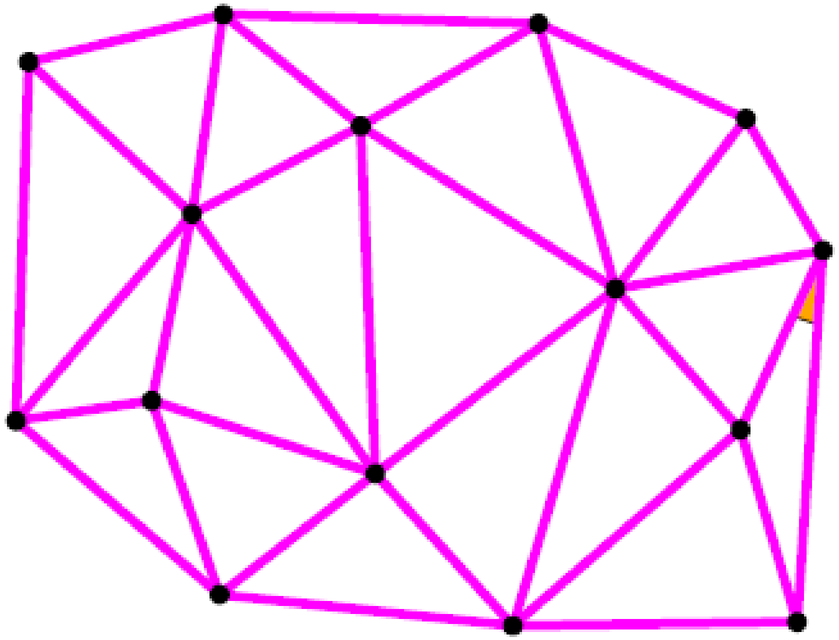
给定平面上一些点，
如何生成三角剖分？

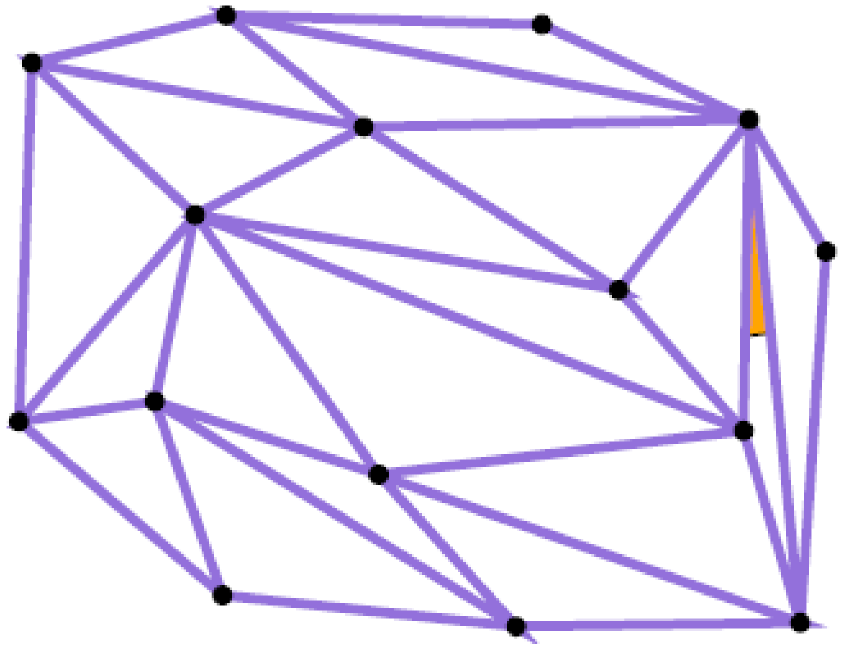


给定平面上一些点，
如何生成三角剖分？



给定平面上一些点，如何生成比较好的三角剖分？

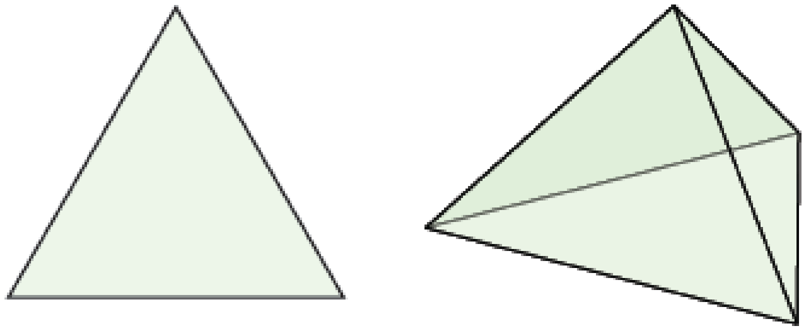




有无“最好”的三角剖分?

Mesh Quality

- What do we mean a “good” mesh/simplex (triangle)?
  - Minimal angle
  - Mean ratio
  - Aspect/radius ratio
  - Singular values
  - …



- It is not easy to define a universal mesh quality acceptable by everyone. But everyone agrees on the "best" simplex: equilateral triangle and tetrahedra.

Delaunay三角化

Boris N. Delaunay

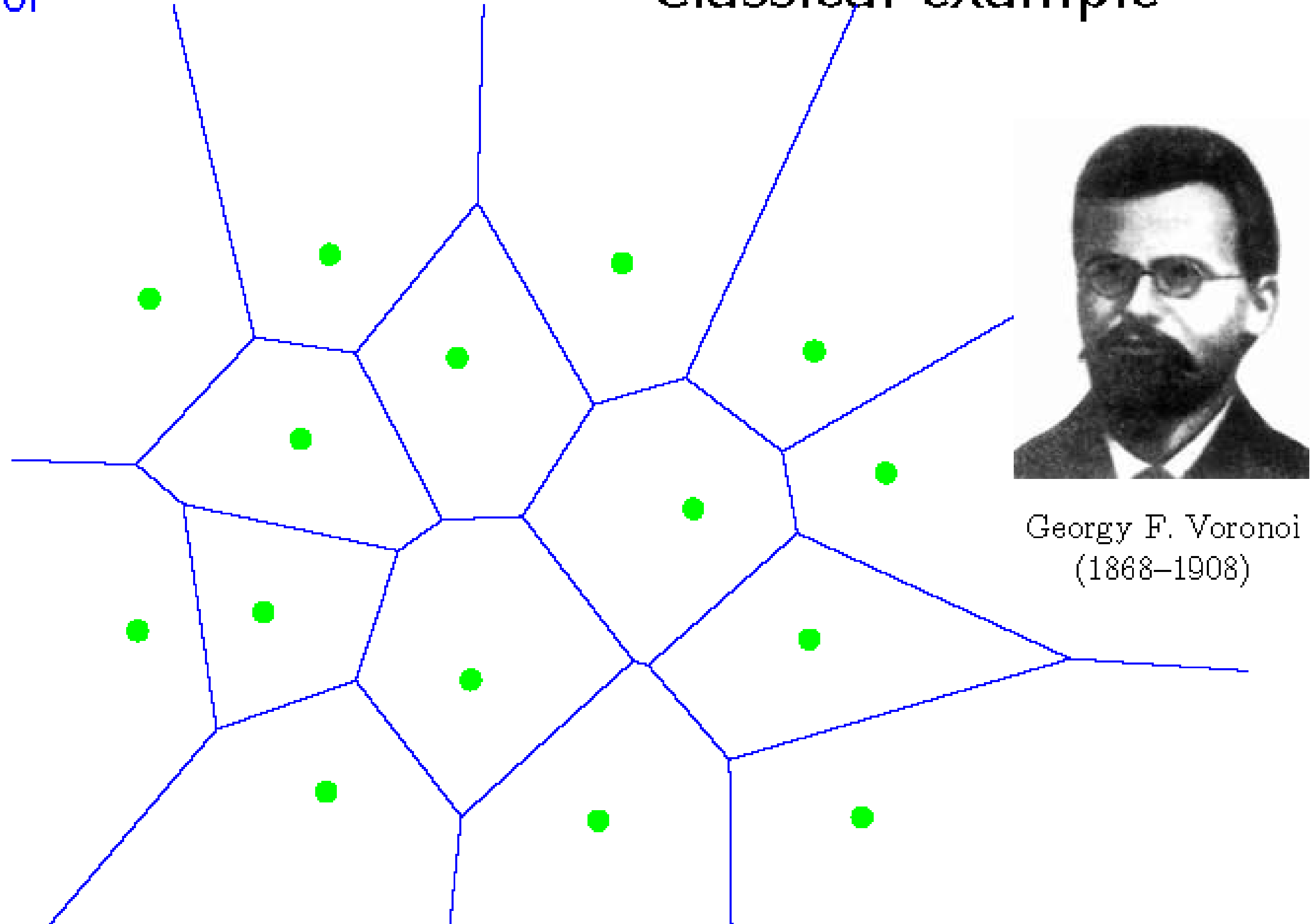
- Russian mathematician
- March 15, 1890 - July 17, 1980
- Introduce Delaunay triangulation in 1934

Georgy F. Voronoi

- Russian mathematician
- April 28, 1868 - November 20, 1908

Voronoi

Classical example



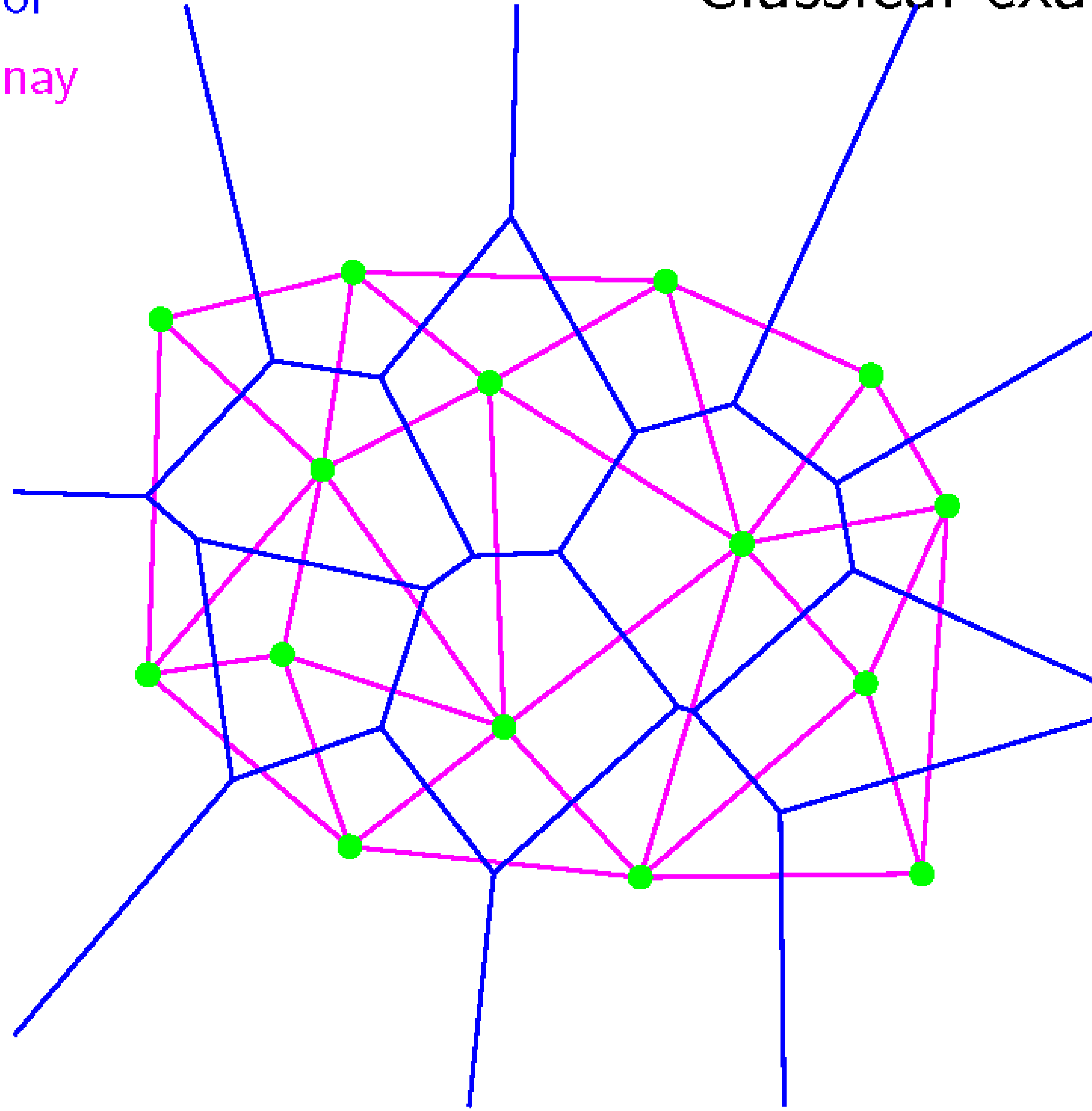
Georgy F. Voronoi
(1868–1908)

$$V_i = \{q; \forall j \neq i \, |qp_i| \leq |qp_j|\}$$

Classical example

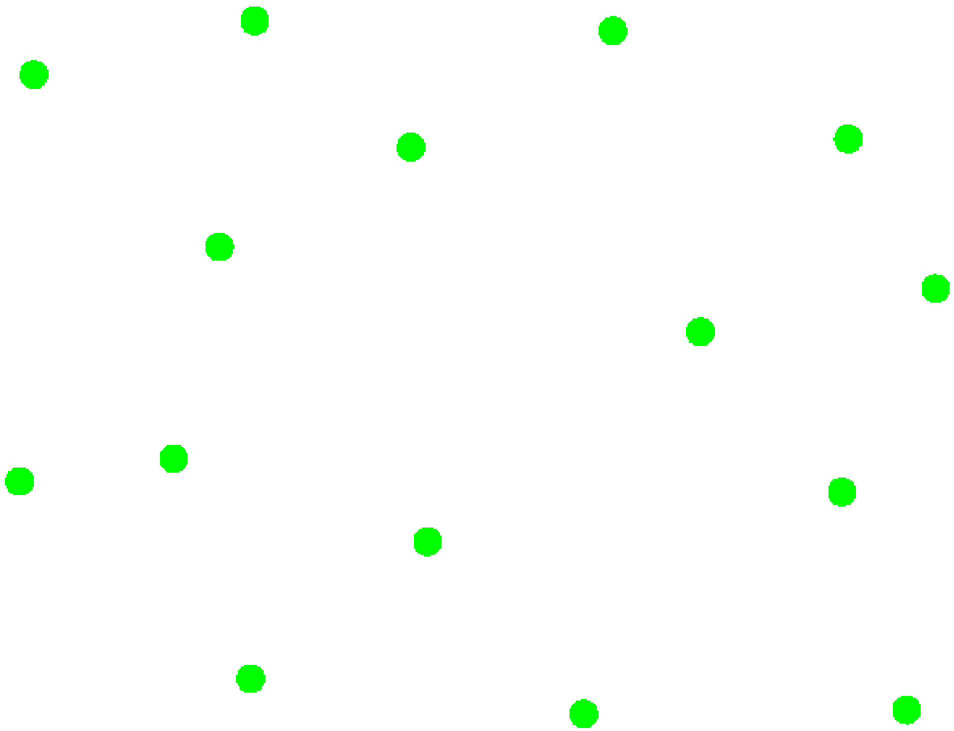
Voronoi

Delaunay



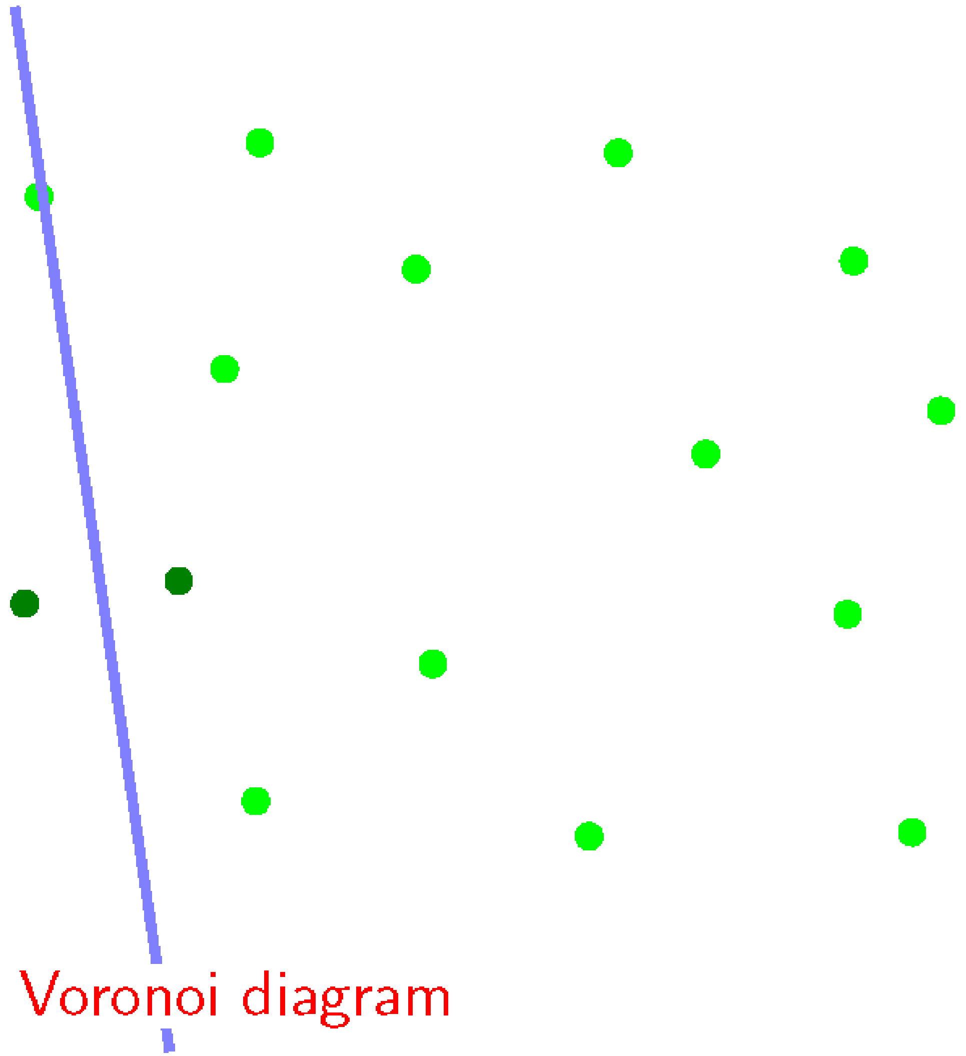
Boris N. Delaunay
(1890–1980)

Voronoi



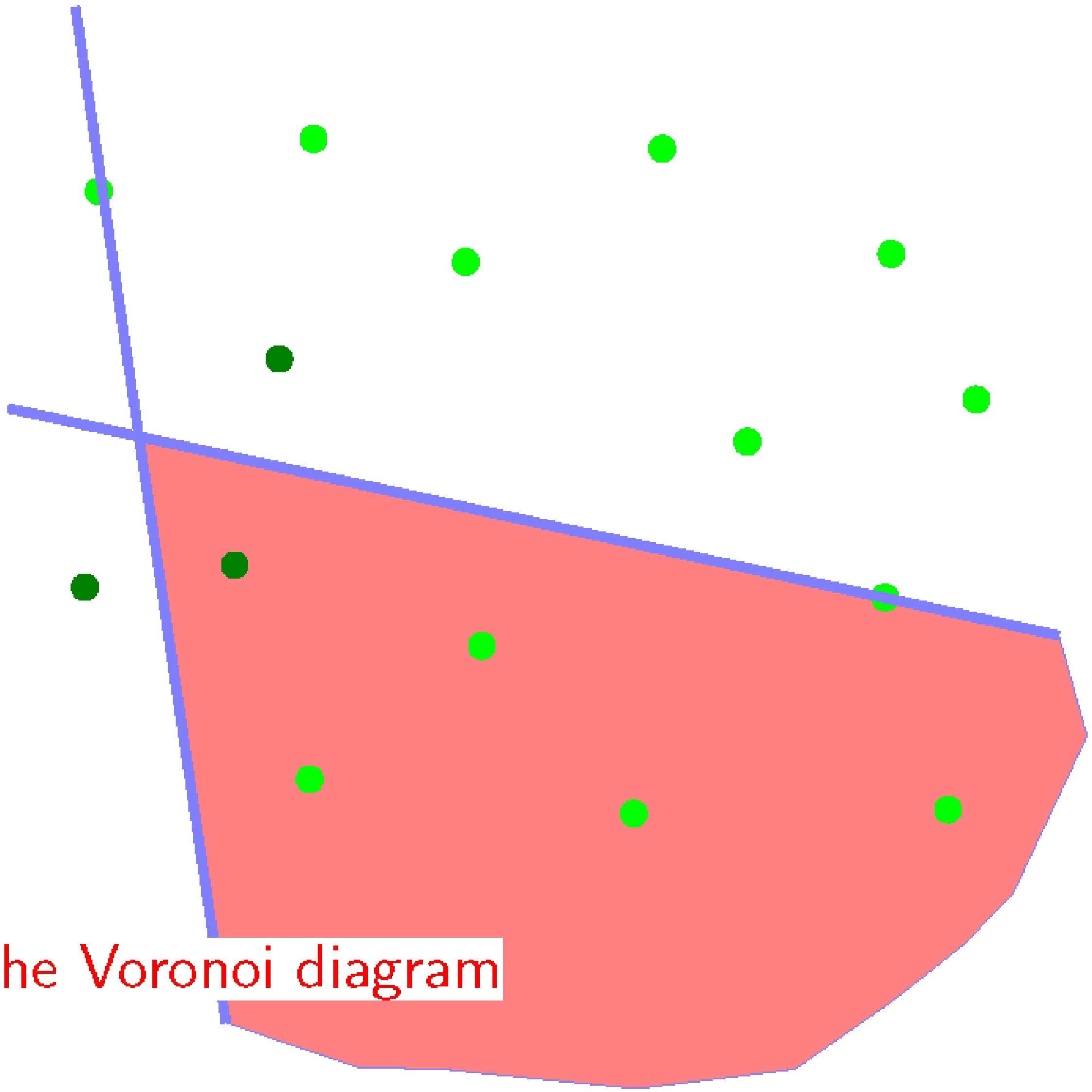
faces of the Voronoi diagram

Voronoi



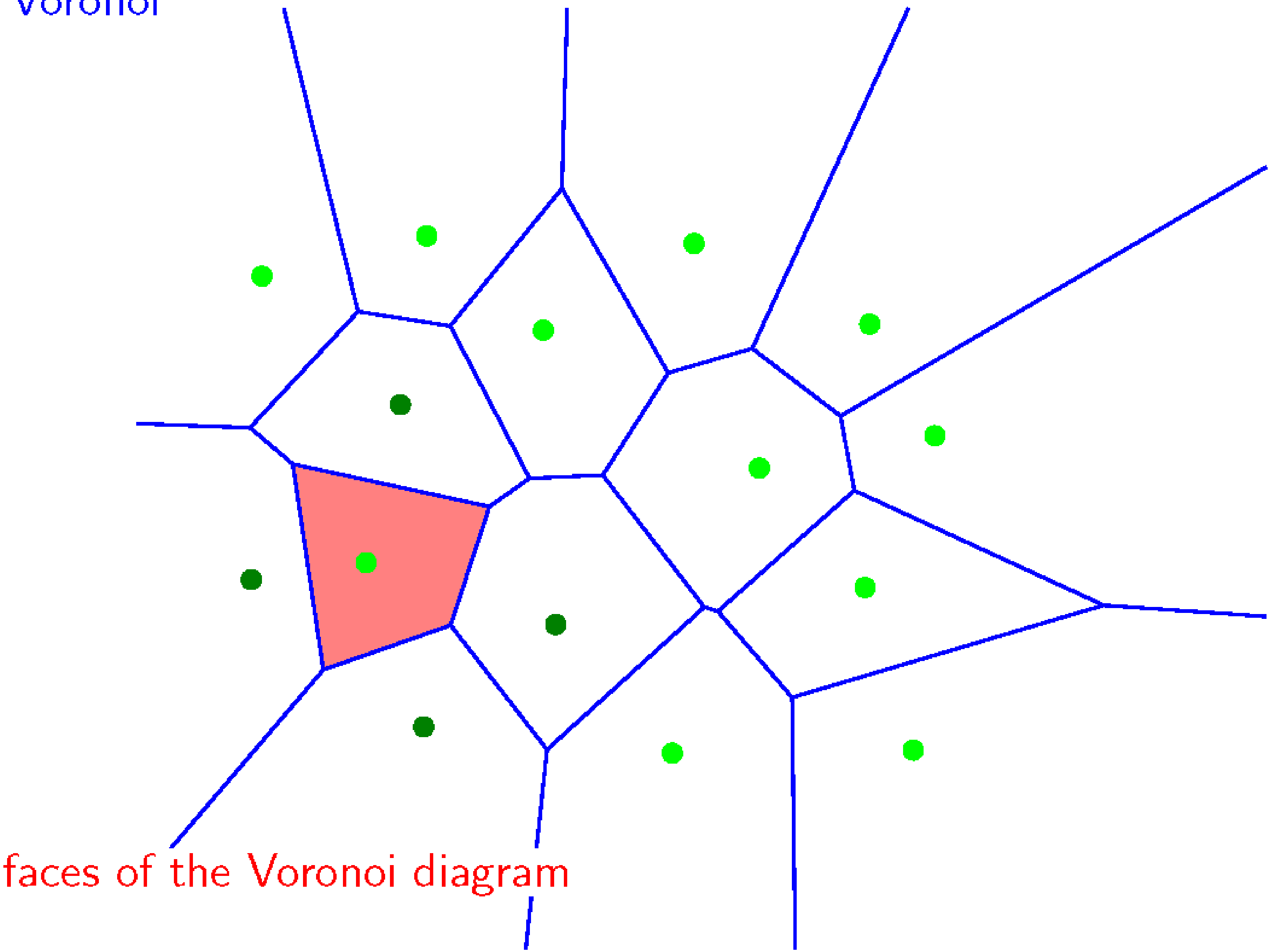
faces of the Voronoi diagram

Voronoi



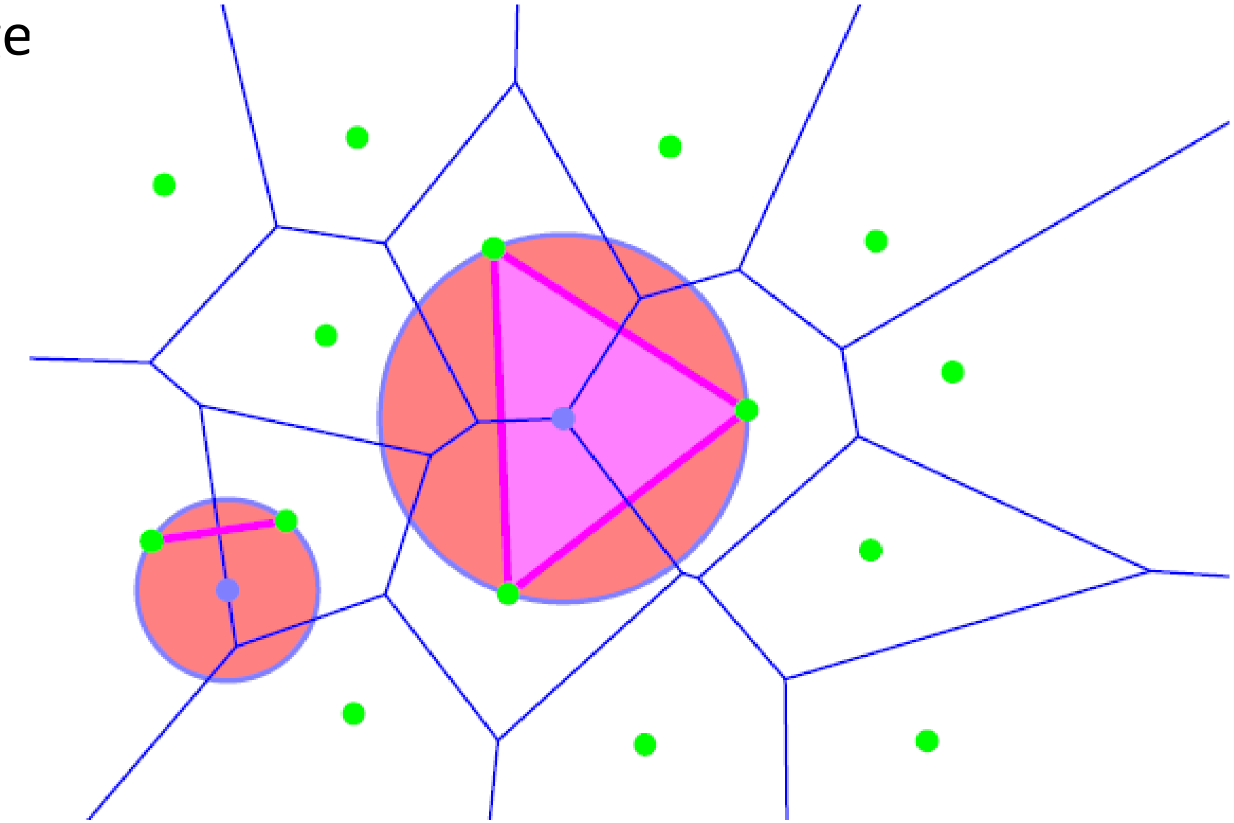
faces of the Voronoi diagram

Voronoi



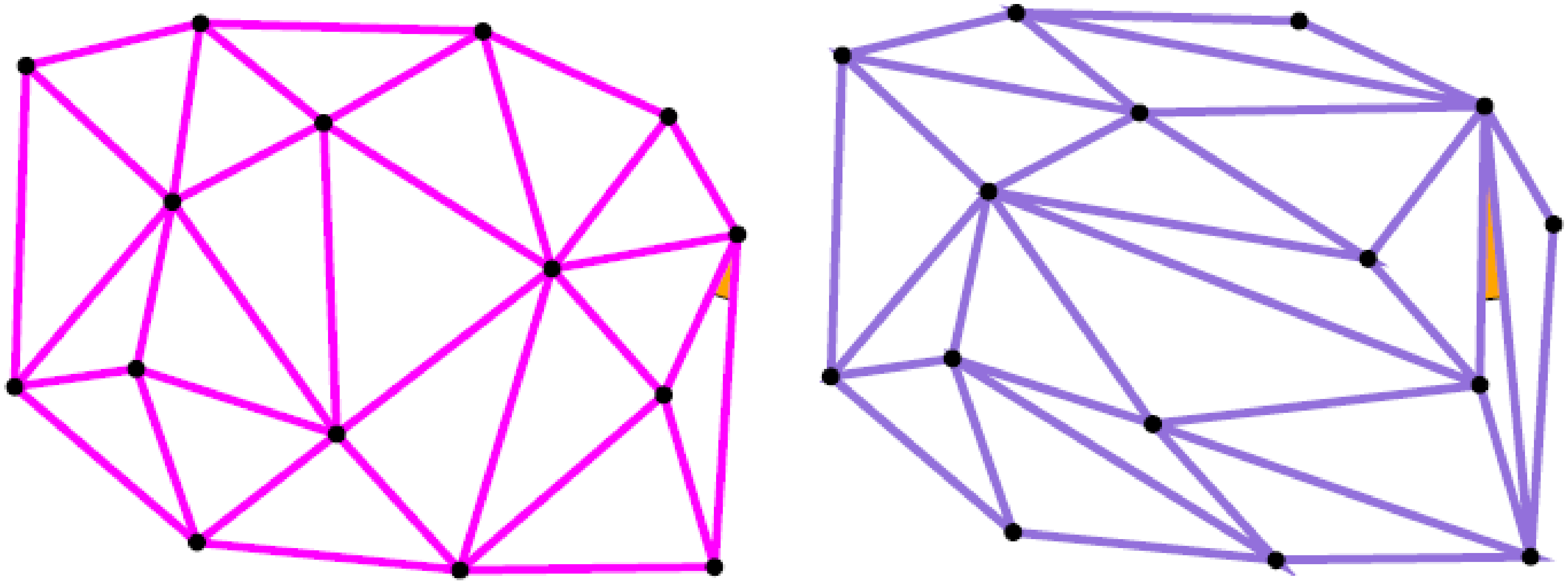
Properties of DT (1)

- Empty sphere property: no points inside the circum-sphere of any simplex
  - Delaunay edge



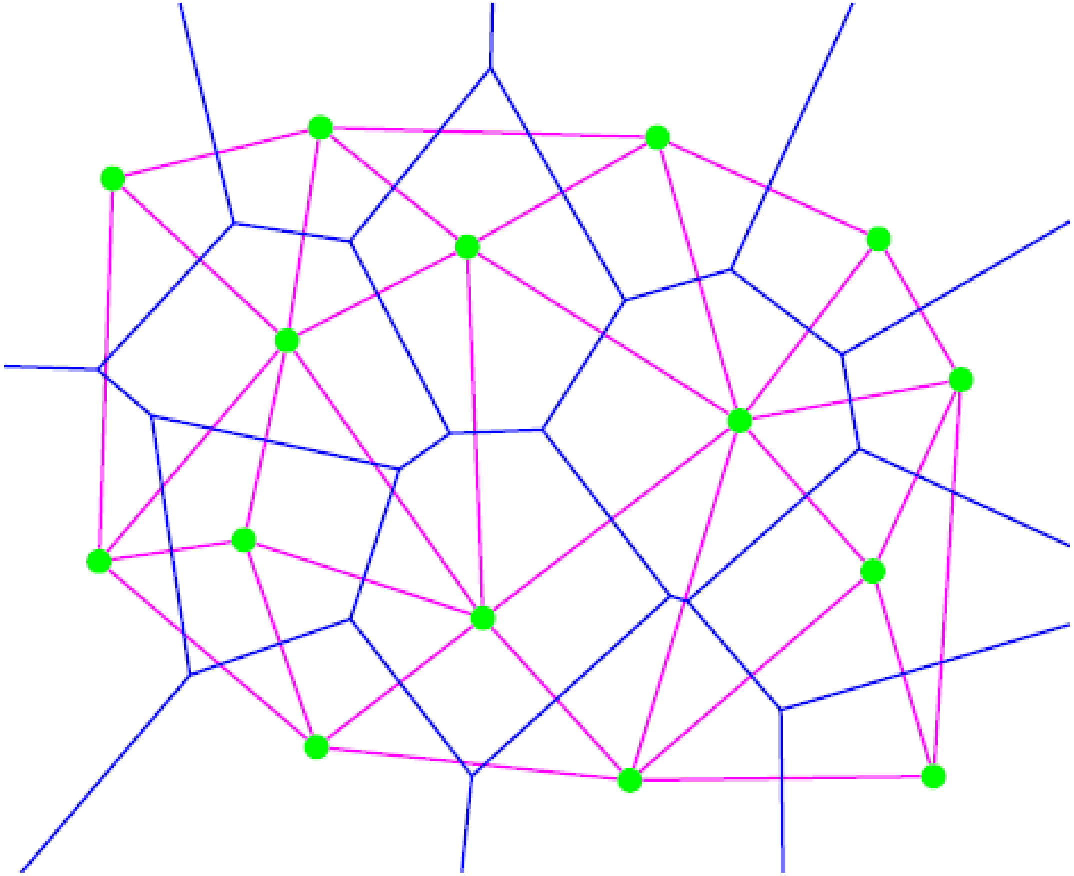
Properties of DT (2)

- DT maximizes the smallest angle
  - [Lawson 1977] and [Sibson 1978]



Properties of DT (3)

- Convex hull: union of all triangles

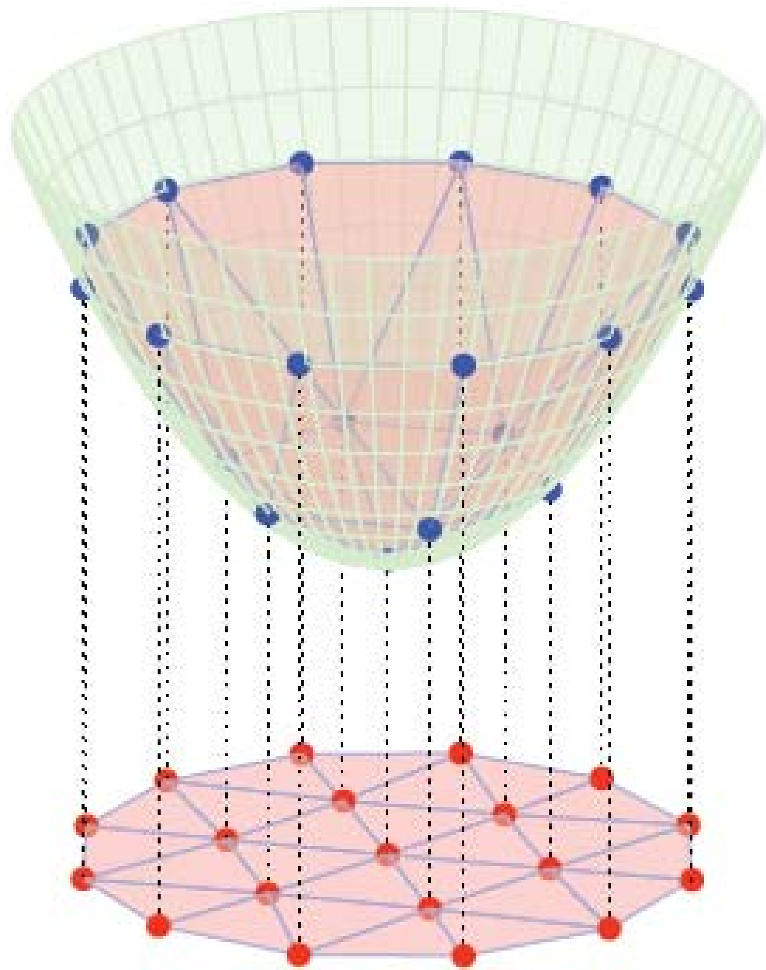


Properties of DT (4)

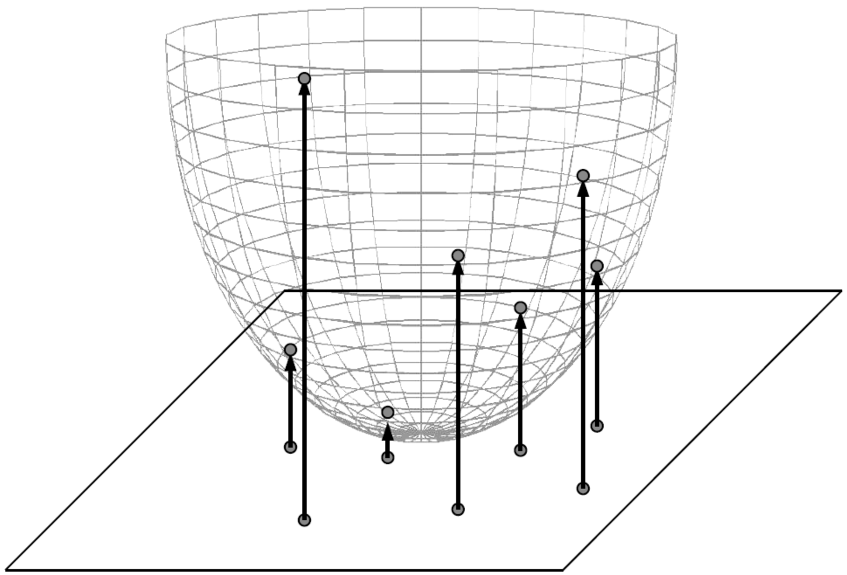
- DT maximizes the arithmetic mean of the radius of inscribed circles of the triangles.
  - [Lambert 1994]
- DT minimizes roughness (the Dirichlet energy of any piecewise-linear scalar function)
  - [Rippa 1990]
- DT minimizes the maximum containing radius (the radius of the smallest sphere containing the simplex)
  - [Azevedo and Simpson 1989], [Rajan 1991]

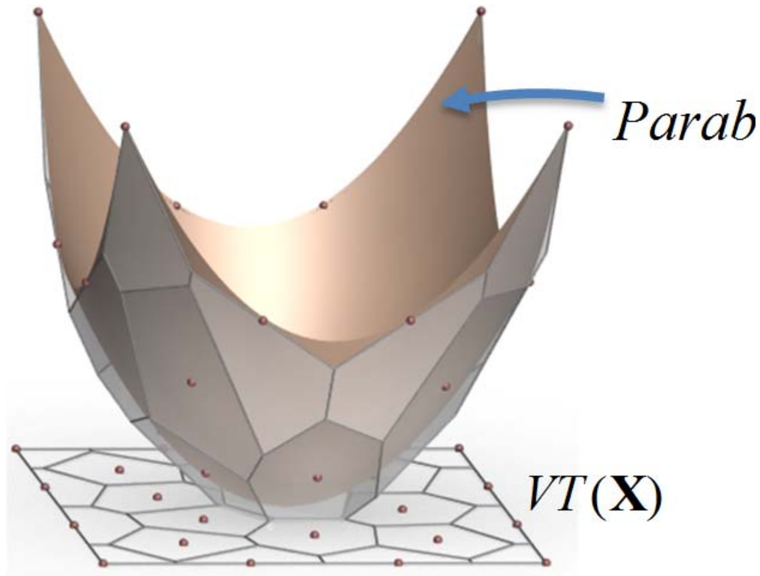
Properties of DT (5)

- The DT in d-dimensional spaces is the projection of the points of convex hull onto a (d+1)-dimensional paraboloid.
  - [Brown 1979]



Properties of DT (5)





Paraboloid: $f(\mathbf{x}) = \| \mathbf{x} \|^2$

Lifting Operator: $\mathbf{x}^* = (\mathbf{x}, \| \mathbf{x} \|^2)$

Intersections of tangent planes

Properties of DT (6)

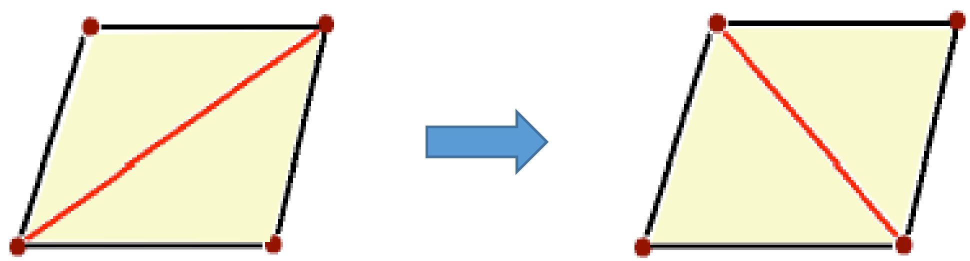
- DT minimizes the spectrum of the geometric Laplacian (spectral characterization)
  - [Chen et al. 2010]

Delaunay Spectral Theorem

The spectrum of the geometric Laplacian obtains its minimum on a Delaunay triangulation. Namely if $\{\lambda_1 = 0, \lambda_2, \ldots, \lambda_n\}$ and $\{\mu_1 = 0, \mu_2, \ldots, \mu_n\}$ are the sequences of non-decreasing eigenvalues of the geometric Laplacian of a Delaunay triangulation and of any other triangulation of the same set of points, respectively, then $\lambda_i \leq \mu_i$ for $i = 1, \ldots, n$.

Simple Method: Edge Swapping/Flipping [Sibson 1978]

- Start with any triangulation
  - 1. find any two adjacent triangles that form a convex quadrilateral that does not satisfy empty sphere condition
  - 2. swap the diagonal of the quadrilateral to be a Deluany triangulation of that four points
  - 3. repeat step 1,2 until stuck.



Convergence? Is it possible to end with an infinite loop?

Algorithms for Voronoi Diagrams

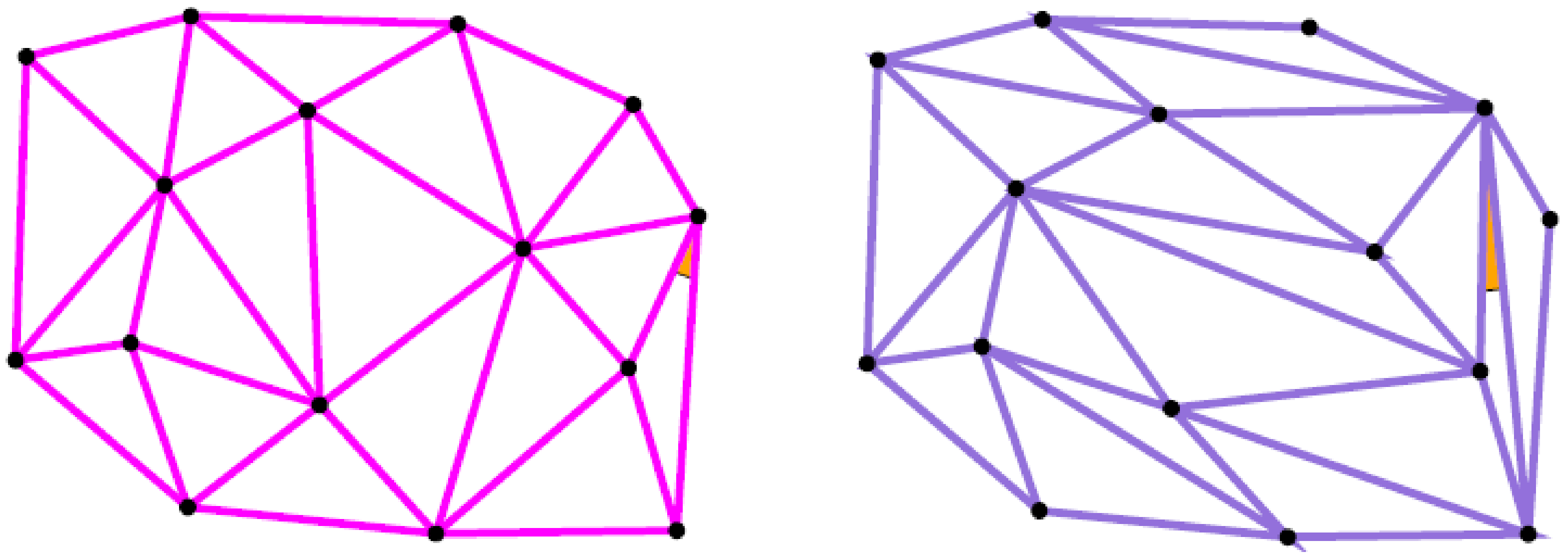
- Compute the intersection of n-1 half-planes for each site, and “merge” the cells into the diagram
- Divide-and-conquer (1975, Shamos & Hoey)
- Plane sweep (1987, Fortune)
- Randomized incremental construction (1992, Guibas, Knuth& Sharir)

Available in open source library CGAL:
http://www.cgal.org

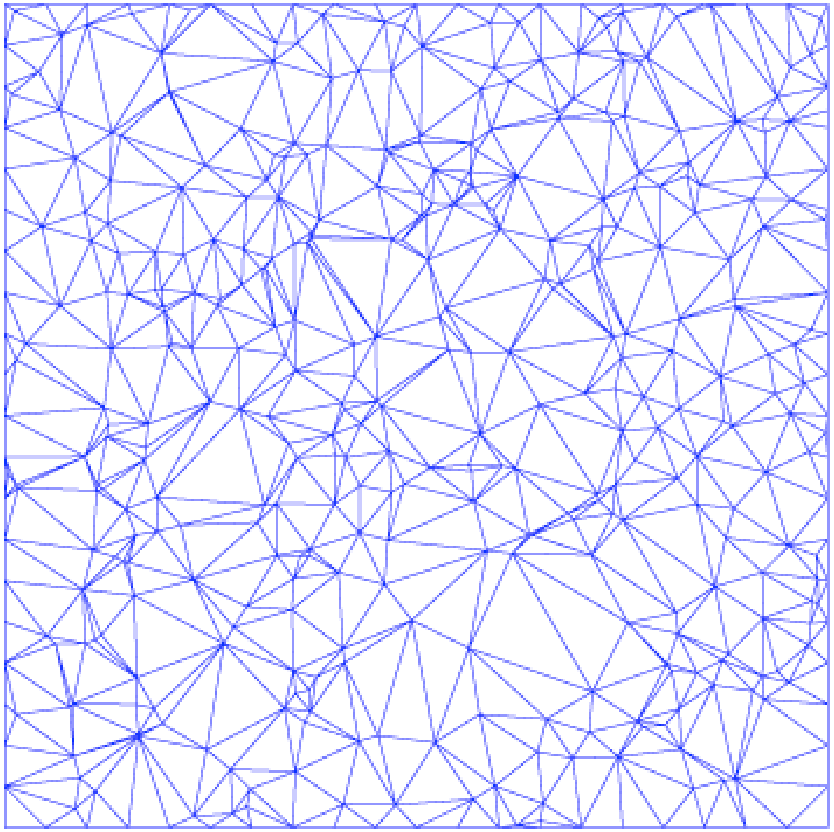
Mesh Generation

Mesh Generation

- Given a fixed point set, Delaunay triangulation will try to make the triangulation more shape regular and thus is considered as a “good” unstructured mesh.



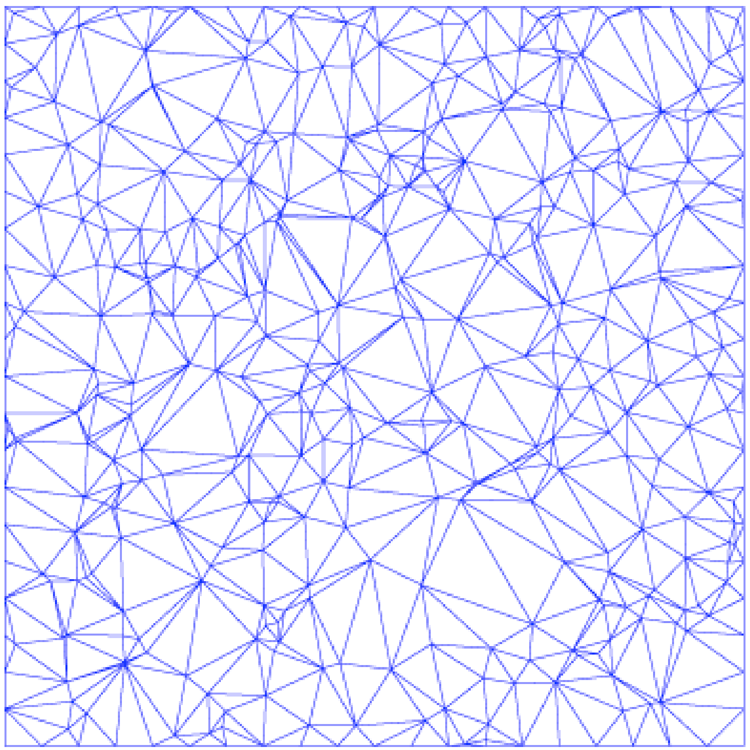
DT is not necessary a good mesh

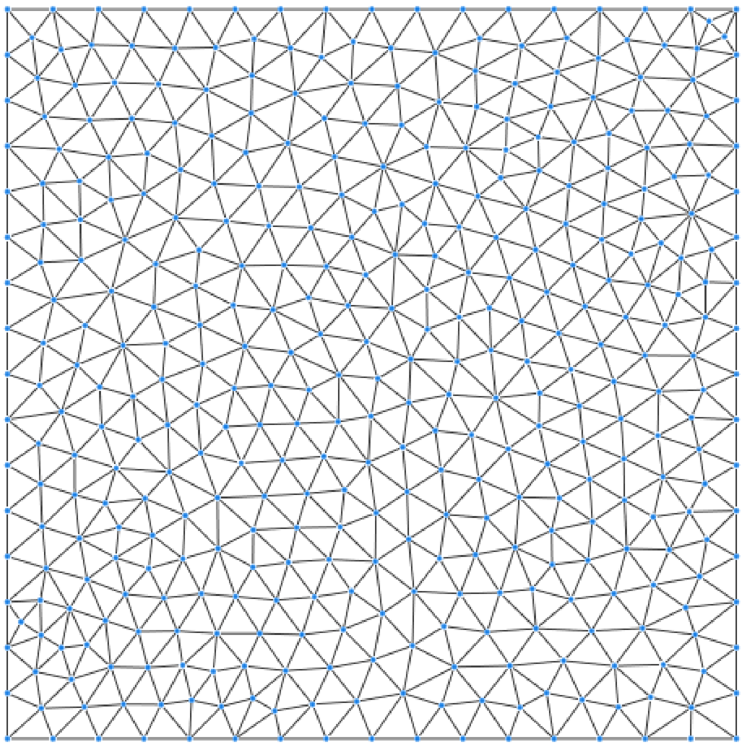


DT only optimize the **connectivity** when points are fixed. The **distribution of points** is more important for a good mesh.

Mesh Generation

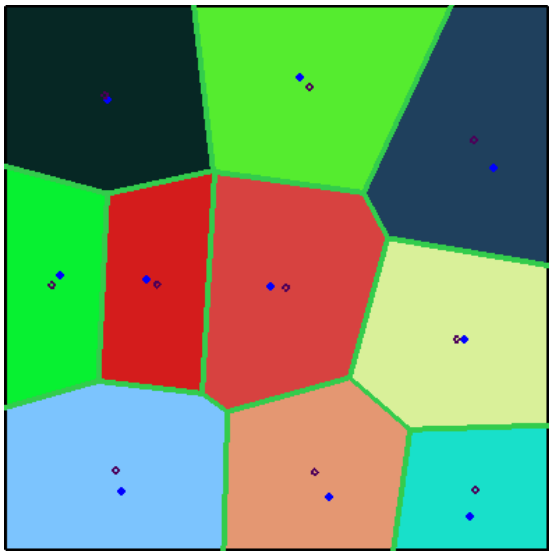
- How to sample points to generate high-quality meshes?



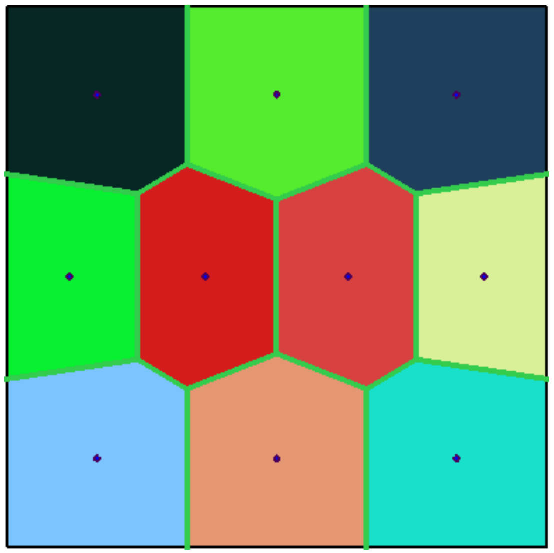


Centroidal Voronoi Tessellation

- *Definition*: The VT is a centroidal Voronoi tessellation (CVT) , if each seed coincides with the centroid of its Voronoi cell

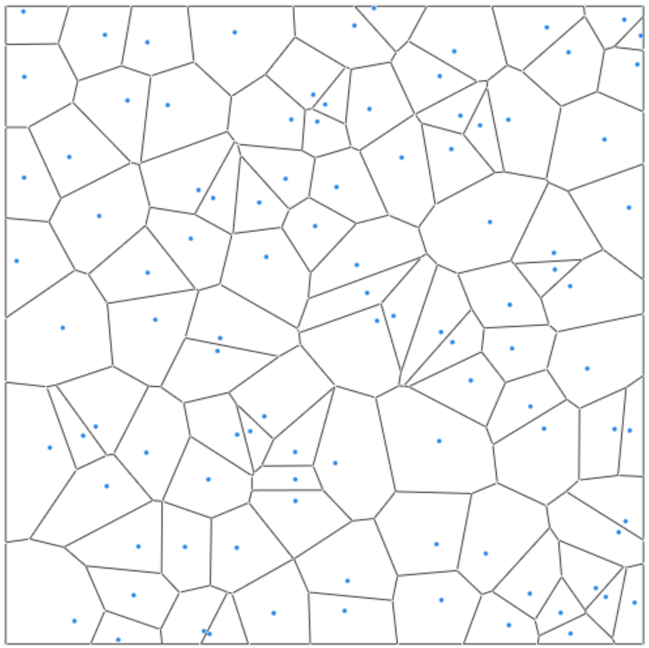


Generic VT

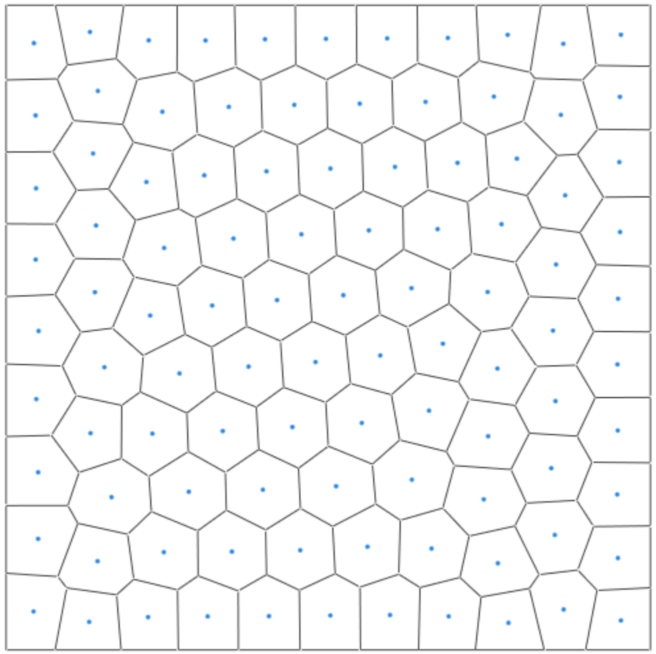


CVT

Centroidal Voronoi Tessellation



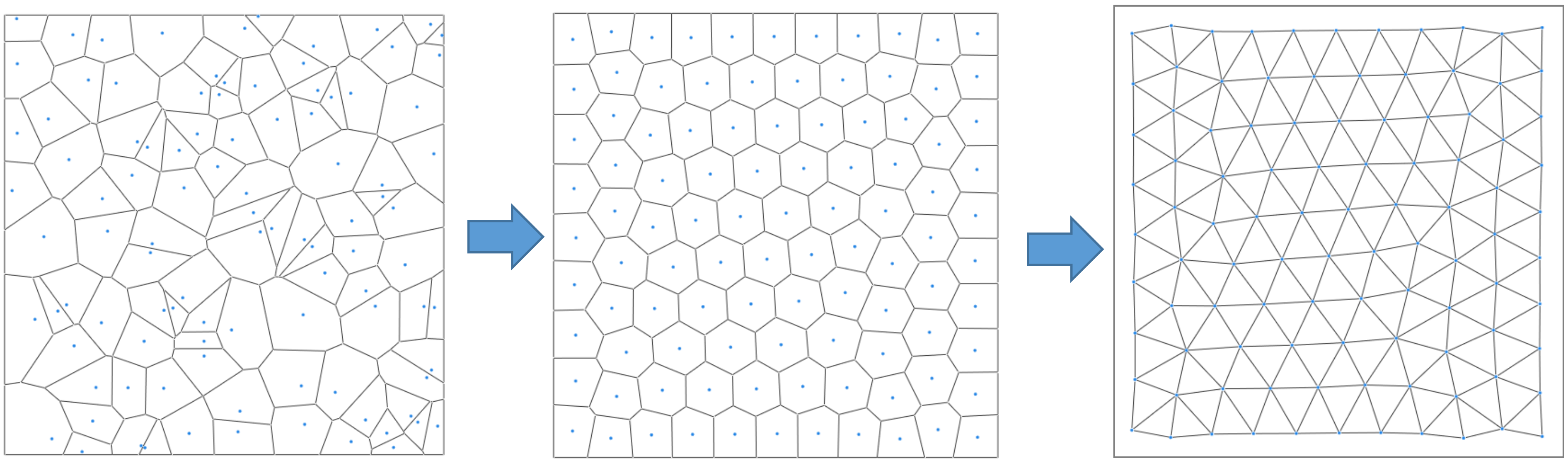
Generic VT



CVT

Lloyd Algorithm

- Construct the VT associated with the points
- Compute the centroids of the Voronoi regions
- Move the points to the centroids
- Iterate until convergent



Centroidal Voronoi Tessellation

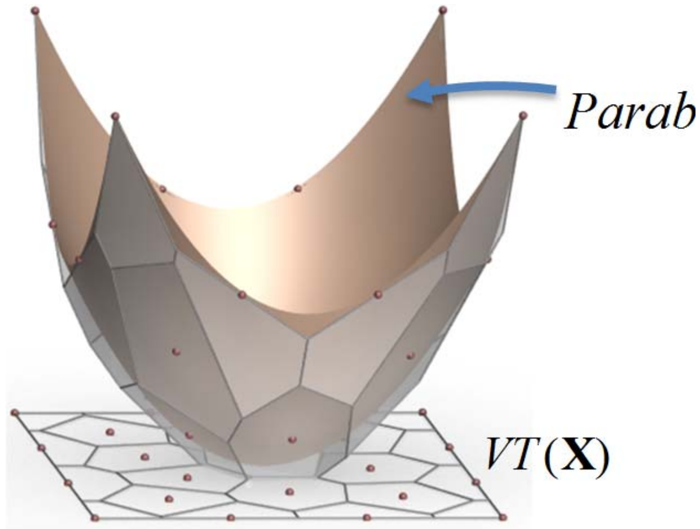
- *Definition* (Variational point of view)
  - CVT energy function:

$$F(X) = \sum_{i=1}^{N} \int_{V_i} \rho(\mathbf{x}) \|\mathbf{x} - \mathbf{x}_i\|^2 d\mathbf{x}$$

- CVT is a critical point of $F(X)$, an *optimal* CVT is a global minimizer of $F(X)$

CVT Energy Function

- Geometric interpretation
  - The CVT energy with $\rho(\mathbf{x})$ identical to *1*, is the volume difference between the circumscribed polytope and the paraboloid.



The Gradient of CVT Energy

- The gradient of *F(**X**)* is [Iri et al. 1984; Asami 1991; Du et al. 1999]:

$$\frac{\partial F}{\partial \mathbf{x}_i} = 2m_i(\mathbf{x}_i - \mathbf{c}_i),$$

where
$$m_i = \int_{\mathbf{x}\in\Omega_i} \rho(\mathbf{x})\,\mathrm{d}\sigma$$

- Lloyd’s method is a gradient descent method, thus has linear convergence

Smoothness of *F(X)*

- Can BFGS method be applied to computing CVT? Or does CVT energy *F(**X**)* have required C^2 smoothness?
- Results:
  - It has been noted that *F(**X**)* is non-smooth [Iri et al. 1984], but without proof
  - It has been proved that *F(**X**)* is C^1 [Cortes et al. 2005]
  - *F(**X**)* is C^2 in a **convex** domain in 2D and 3D [Liu et al. 2009]

C^2 Continuity of $F(X)$ – Illustration

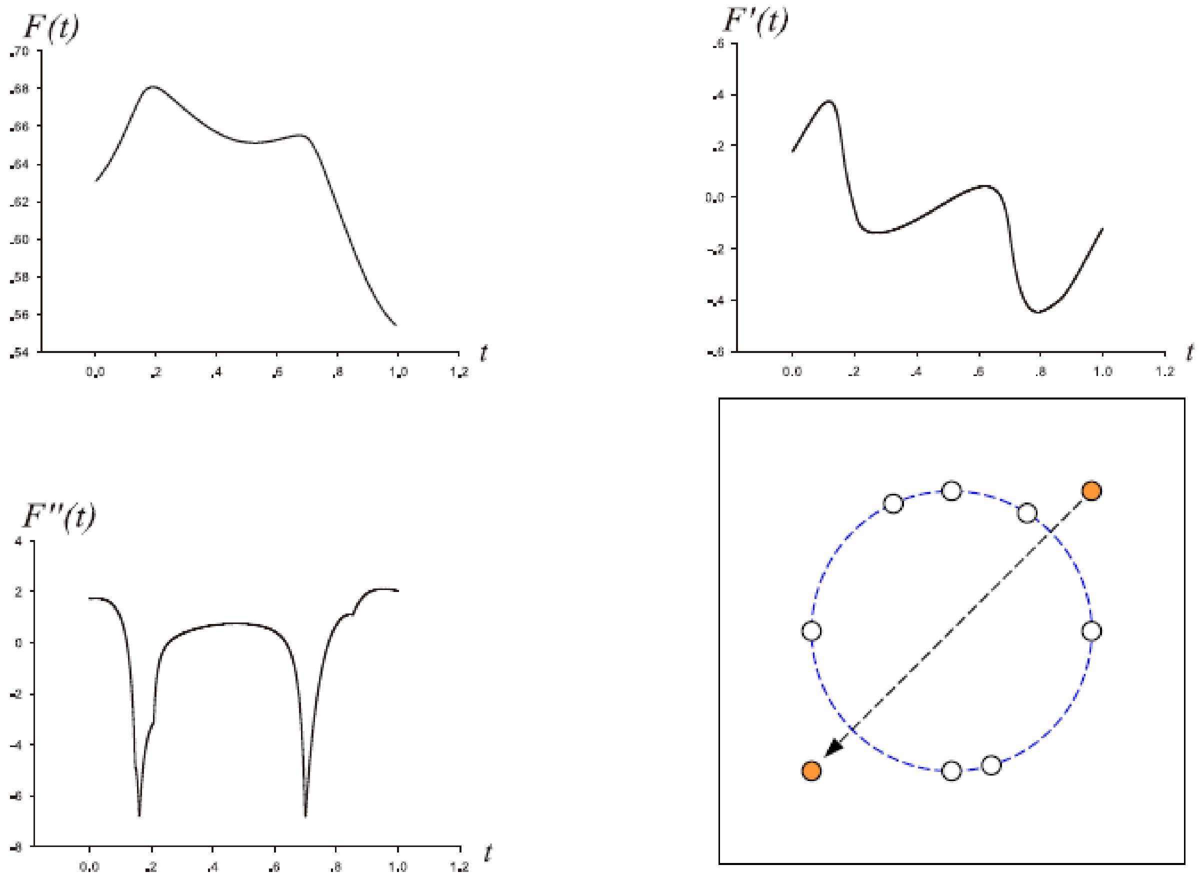
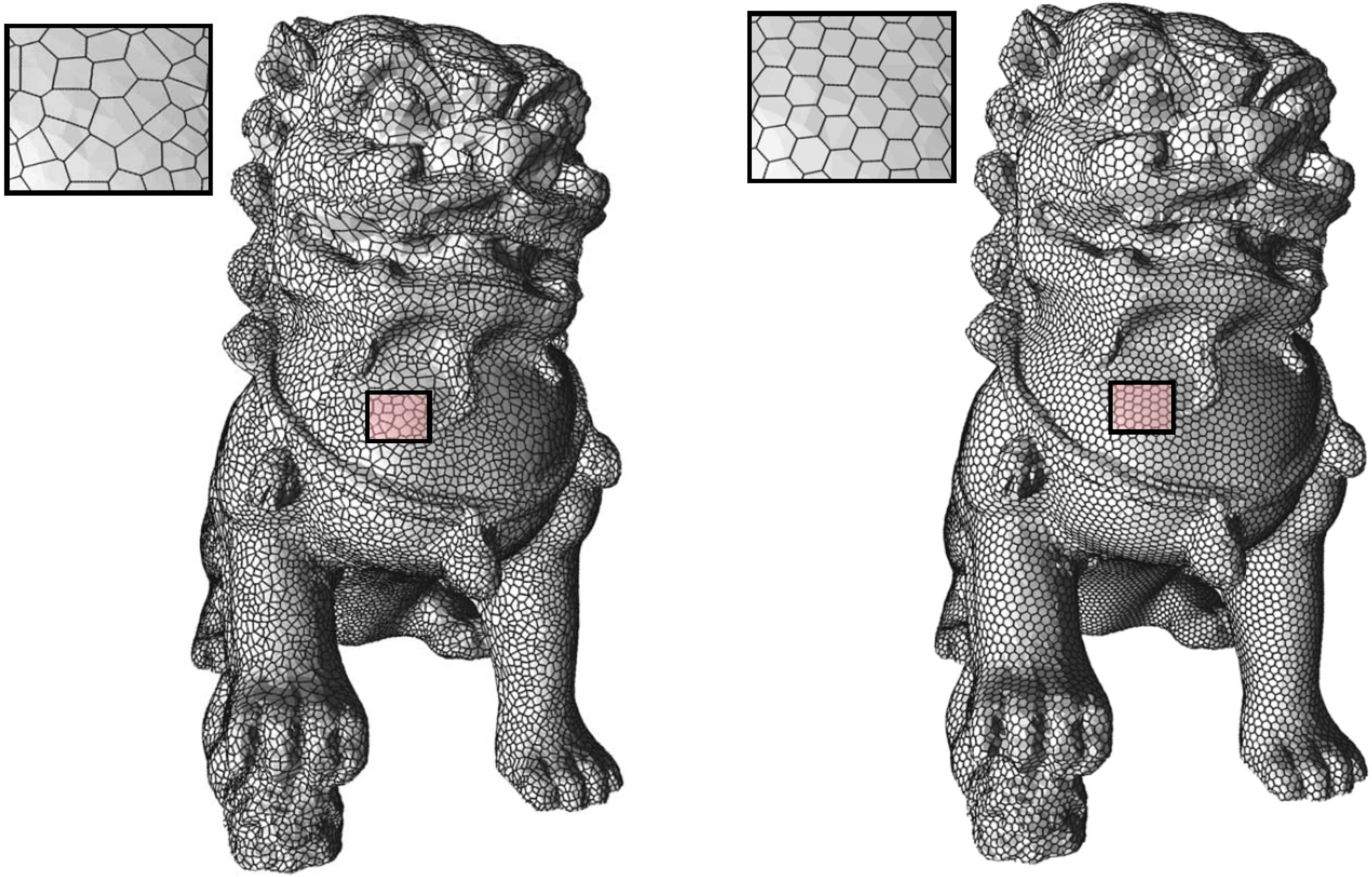


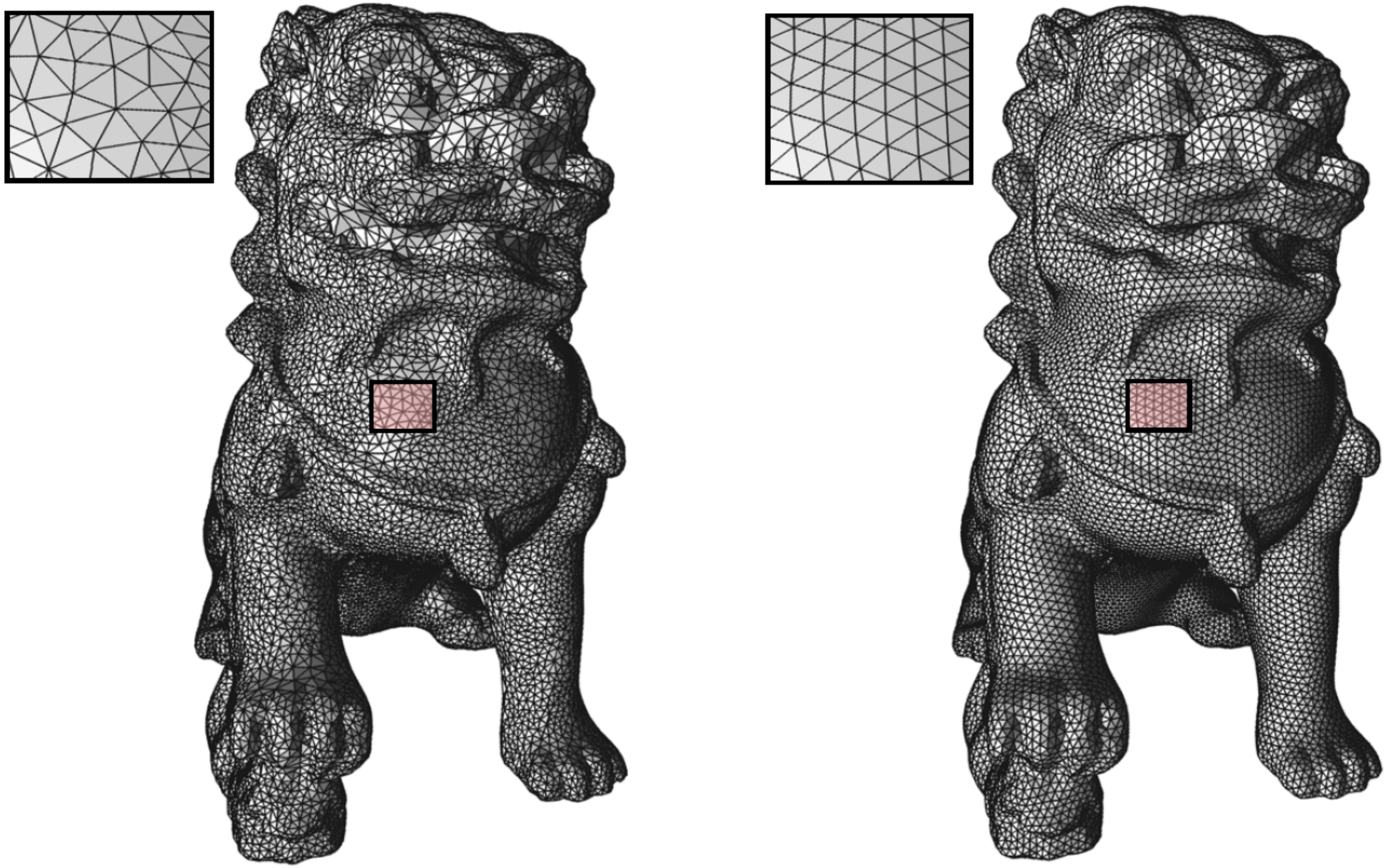
Figure: Illustration of C^2 smoothness of CVT energy in 2D

CVT on Surface



Left: initial Voronoi; Right: CVT

CVT for Remeshing



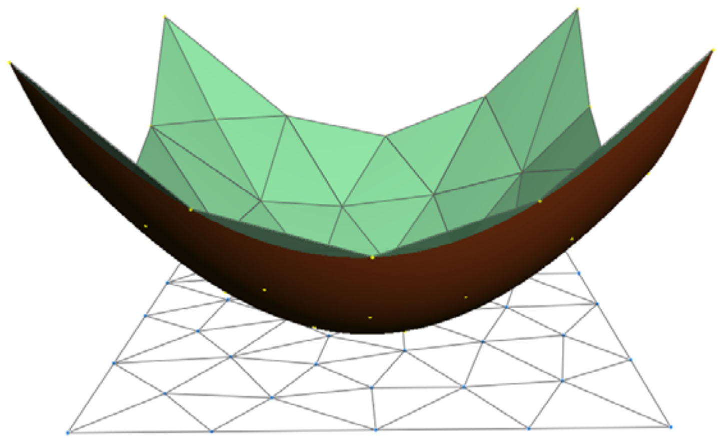
Left: initial mesh; Right: dual mesh of CVT

Optimal Delaunay Triangulation

- ODT energy function:

$$
\begin{aligned}
E(X) &= \|f - f_{I,\mathcal{T}}\|_{L^1(\Omega)} \\
&= \sum_{\tau \in \mathcal{T}} \int_{\tau} f_I(\mathbf{x}) d\mathbf{x} - \int_{\Omega} f(\mathbf{x}) d\mathbf{x}
\end{aligned}
$$

Volume between the lift-up of DT and the paraboloid



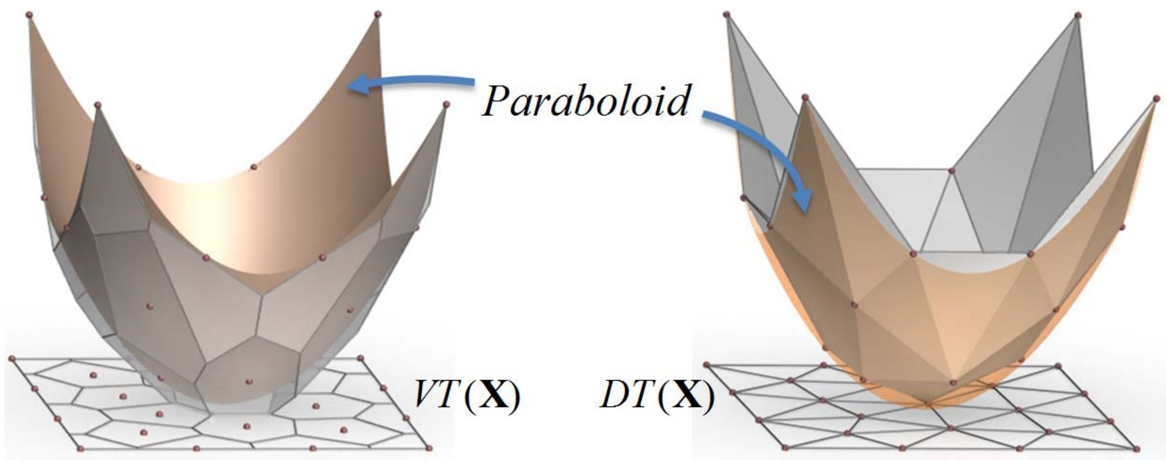
CVT & ODT Energy

- CVT energy

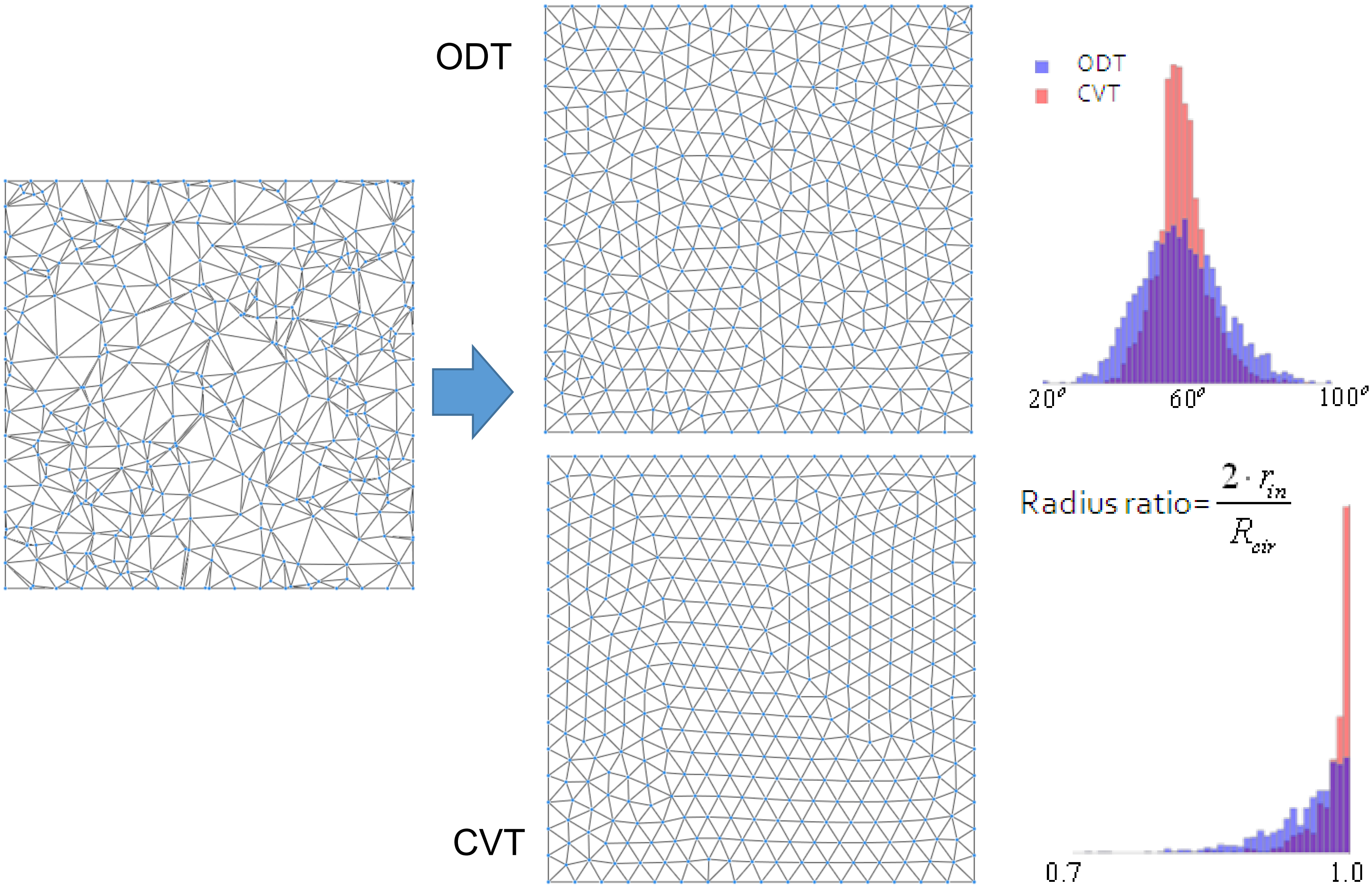
$$F(X) = \sum_{i=1}^{N} \int_{V_i} \|\mathbf{x} - \mathbf{x}_i\|^2 d\mathbf{x}$$

- ODT energy

$$E(X) \;=\; \sum_{\tau \in \mathcal{T}} \int_{\tau} f_I(\mathbf{x}) d\mathbf{x} - \int_{\Omega} f(\mathbf{x}) d\mathbf{x}$$



Compare ODT and CVT



Triangle Library

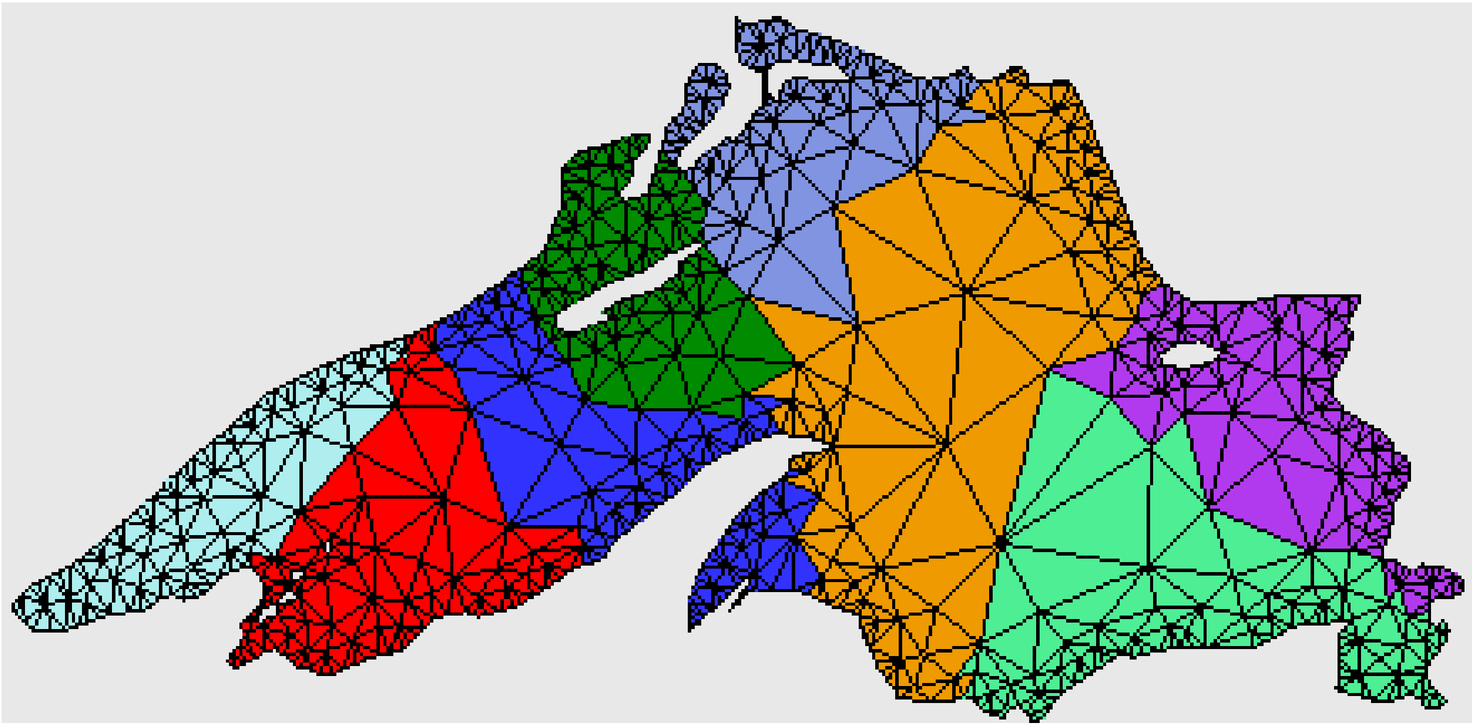
A Two-Dimensional Quality Mesh Generator and
Delaunay Triangulator

Use the library “Triangle”!

Triangle

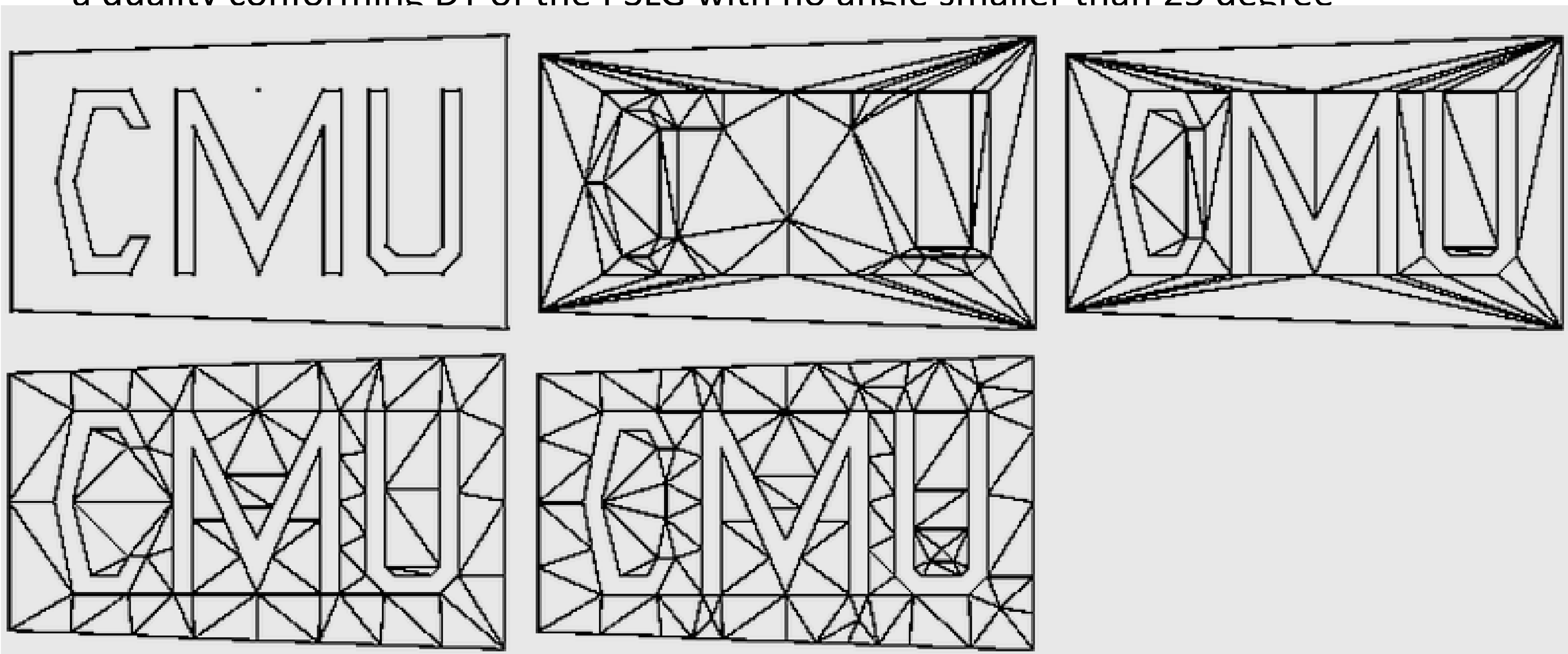
A Two-Dimensional Quality Mesh Generator and Delaunay Triangulator

- http://www.cs.cmu.edu/~quake/triangle.html



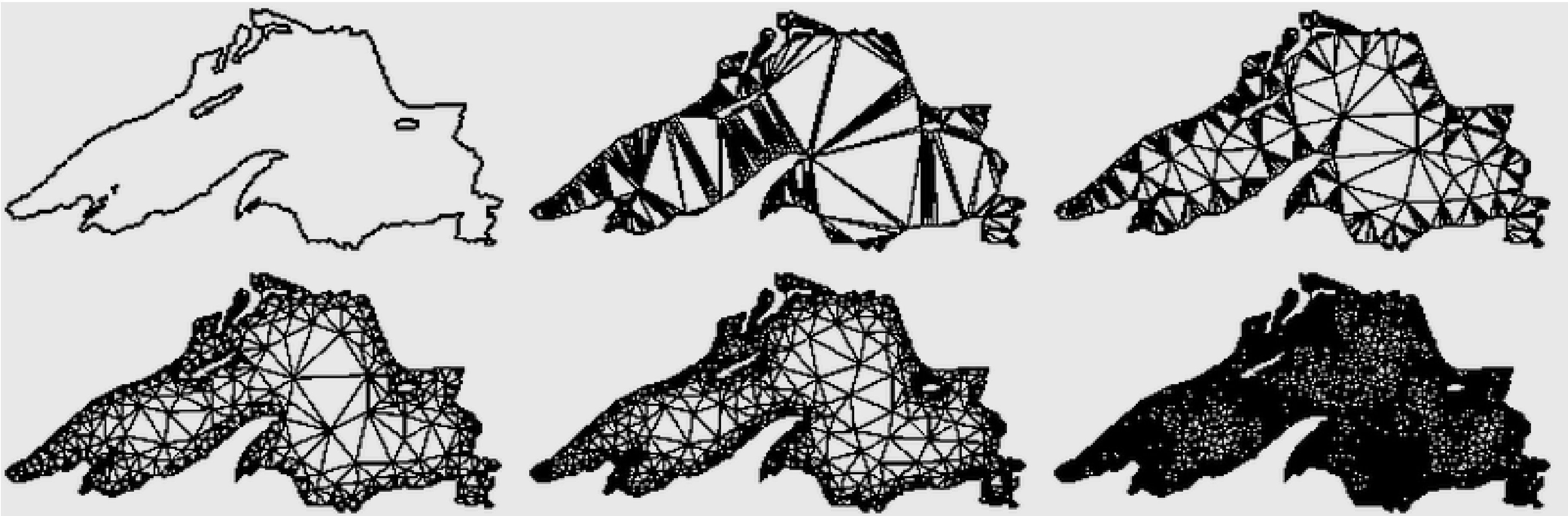
Examples of “Triangle”

- Planar Straight Line Graph (PSLG)
- a Delaunay triangulation of its vertices
- a constrained Delaunay triangulation of the PSLG
- a conforming Delaunay triangulation of the PSLG
- a quality conforming DT of the PSLG with no angle smaller than 25 degree



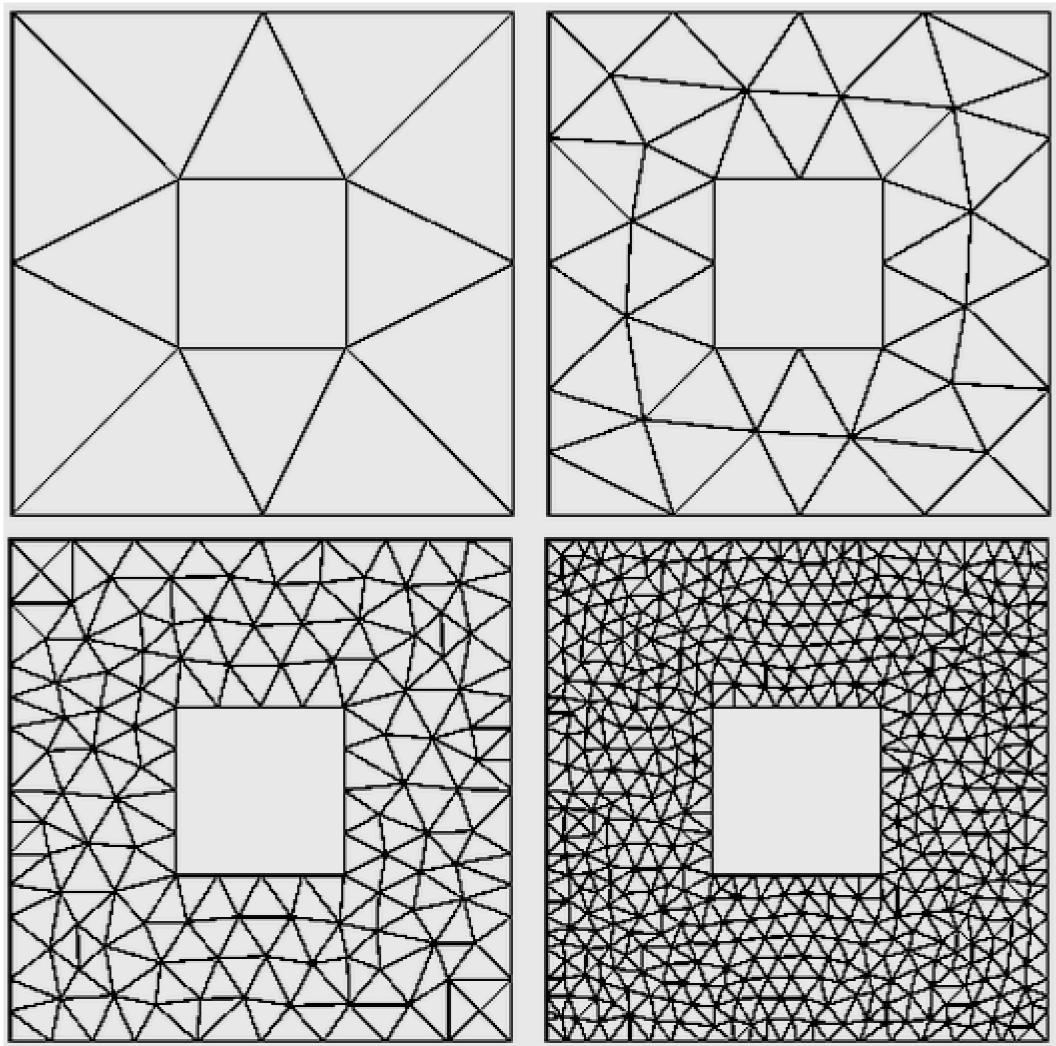
Examples of “Triangle”

- a PSLG of Lake Superior
- triangulations having minimum angles of 0, 5, 15, 25, and 33.8 degrees.



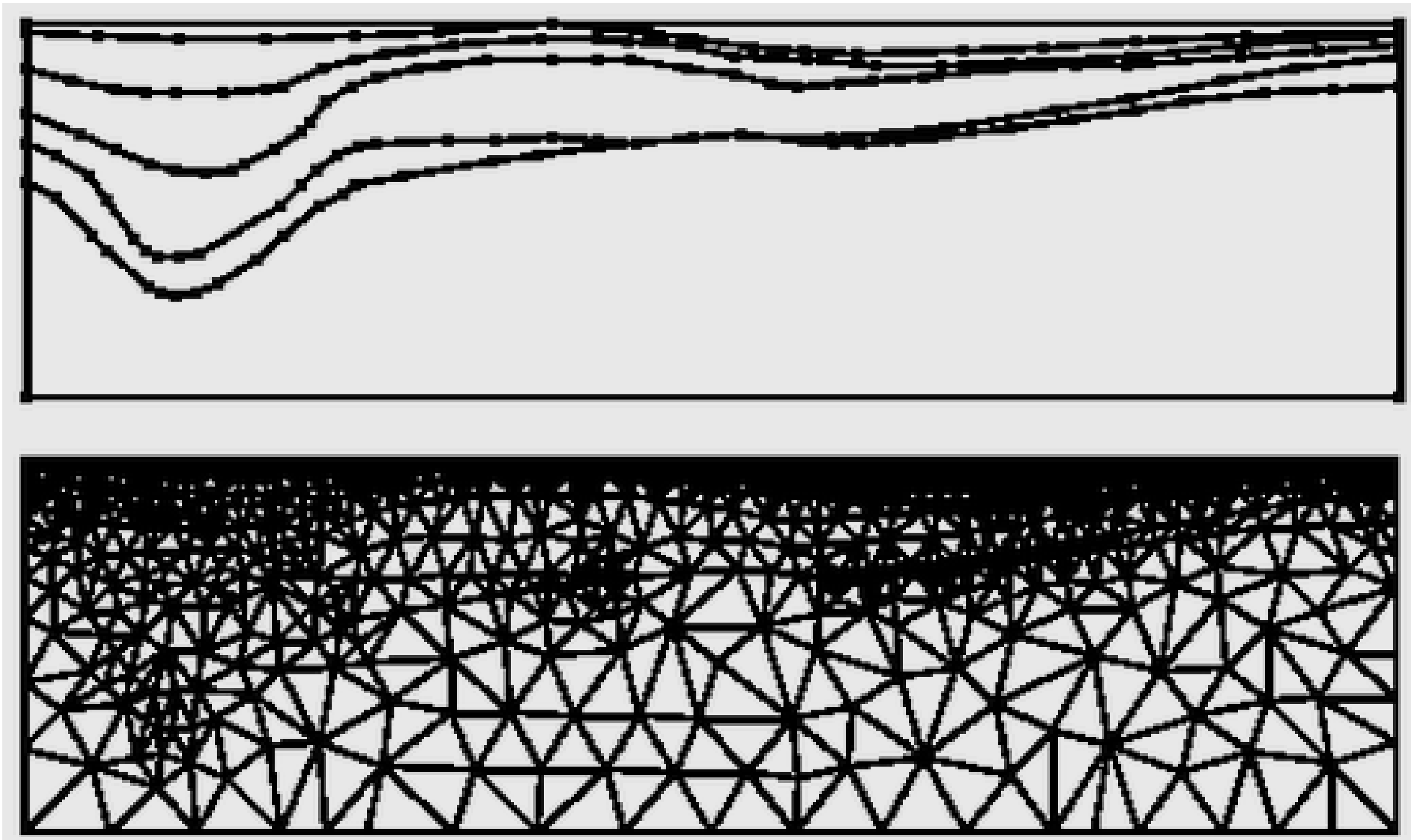
Examples of "Triangle"

- Maximum area constraints

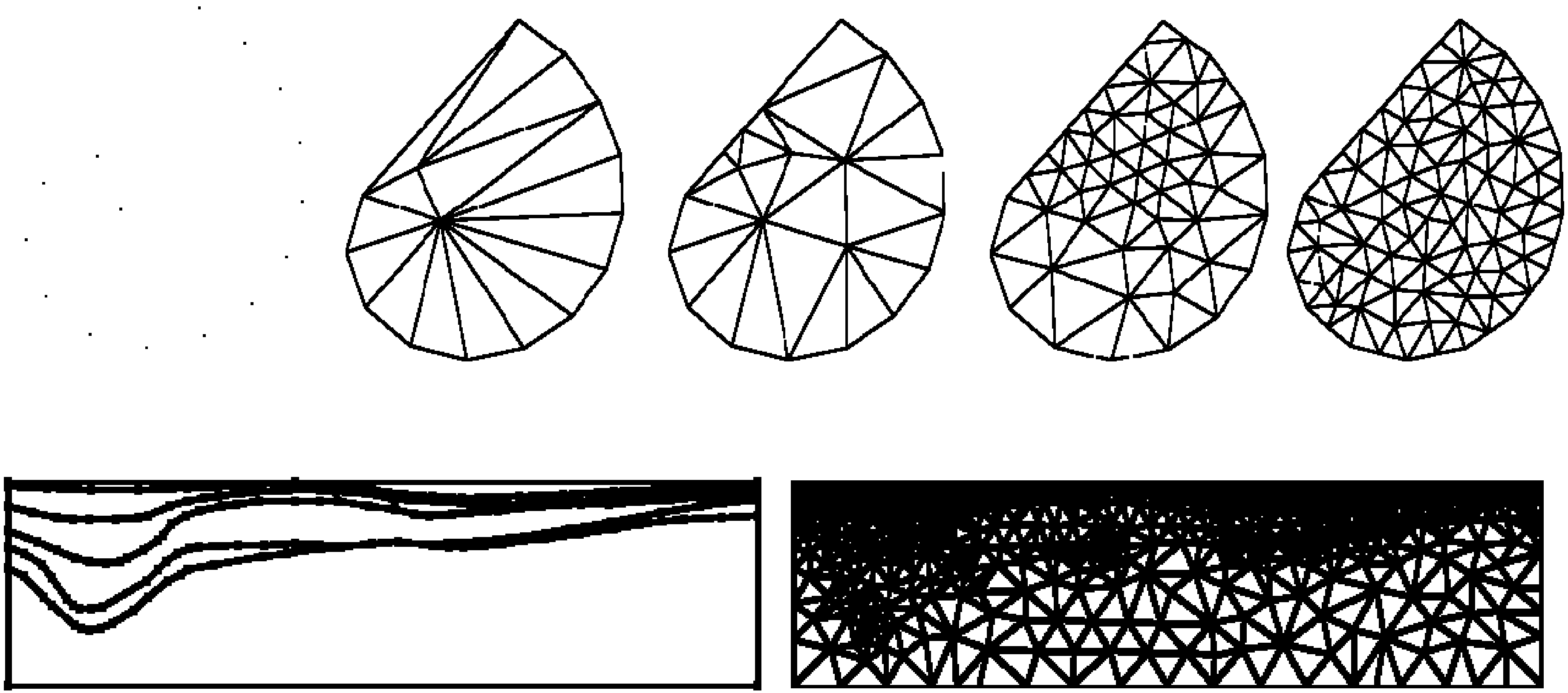


Examples of “Triangle”

- Different maximum triangle area constraints

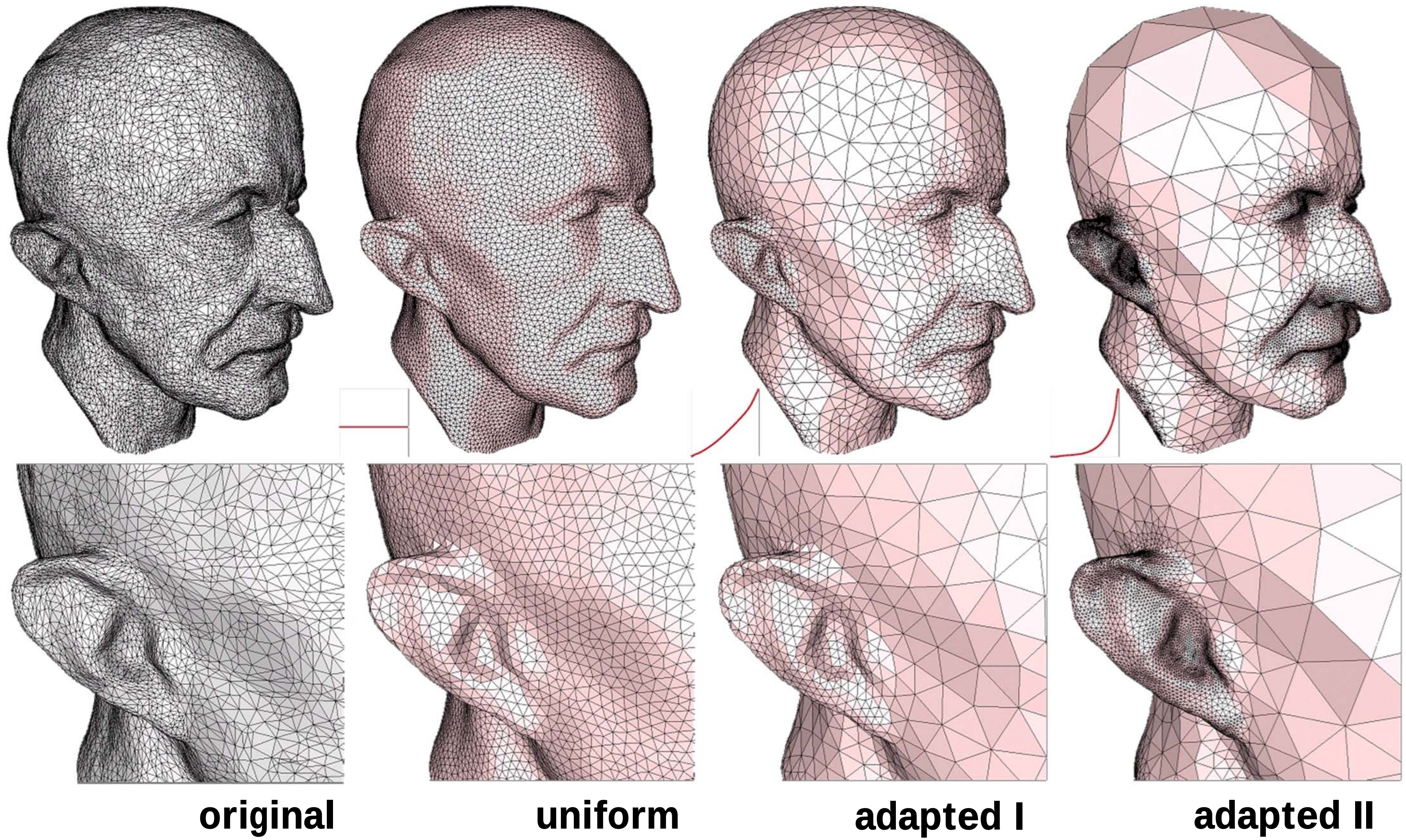


Examples of “Triangle”

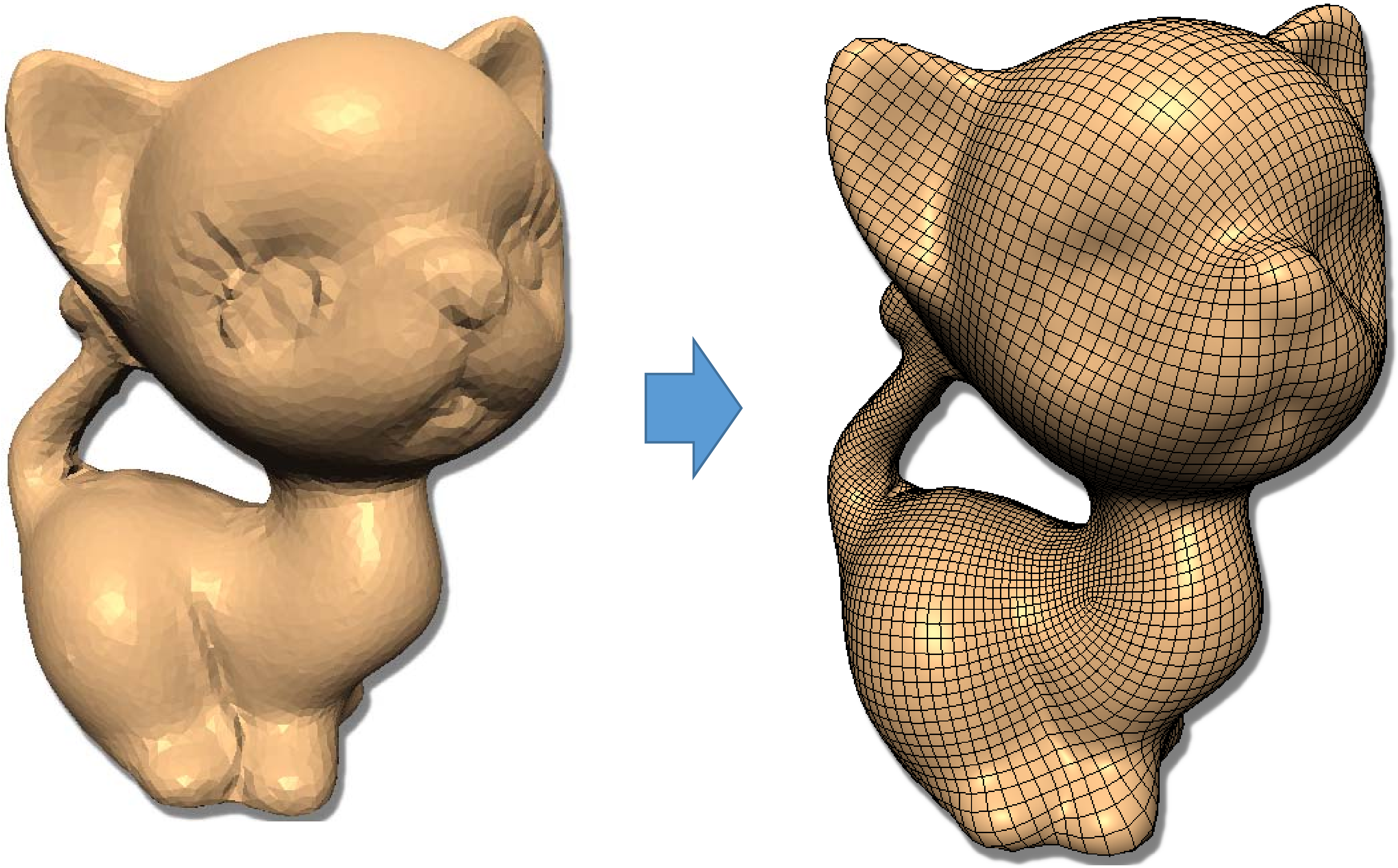


高维几何对象的
采样与剖分

二维流形曲面的采样与网格化

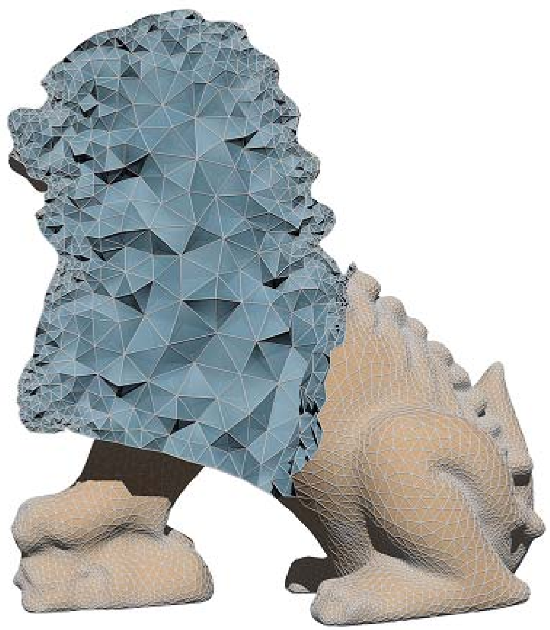


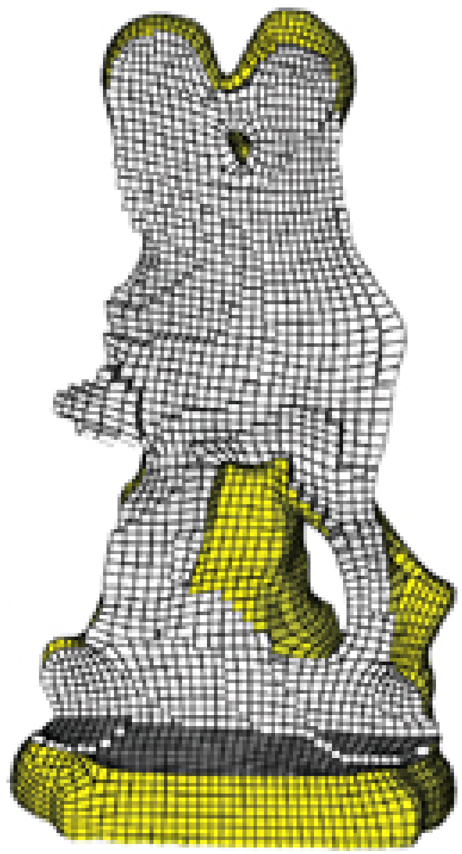
二维流形曲面的四边形网格化

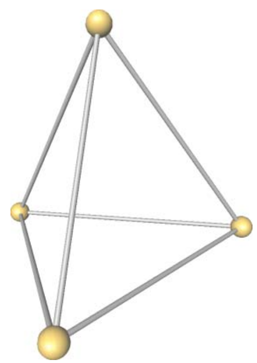


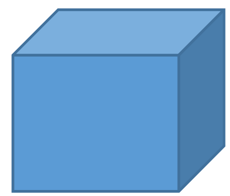
空间体的采样与剖分

- Interior of 3D shapes
  - FEM
  - Simulation
  - …
- Two typical types
  - Tetrahedral meshes
  - Hexahedral meshes

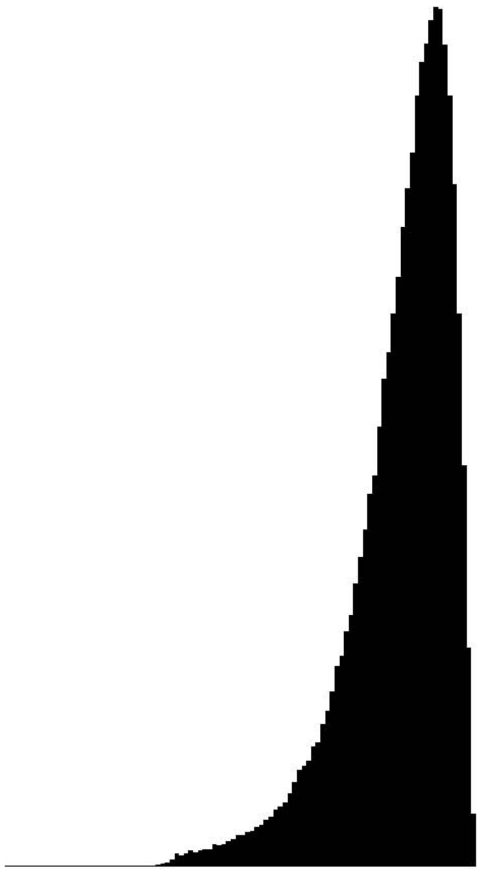


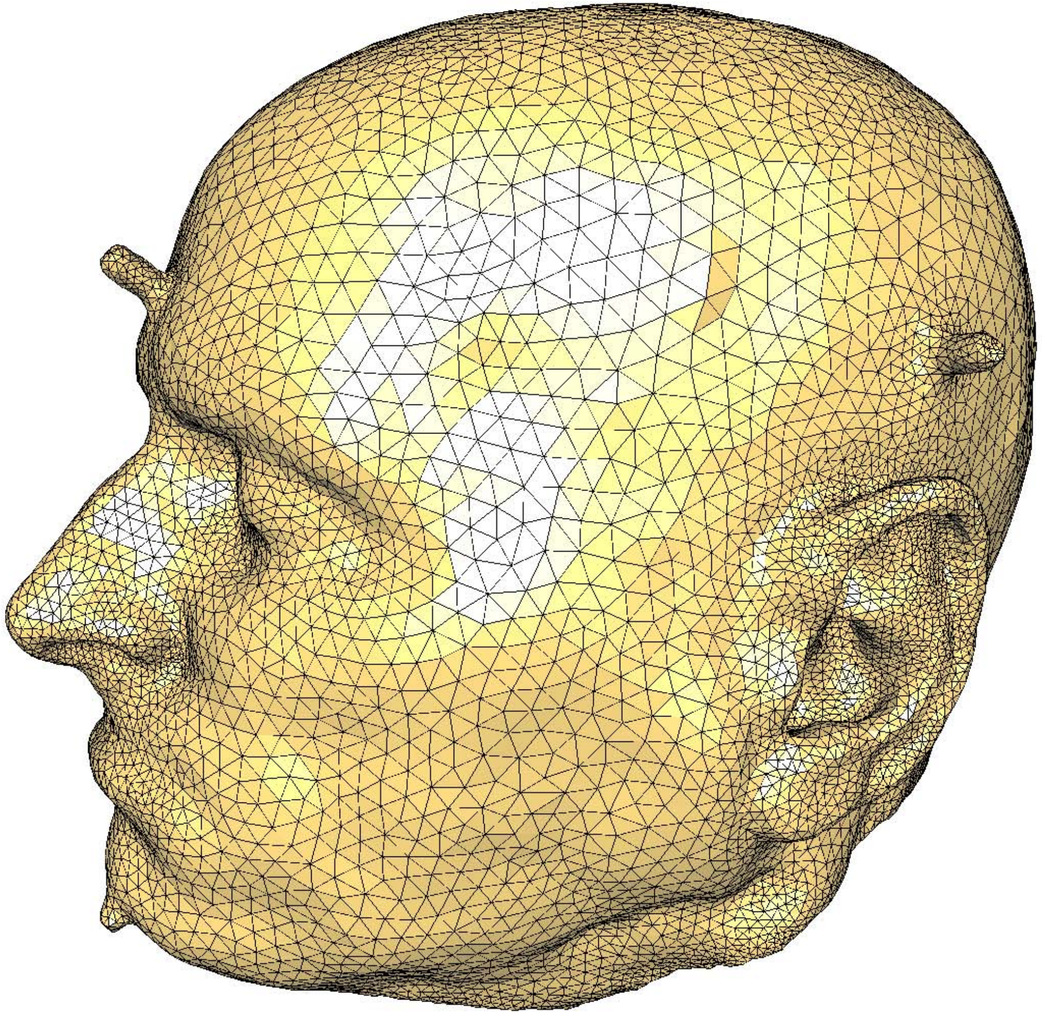






空间体的四面体网格





TETGEN (Tetrahedron Generation)

- **A Quality Tetrahedral Mesh Generator and 3D Delaunay Triangulator**

- https://people.sc.fsu.edu/~jburkardt/examples/tetgen/tetgen.html

- Author: Hang Si

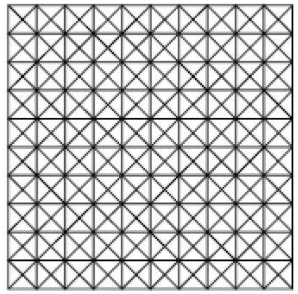
Meshing is still hard…

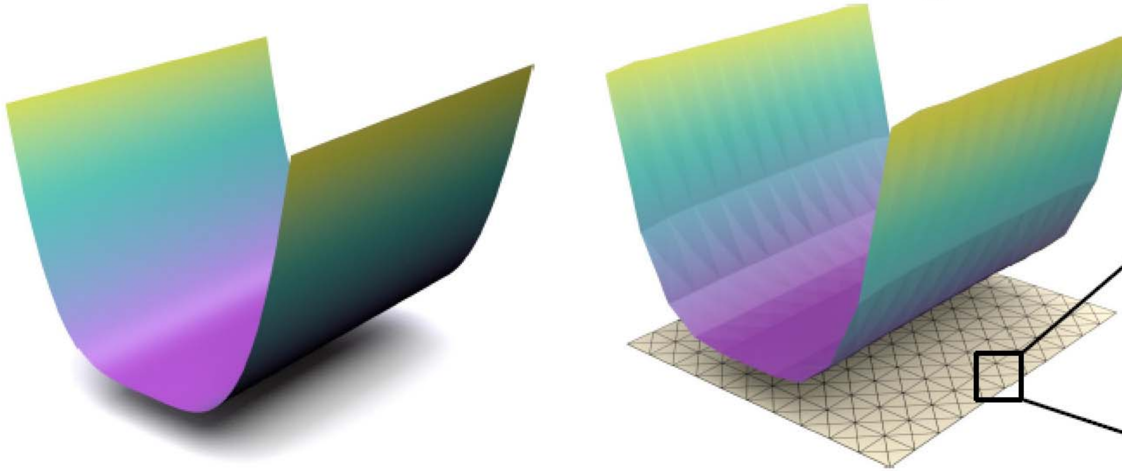
Solving PDE (FEM)

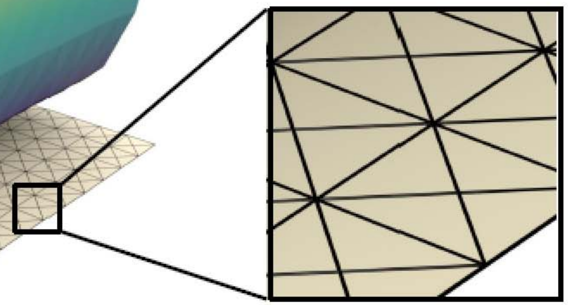
- Resolution, basis order, element quality…

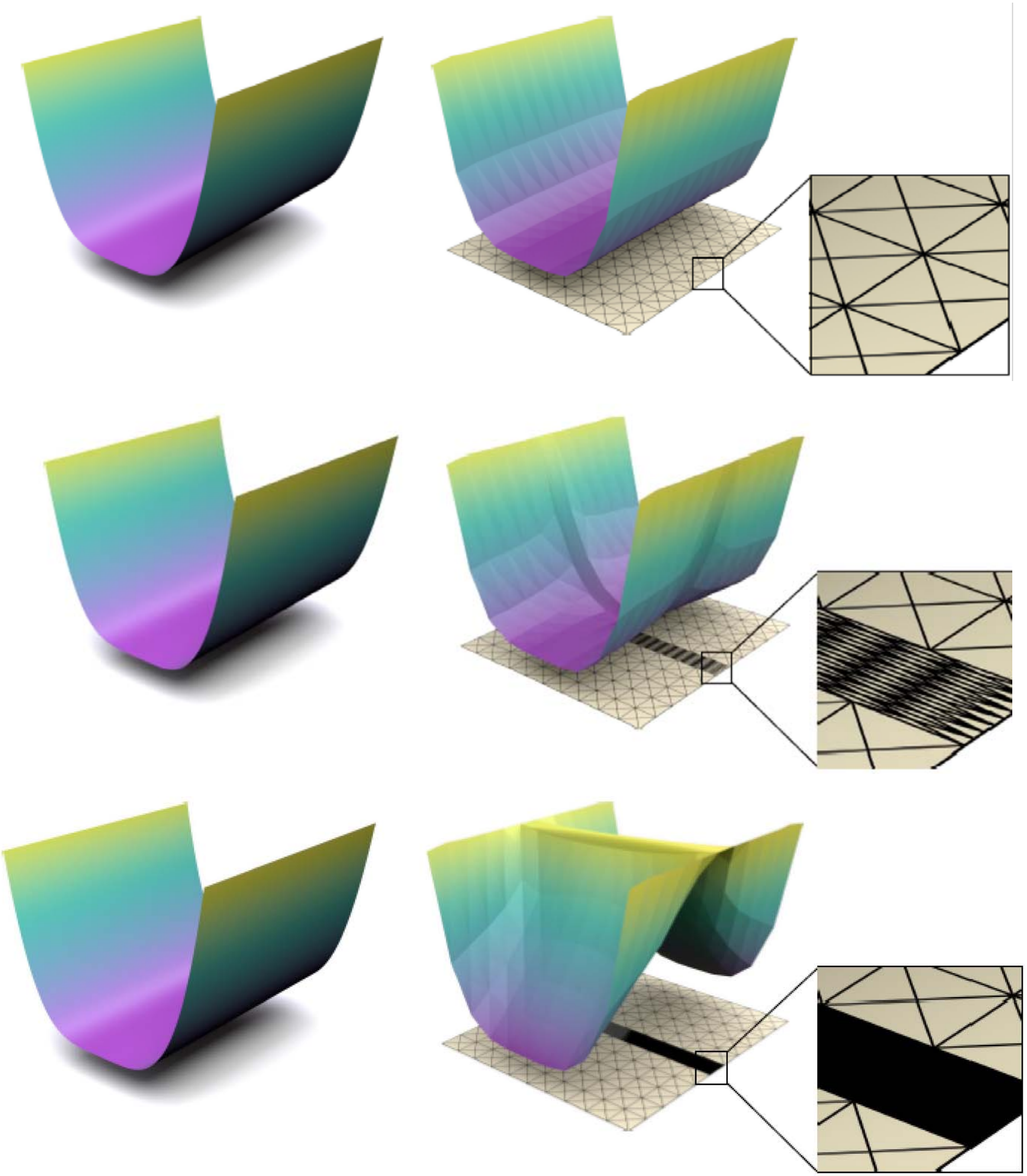
$$\Delta u = f$$

$$u = x^4 \quad \approx \quad U = \sum_{i=1}^{n} u_i \phi_i$$









作业8

- 任务：
  - 实现平面点集CVT的Lloyd算法
- 目的
  - 学习Voronoi算法、使用相关数学库（如Triangle、CGAL等）
- 步骤
  - 在给定的正方形区域内随机生成若干采样点
  - 生成这些点的Voronoi剖分
  - 计算每个剖分的重心，将采样点的位置更新到该重心
  - 迭代步骤2和3
- 思考（非必要、可选）
  - 如何在曲面上采样并生成CVT（如何将算法推广到流形曲面）？
- Deadline：2020 年12 月19日晚

中国科学技术大学
University of Science and Technology of China

谢　谢　！