中国科学技术大学
University of Science and Technology of China

GAMES102

GAMES 102在线课程

几何建模与处理基础

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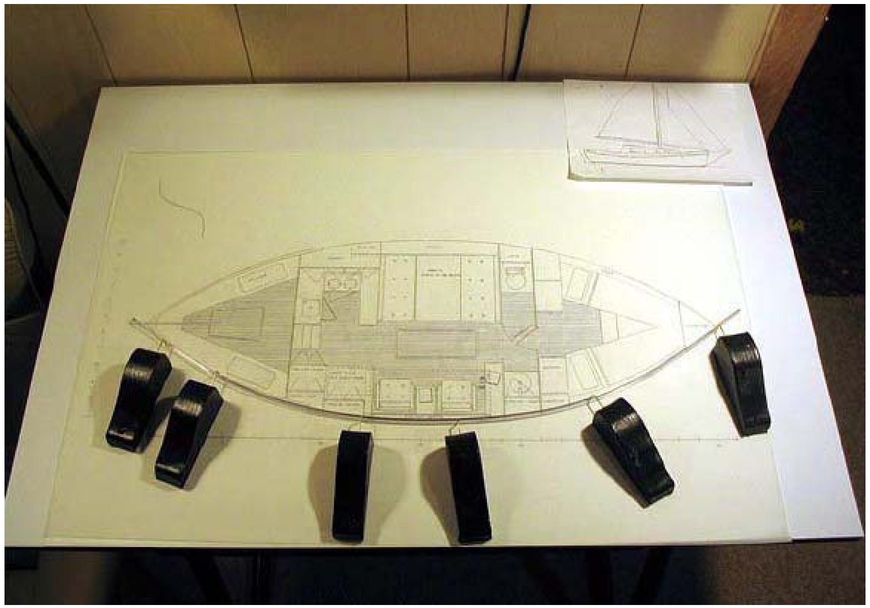
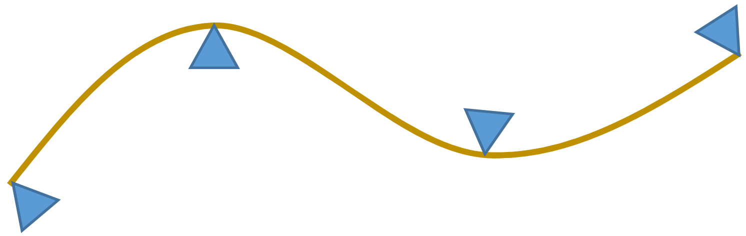
GAMES 102在线课程：几何建模与处理基础

Bezier曲线

回顾：函数/曲线拟合

- **逆向工程**中的建模问题：给定产品，用测量的方法得到产品外形上的一些采样点，然后通过拟合的方法得到产品外形的表达。





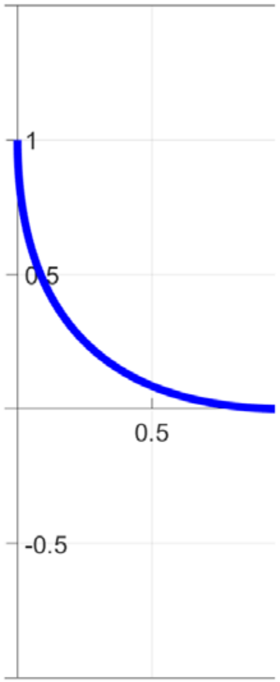
回顾：函数/曲线拟合

- 从**代数观点**来看：从一组基函数所张成的函数空间中，找一个“好”的函数来拟合给定的采样点。
- 比如幂基 $\{1, x, x^2, \ldots, x^n\}$
  - $(n = 2)$ 二次多项式：$f(t) = at^2 + bt + c$
- 参数曲线形式：$\boldsymbol{f}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

使用幂基来表达曲线

- 二次多项式曲线（抛物线）：

$$\boldsymbol{f}(t) = \boldsymbol{a}t^2 + \boldsymbol{b}t + \boldsymbol{c}$$



$$\boldsymbol{f}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} t^2 + \boxed{\begin{pmatrix} -2 \\ 0 \end{pmatrix}} t + \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

几何观点：系数顶点

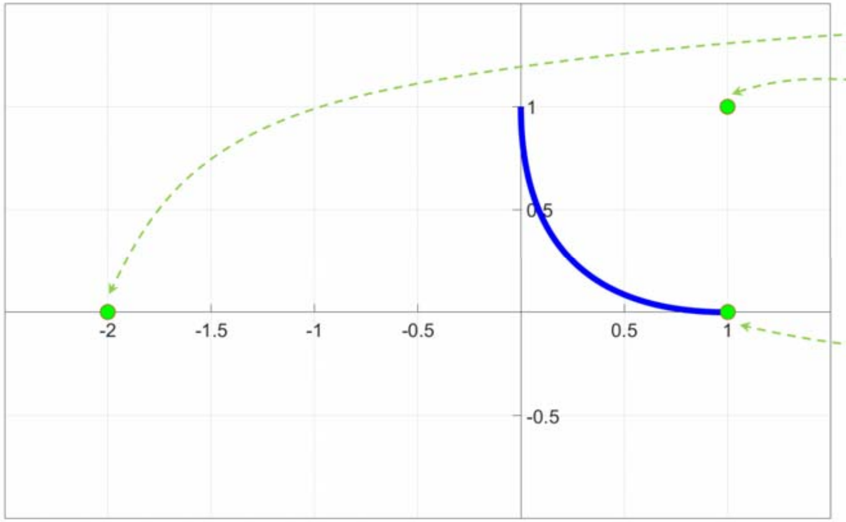
基函数为这些顶点的组合权系数

Courtesy of Renjie Chen

使用幂基来表达曲线

- 二次多项式曲线（抛物线）：

$$\boldsymbol{f}(t) = \boldsymbol{a}t^2 + \boldsymbol{b}t + \boldsymbol{c}$$



$$\mathbf{f}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

系数顶点与曲线本身无直观的联系：无几何意义！

不利于用户来交互修改曲线：设计建模

建模的两种形式

$$f(t) = at^2 + bt + c$$

- 1. 重建（Reconstruction）
  - 逆向工程：形状已有，要将其“猜”出来
  - 采样→拟合：需要函数空间足够丰富（表达能力够）
  - **代数观点**：$\{a, b, c\}$作为基函数的组合权系数
- 2. 设计（Design）
  - 自由设计：凭空产生，或从一个简单的形状编辑得到
  - 交互式编辑：几何直观性要好
  - **几何观点**：基函数$\{t^2, t, 1\}$作为控制点的组合权系数

使用Bernstein基函数表达

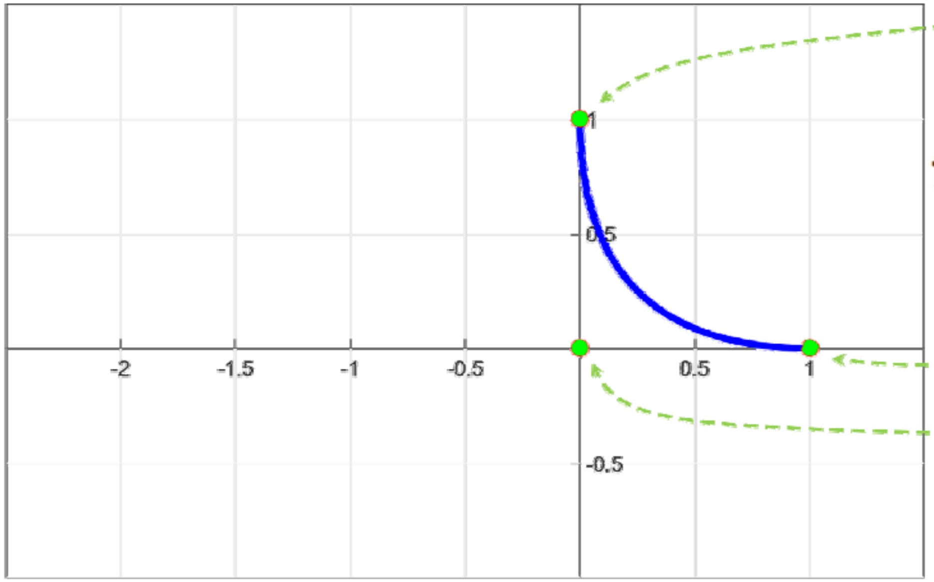
- 使用Bernstein基函数来改写

$$\boldsymbol{f}(t) = \binom{1}{1} t^2 + \binom{-2}{0} t + \binom{1}{0}$$

$$\boldsymbol{f}(t) = \binom{1}{0} \underline{(1-t)^2} + \binom{0}{0} \underline{2t(1-t)} + \binom{0}{1} \underline{t^2}$$

使用Bernstein基函数表达

- 系数顶点与曲线关联性强，具有很好的几何意义
- 对于交互式曲线设计更直观



$$f(t) = \begin{pmatrix}1\\0\end{pmatrix}(1-t)^2 + \begin{pmatrix}0\\0\end{pmatrix}2t(1-t) + \begin{pmatrix}0\\1\end{pmatrix}t^2$$

Bernstein基函数

- n次Bernstein基函数: $B = \{B_0^{(n)}, B_1^{(n)}, \dots, B_n^{(n)}\}$

$$B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i} = B_{i\text{–th basis function}}^{(\text{degree})}$$

where the binomial coefficients are given by:

$$\binom{n}{i} = \begin{cases} \dfrac{n!}{(n-i)!\,i!} & \text{for } 0 \le i \le n \\ 0 & \text{otherwise} \end{cases}$$

Examples: The first few

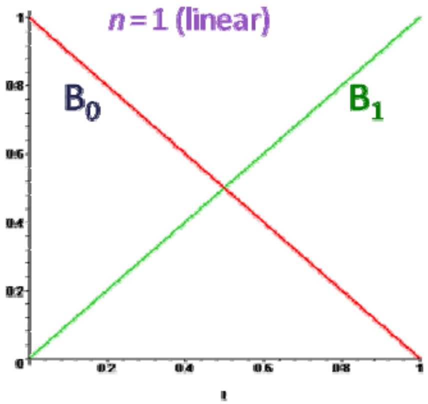
$$B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

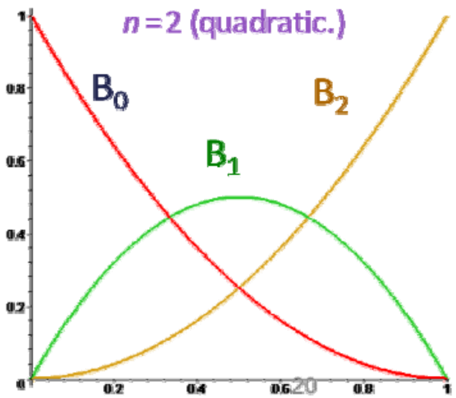
$$B_0^{(0)} \coloneqq 1$$

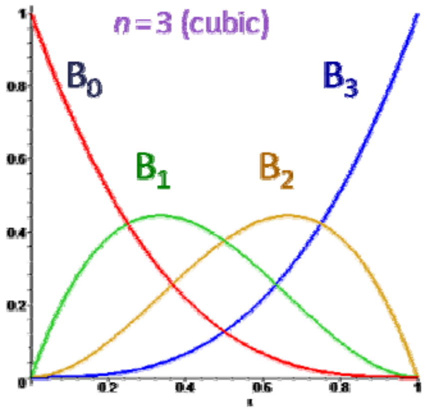
$$B_0^{(1)} \coloneqq 1 - t$$
$$B_1^{(1)} \coloneqq t$$

$$B_0^{(2)} \coloneqq (1-t)^2$$
$$B_1^{(2)} \coloneqq 2t(1-t)$$
$$B_2^{(2)} \coloneqq t^2$$

$$B_0^{(3)} \coloneqq (1-t)^3$$
$$B_1^{(3)} \coloneqq 3t(1-t)^2$$
$$B_2^{(3)} \coloneqq 3t^2(1-t)$$
$$B_3^{(3)} \coloneqq t^3$$

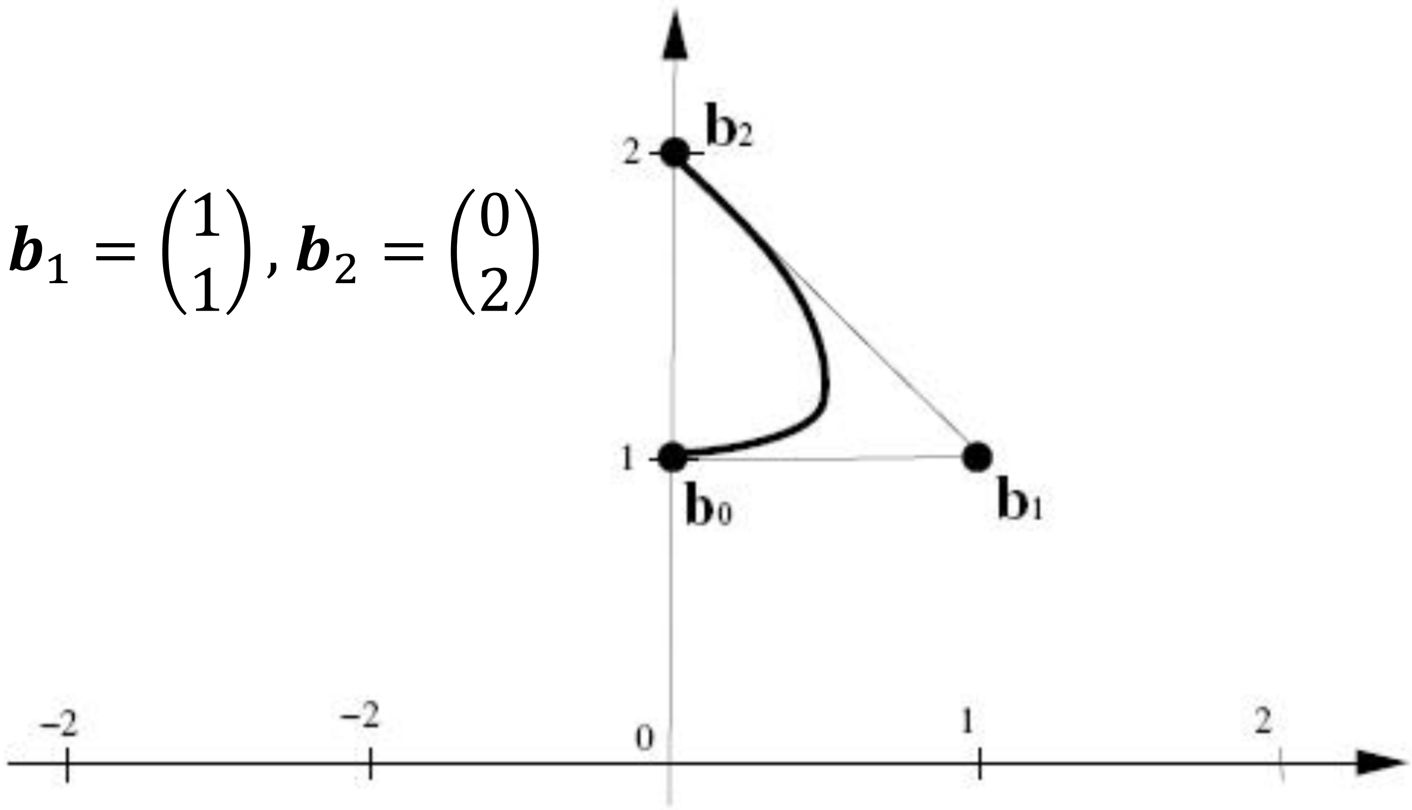






另一个例子

$$\boldsymbol{b}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \boldsymbol{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \boldsymbol{b}_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$



用Bernstein基函数所表达的曲线具有非常好的几何意义！

Notations

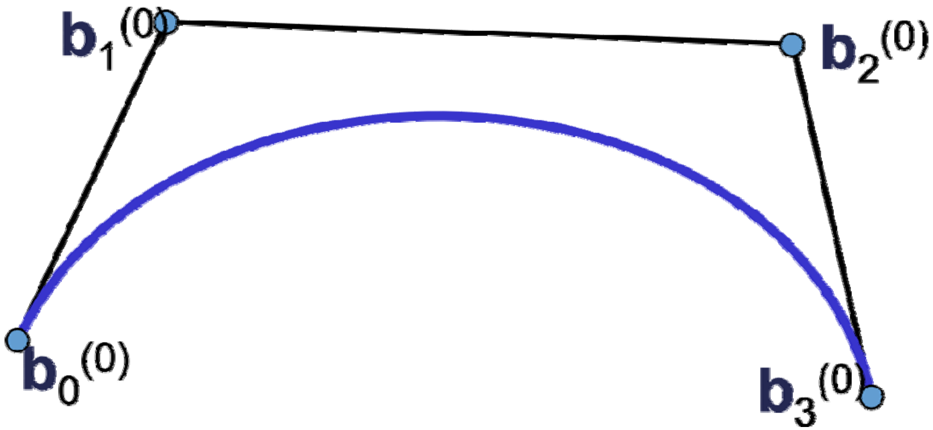
Curve **basis function** **control points**

$$\boldsymbol{f}(t) = \sum_{i=1}^{n} b_i(t)\boldsymbol{p}_i$$

Bezier曲线

- n次Bezier曲线： $n+1$个控制顶点

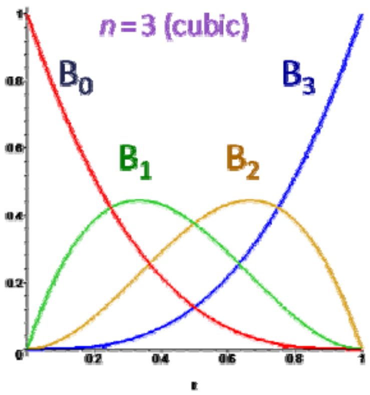
$$x(t)=\sum_{i=0}^{n} B_i^n(t)\cdot b_i$$



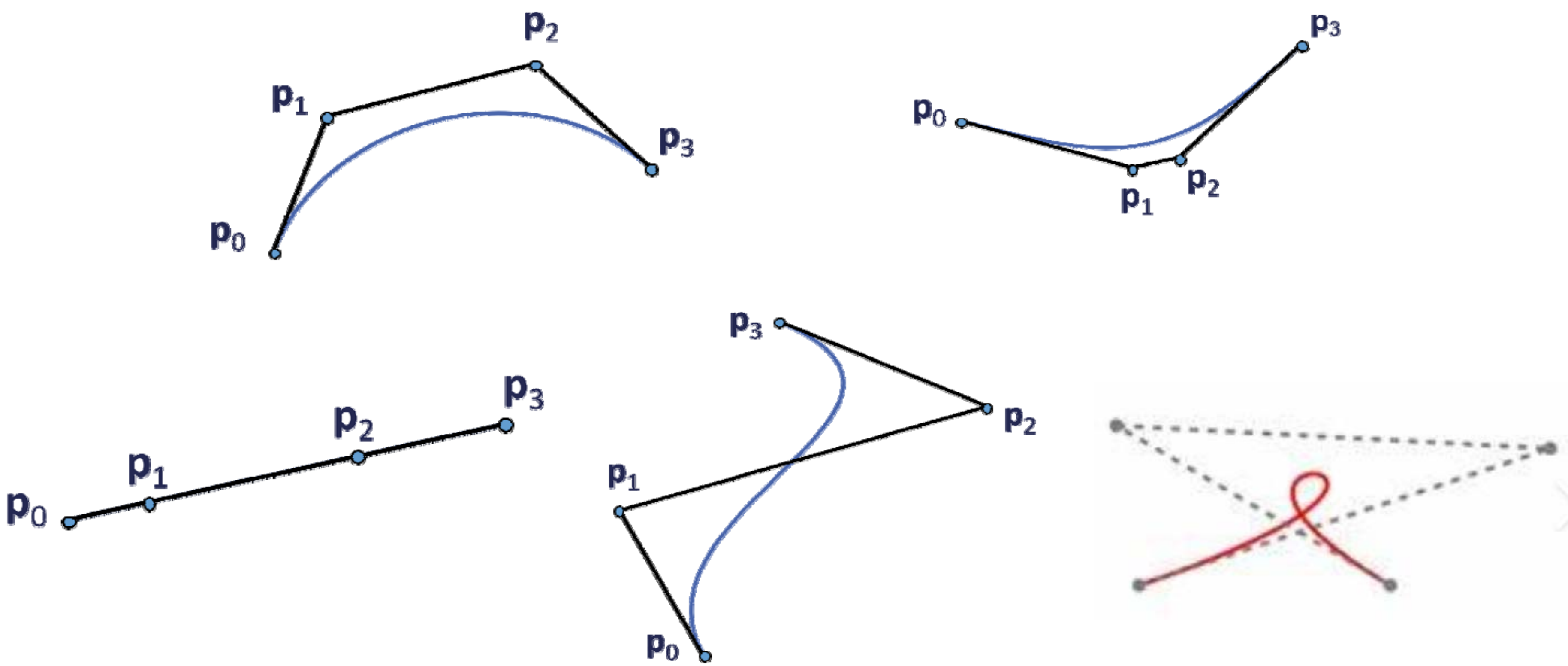
控制顶点
控制多边形

Bezier曲线的性质来源于Bernstein基函数的性质
（曲线是控制顶点的线性组合构成的，基函数提供了组合系数）

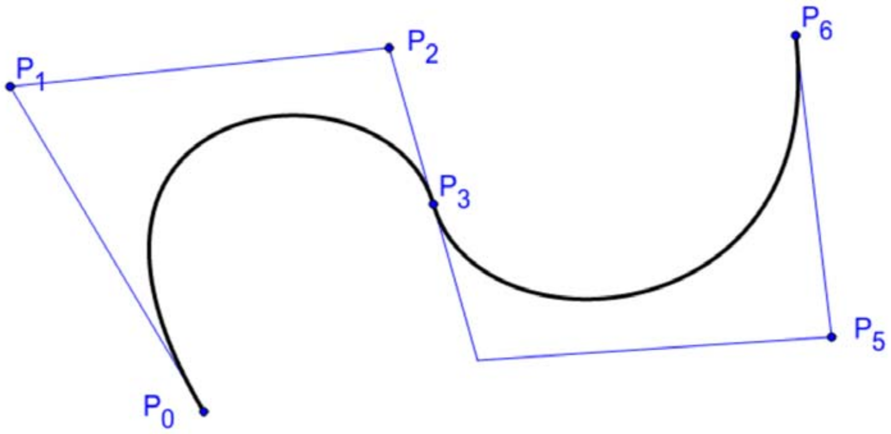
例子：3次Bezier曲线

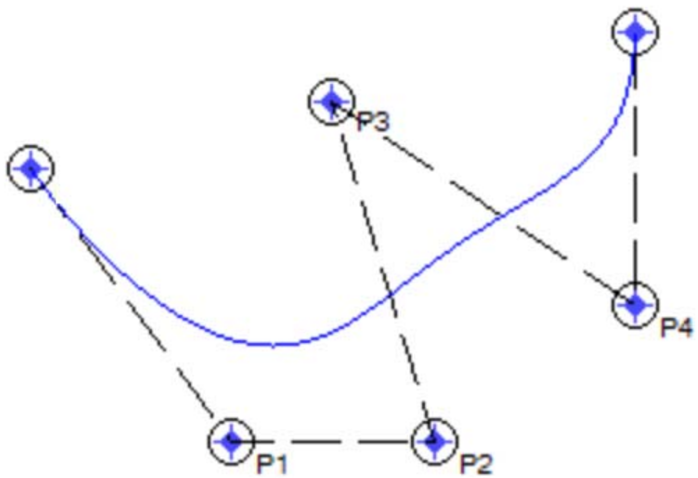


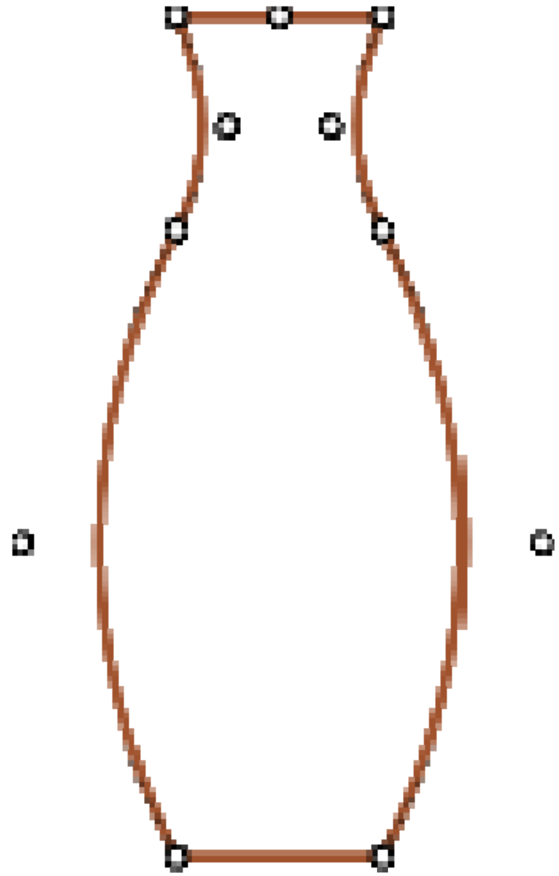
- $\boldsymbol{f}(t) = \sum_{i=1}^{3} B_i^3 \boldsymbol{p}_i \,, t \in [0,1]$



例子：更复杂的Bezier曲线



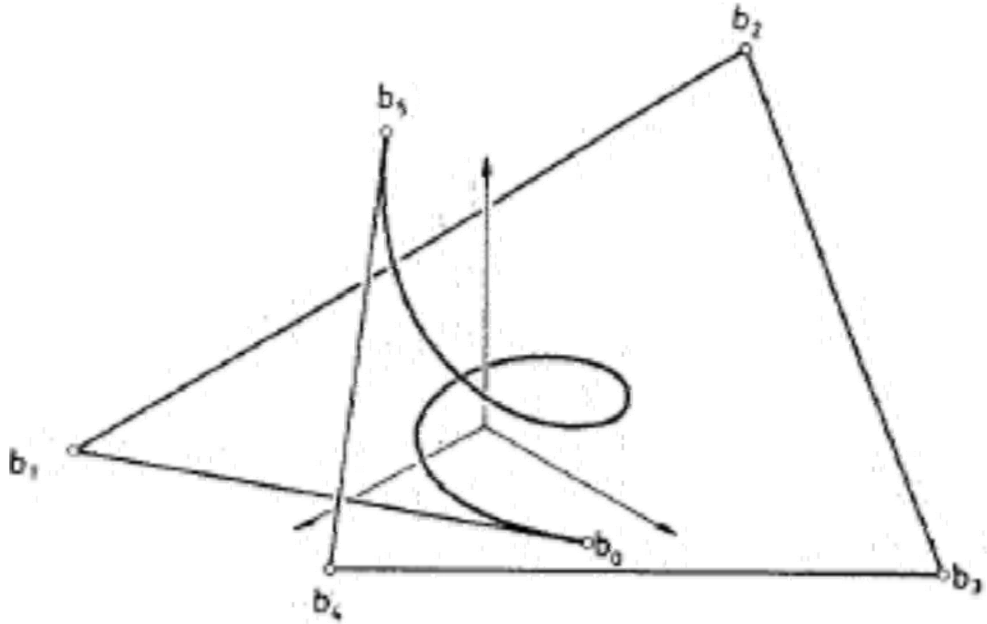




分段Bezier曲线

3D空间的Bezier曲线（单参数）

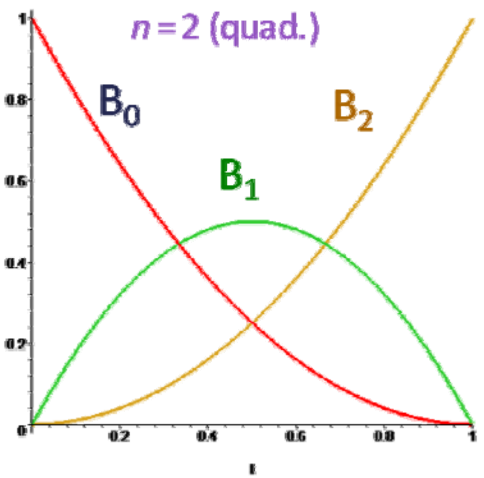
- $\boldsymbol{f}(t) = \sum_{i=1}^{n} B_i^n \boldsymbol{p}_i \, , t \in [0,1]$

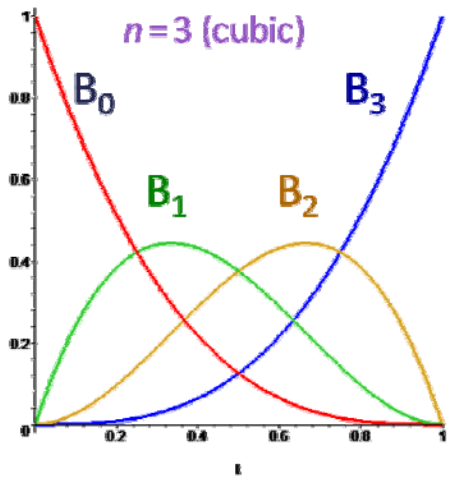


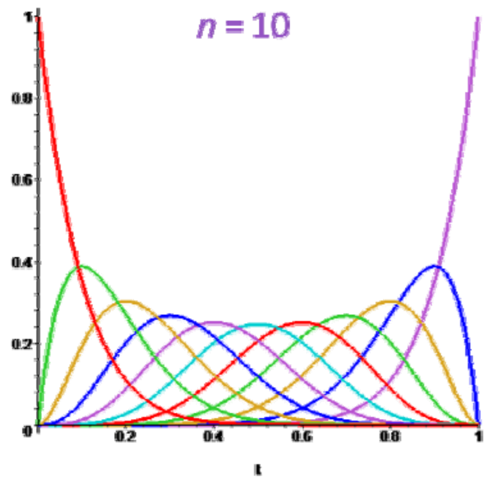
Bernstein基函数及Bezier曲线的性质

Bernstein基函数

- Bernstein基函数: $B = \left\{B_0^{(n)}, B_1^{(n)}, \dots, B_n^{(n)}\right\}$
  - n次（$n+1$阶）Bernstein基函数: $B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i} = B_{i\text{-th basis function}}^{(\text{degree})}$
    - 对称性: $B_i^n(t) = B_{n-i}^n(1-t)$
    - $B_i^{(n)}(t)$在$t = \frac{i}{n}$达到最大值

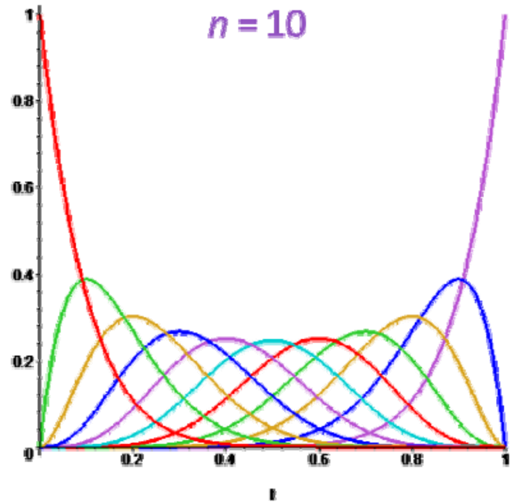


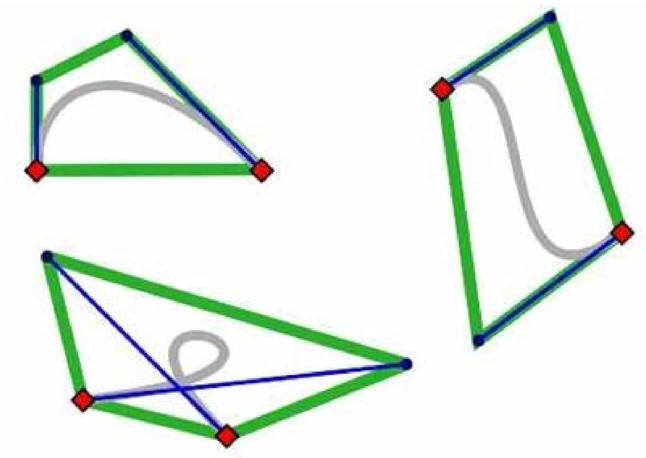


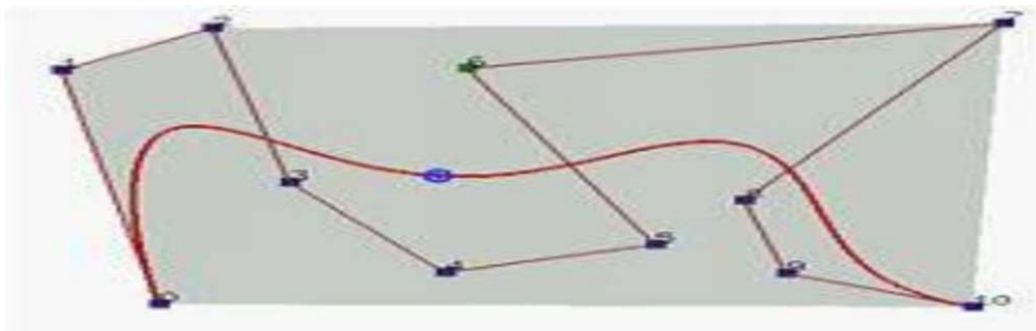


性质1. 正权性

- 正性（非负性）+ 权性
  - $B_i^{(n)}(t) \geq 0$， $\forall t \in [0,1]$
  - $\sum_{i=1}^{n} B_i^{(n)}(t) = 1, \forall t \in [0,1]$
- Bezier曲线的凸包性







性质2. 基性

- $B=\left\{B_0^{(n)},B_1^{(n)},\dots,B_n^{(n)}\right\}$是次数不高于$n$的多项式集合（空间）的一组基

- 与幂基可以相互线性表达：

$$\begin{bmatrix} B_{0,n}(t) & B_{1,n}(t) & \cdots & B_{n,n}(t) \end{bmatrix} = \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix} \begin{bmatrix} b_{0,0} & 0 & 0 & \cdots & 0 \\ b_{1,0} & b_{1,1} & 0 & \cdots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n,0} & b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix}$$

性质3. 递推公式

- 基函数的递推公式

$$B_i^n(t) = (1-t)B_i^{(n-1)}(t) + tB_{i-1}^{(n-1)}(1-t)$$

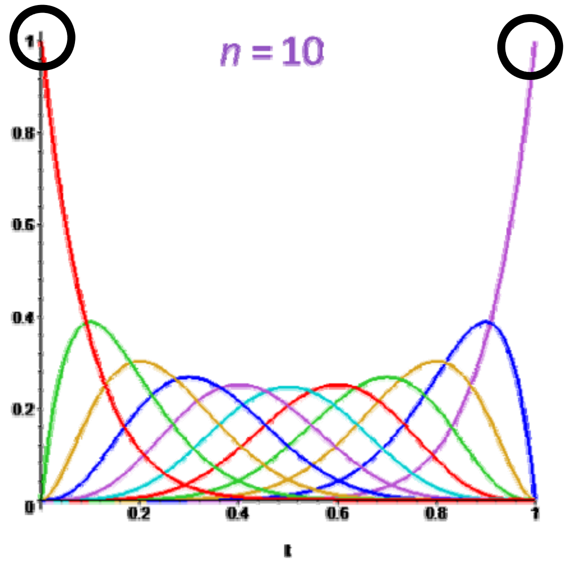
$$\text{with } B_0^0(t) = 1, B_i^n(t) = 0 \text{ for } i \notin \{0 \ldots n\}$$

  - 由 $\binom{n-1}{i} + \binom{n-1}{i-1} = \binom{n}{i}$ 可推导得到

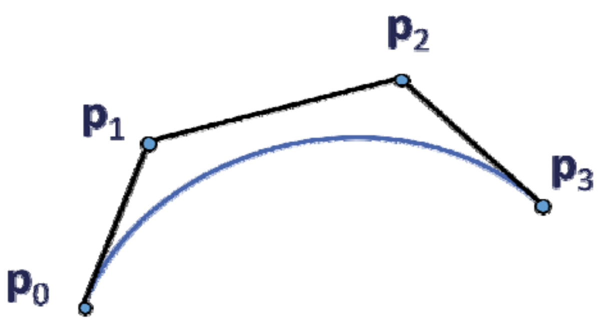
- 高阶的基函数由2个低阶的基函数“升阶”得到
  - 利于保持一些良好的性质

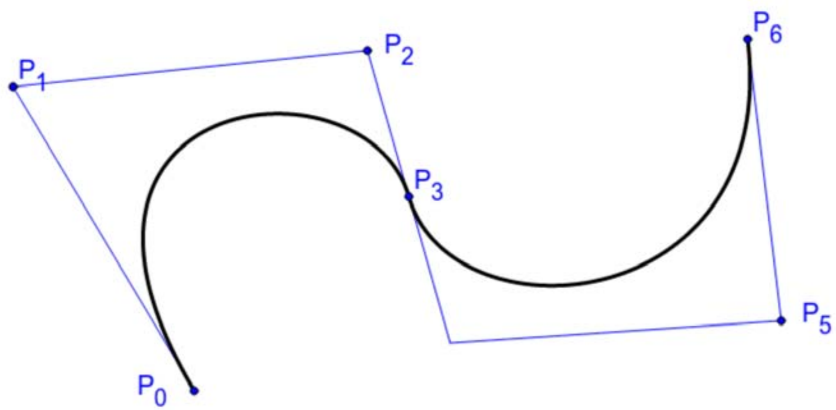
性质4. 端点插值性

- $B_0^n(0) = 1,\ \ B_1^n(0) = \cdots = B_n^n(0) = 0$
- $B_0^n(1) = \cdots = B_{n-1}^n(1) = 0,\ B_n^n(0) = 1$



- Bezier曲线经过首末两个控制顶点$\boldsymbol{p}_0, \boldsymbol{p}_n$





性质5. 导数

- $\frac{d}{dt}B_i^{(n)}(t) = n\left[B_{i-1}^{(n-1)}(t) - B_i^{(n-1)}(t)\right]$
- $\frac{d^2}{dt^2}B_i^{(n)}(t) = n(n-1)\left[B_{i-2}^{(n-2)}(t) - 2B_{i-1}^{(n-2)}(t) + B_i^{(n-2)}(t)\right]$

- Bezier曲线的导数（切线）
  - Given: $\boldsymbol{p}_0, \ldots, \boldsymbol{p}_n, \boldsymbol{f}(t) = \sum_{i=0}^{n} B_i^n(t)\,\boldsymbol{p}_i$
    - $\boldsymbol{f}'(t) = n\sum_{i=0}^{n-1} B_i^{n-1}(t)\,(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i)$
    - $\boldsymbol{f}^{[r]}(t) = \frac{n!}{(n-r)!} \cdot \sum_{i=0}^{n-r} B_i^{n-r}(t) \cdot \Delta^r \boldsymbol{p}_i$

Bezier曲线的端点性质

- 端点插值:

$$\boldsymbol{f}(0) = \boldsymbol{p}_0$$
$$\boldsymbol{f}(1) = \boldsymbol{p}_n$$

- 端点的切线方向与边相同:

$$\boldsymbol{f}'(0) = n[\boldsymbol{p}_1 - \boldsymbol{p}_0]$$
$$\boldsymbol{f}'(1) = n[\boldsymbol{p}_{n-1} - \boldsymbol{p}_n]$$

- 端点的2阶(k)切线与3点(k+1)相关:

$$\boldsymbol{f}''(0) = n(n-1)[\boldsymbol{p}_2 - \boldsymbol{2}\boldsymbol{p}_1 + \boldsymbol{p}_0]$$
$$\boldsymbol{f}''(1) = n(n-1)[\boldsymbol{p}_n - 2\boldsymbol{p}_{n-1} + \boldsymbol{p}_{n-2}]$$

结合几何意义来理解

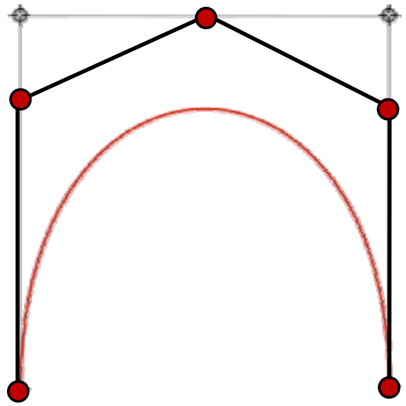
性质6. 升阶

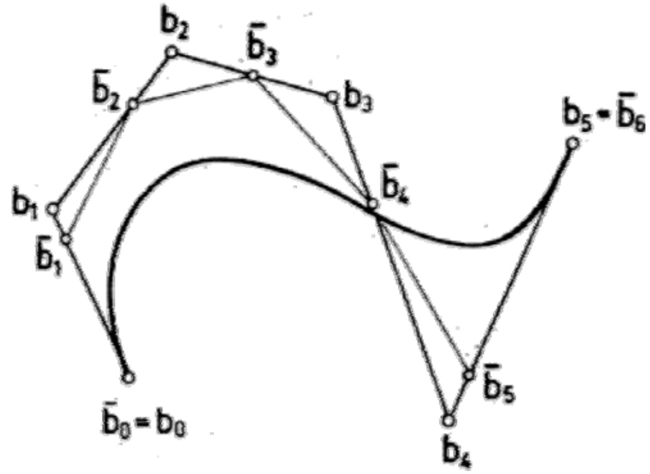
$$(1-t)B_i^n(t) = (1-\frac{i}{n+1})B_i^{n+1}(t)$$

$$tB_i^n(t) = \frac{i+1}{n+1}B_i^{n+1}(t)$$

- Bezier曲线的升阶

$$\boldsymbol{f}(t) = \sum_{i=0}^{n+1} B_i^{n+1}(t)\left[\frac{n+1-i}{n+1}\boldsymbol{P}_i + \frac{i}{n+1}\boldsymbol{P}_{i-1}\right]$$





Bezier曲线的
de Casteljau算法

（Bezier曲线的作图算法与细分）

De Casteljau algorithm

- Algorithm description
  - Input: points $\boldsymbol{b}_0, \boldsymbol{b}_1, \dots \boldsymbol{b}_n \in \mathbb{R}^3$
  - Output: curve $\boldsymbol{x}(t), t \in [0,1]$

  - Geometric construction of the points $\boldsymbol{x}(t)$ for given t:

$$\boldsymbol{b}_i^0(t) = \boldsymbol{b}_i, \qquad i = 0, \dots, n$$

$$\boldsymbol{b}_i^r(t) = (1-t)\boldsymbol{b}_i^{r-1}(t) + t\, \boldsymbol{b}_{i+1}^{r-1}(t)$$

$$r = 1, \dots, n \qquad i = 0, \dots, n-r$$

  - Then $\boldsymbol{b}_0^n(t)$ is the searched curve point $\boldsymbol{x}(t)$ at the parameter value t

De Casteljau algorithm

- Repeated convex combination of control points

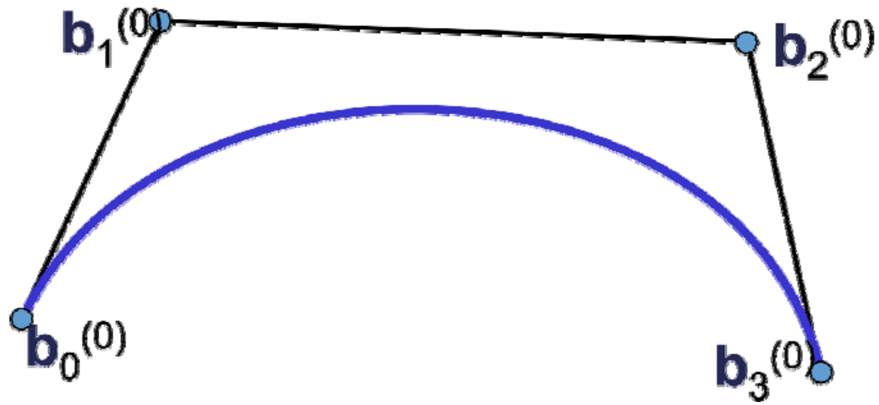
$$\boldsymbol{b}_i^{(r)} = (1 - t)\boldsymbol{b}_i^{(r-1)} + t\boldsymbol{b}_{i+1}^{(r-1)}$$

$\boldsymbol{b}_0^{(0)}$

$\boldsymbol{b}_1^{(0)}$

$\boldsymbol{b}_2^{(0)}$

$\boldsymbol{b}_3^{(0)}$

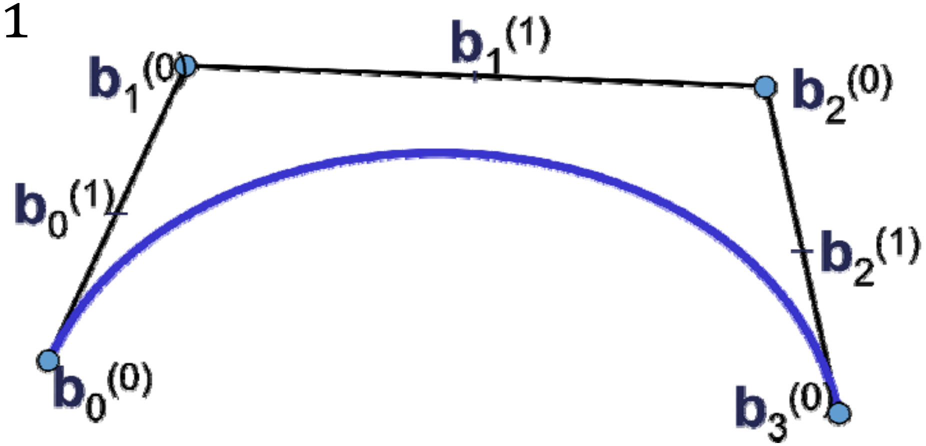


De Casteljau algorithm

- Repeated convex combination of control points

$$\boldsymbol{b}_i^{(r)} = (1-t)\boldsymbol{b}_i^{(r-1)} + t\boldsymbol{b}_{i+1}^{(r-1)}$$

$$\begin{tikzcd}
\boldsymbol{b}_0^{(0)} \arrow[dr, "1-t"] & \\
\boldsymbol{b}_1^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_0^{(1)} \\
\boldsymbol{b}_2^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_1^{(1)} \\
\boldsymbol{b}_3^{(0)} \arrow[r, "t"] & \boldsymbol{b}_2^{(1)}
\end{tikzcd}$$

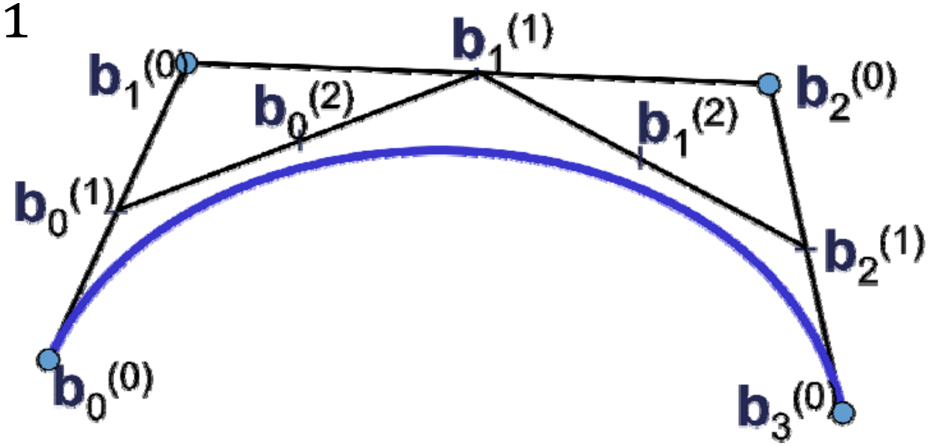


De Casteljau algorithm

- Repeated convex combination of control points

$$\boldsymbol{b}_i^{(r)} = (1-t)\boldsymbol{b}_i^{(r-1)} + t\boldsymbol{b}_{i+1}^{(r-1)}$$

$$\begin{tikzcd}
\boldsymbol{b}_0^{(0)} \arrow[dr, "1-t"] & & \\
\boldsymbol{b}_1^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_0^{(1)} \arrow[dr, "1-t"] & \\
\boldsymbol{b}_2^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_1^{(1)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_0^{(2)} \\
\boldsymbol{b}_3^{(0)} \arrow[r, "t"] & \boldsymbol{b}_2^{(1)} \arrow[r, "t"] & \boldsymbol{b}_1^{(2)}
\end{tikzcd}$$



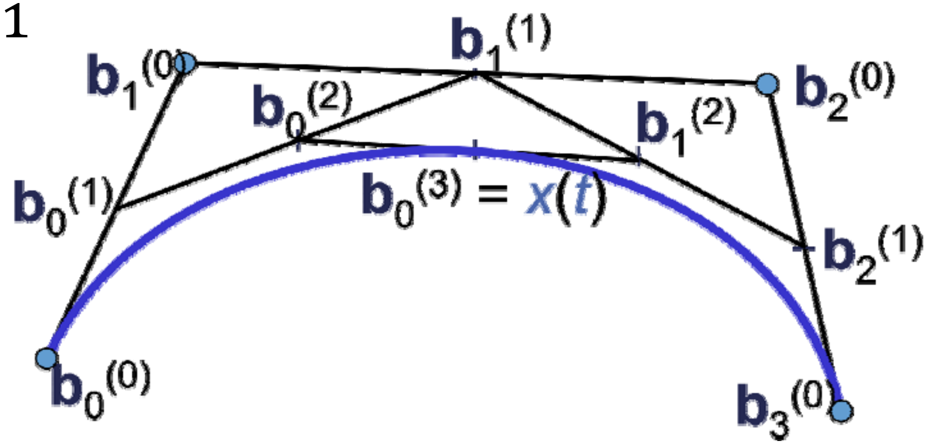
De Casteljau algorithm

- Repeated convex combination of control points

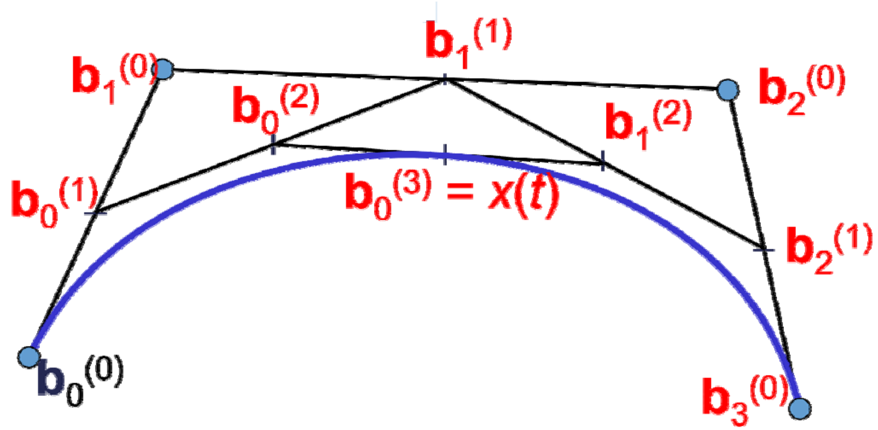
$$\boldsymbol{b}_i^{(r)} = (1-t)\boldsymbol{b}_i^{(r-1)} + t\boldsymbol{b}_{i+1}^{(r-1)}$$

$$\begin{tikzcd}
\boldsymbol{b}_0^{(0)} \arrow[dr, "1-t"] & & & \\
\boldsymbol{b}_1^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_0^{(1)} \arrow[dr, "1-t"] & & \\
\boldsymbol{b}_2^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_1^{(1)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_0^{(2)} \arrow[dr, "1-t"] & \\
\boldsymbol{b}_3^{(0)} \arrow[r, "t"] & \boldsymbol{b}_2^{(1)} \arrow[r, "t"] & \boldsymbol{b}_1^{(2)} \arrow[r, "t"] & \boldsymbol{b}_0^{(3)} = x(t)
\end{tikzcd}$$

De Casteljau scheme



De Casteljau algorithm



Algorithm:

for r=1..n

for i=0..n-r

$$b_i^{(r)} = (1-t)\, b_i^{(r-1)} + t\, b_{i+1}^{(r-1)}$$

end

end

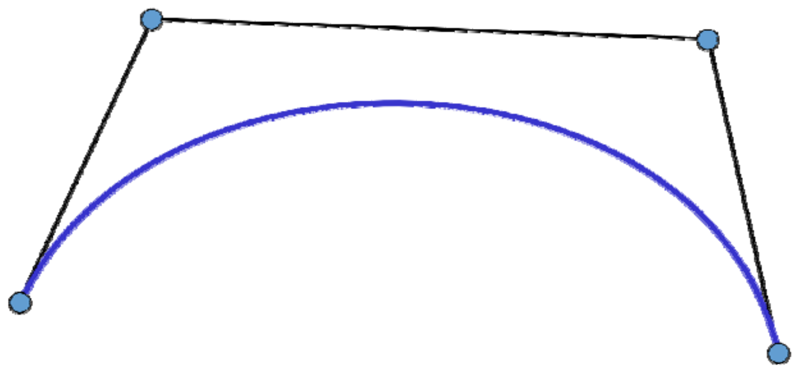
return $b_0^{(n)}$

只有线性运算，计算稳定

复杂度：$O(n^2)$ time

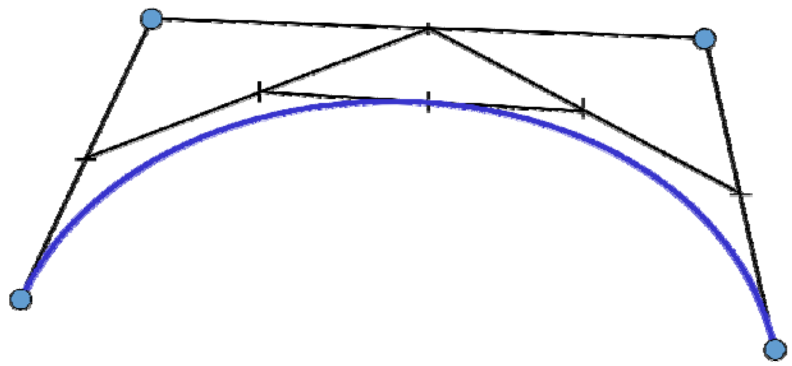
$O(n)$ memory

De Casteljau algorithm



- 计算Bezier曲线$x(t)$上参数为 t 的点
  - Bisect control polygon in ratio $t: (1-t)$
- 良好的几何意义：该点将曲线一分两条子Bezier曲线，其控制顶点是中间生成的点
- 可用于Bezier曲线的离散及求根等许多应用

De Casteljau algorithm



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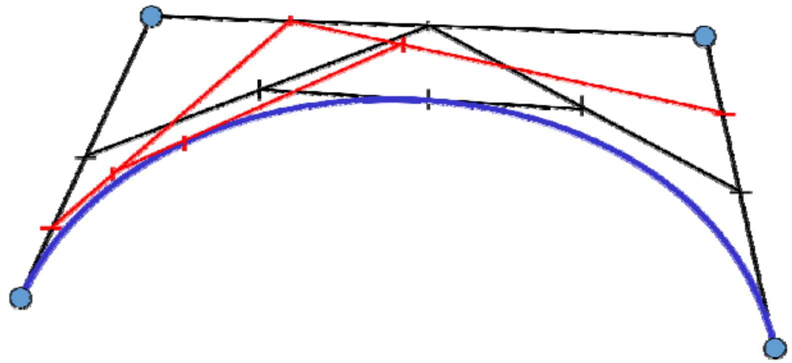

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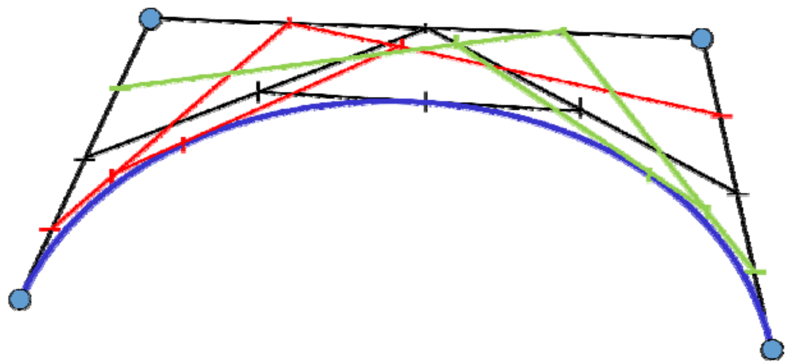

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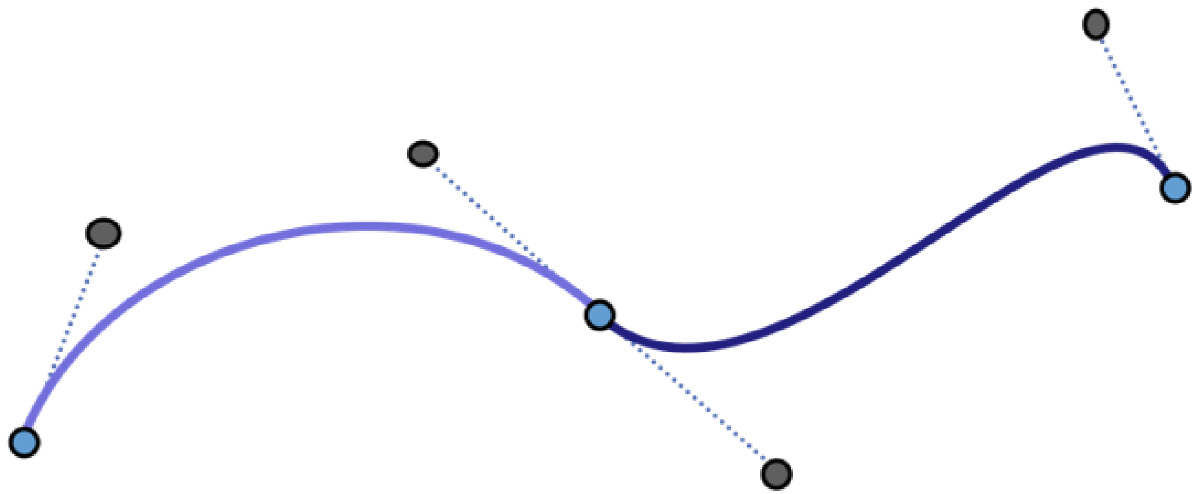
几何样条曲线

用分段Bezier曲线来插值型值点

- 给定型值点:

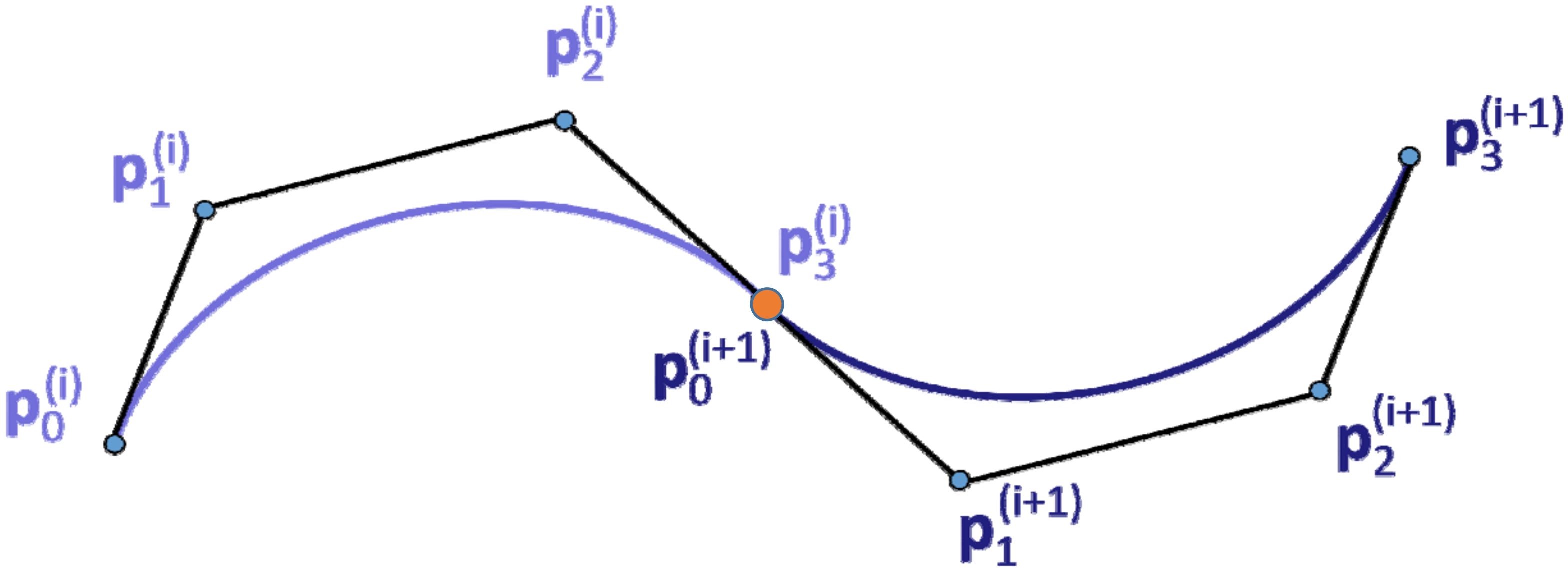
$$\boldsymbol{k}_0, \ldots, \boldsymbol{k}_n \in \mathbb{R}^3$$

- 每两点间生成一段Bezier曲线，使得整体曲线满足一定的连续性(C^0, C^1, C^2)



问题：两Bezier曲线的拼接条件

- C^0, C^1, C^2?



回顾：Bezier曲线的端点性质

- 端点插值：

$$\boldsymbol{f}(0) = \boldsymbol{p}_0$$
$$\boldsymbol{f}(1) = \boldsymbol{p}_n$$

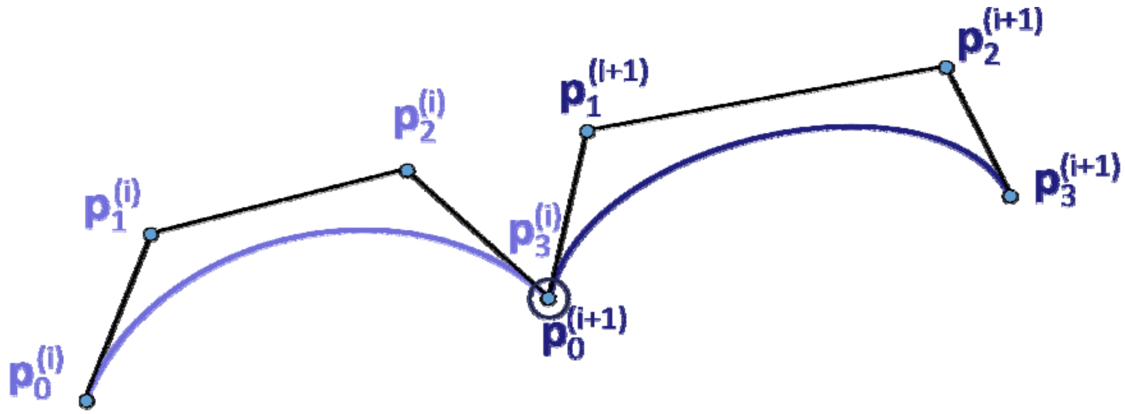
- 端点的切线方向与边相同：

$$\boldsymbol{f}'(0) = n[\boldsymbol{p}_1 - \boldsymbol{p}_0]$$
$$\boldsymbol{f}'(1) = n[\boldsymbol{p}_{n-1} - \boldsymbol{p}_n]$$

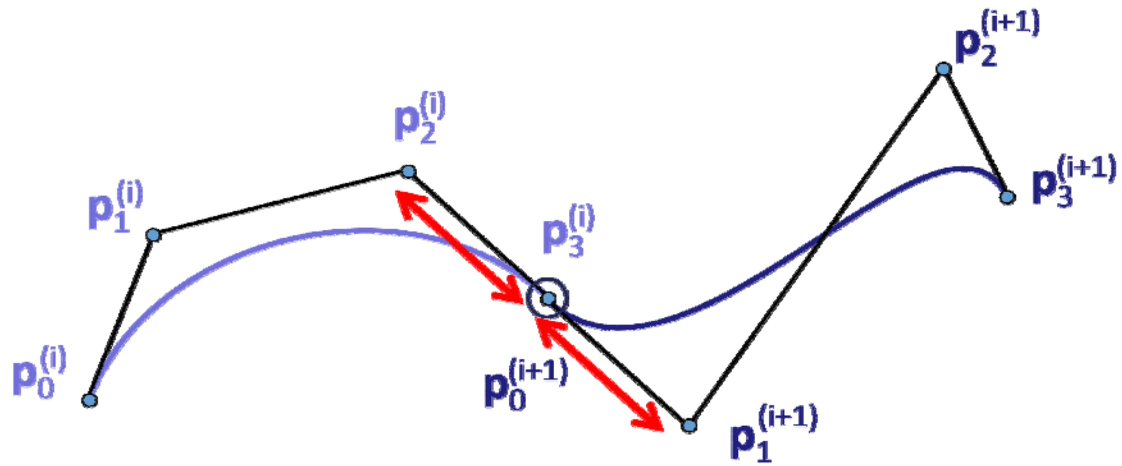
- 端点的2阶(*k*)切线与3点(*k*+1)相关：

$$\boldsymbol{f}''(0) = n(n-1)[\boldsymbol{p}_2 - \boldsymbol{2}\boldsymbol{p}_1 + \boldsymbol{p}_0]$$
$$\boldsymbol{f}''(1) = n(n-1)[\boldsymbol{p}_n - 2\boldsymbol{p}_{n-1} + \boldsymbol{p}_{n-2}]$$

两Bezier曲线的拼接条件



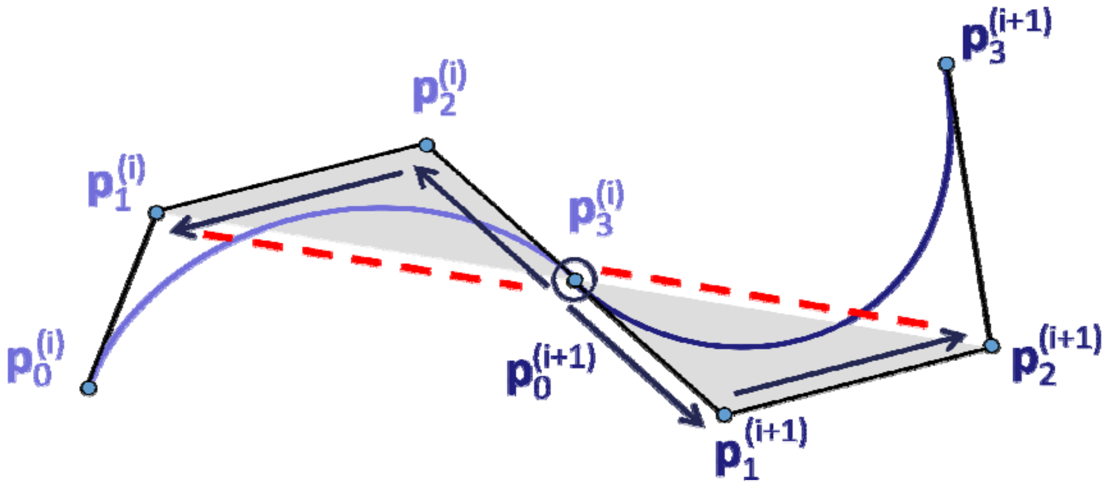
C^0连续



G^1连续：三点共线

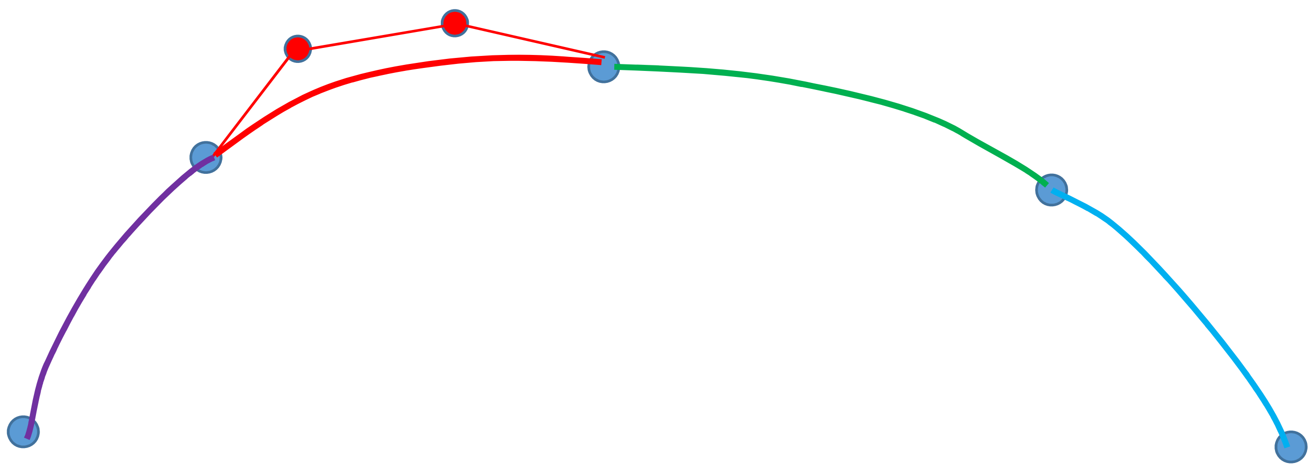
C^1连续：三点共线且等长

两Bezier曲线的拼接条件



- C^2 连续
  - d^2/dt^2 为 $(\boldsymbol{p}_2 - 2\boldsymbol{p}_1 + \boldsymbol{p}_0), (\boldsymbol{p}_n - 2\boldsymbol{p}_{n-1} + \boldsymbol{p}_{n-2})$
  - 阴影三角形相似
- G^2 连续?

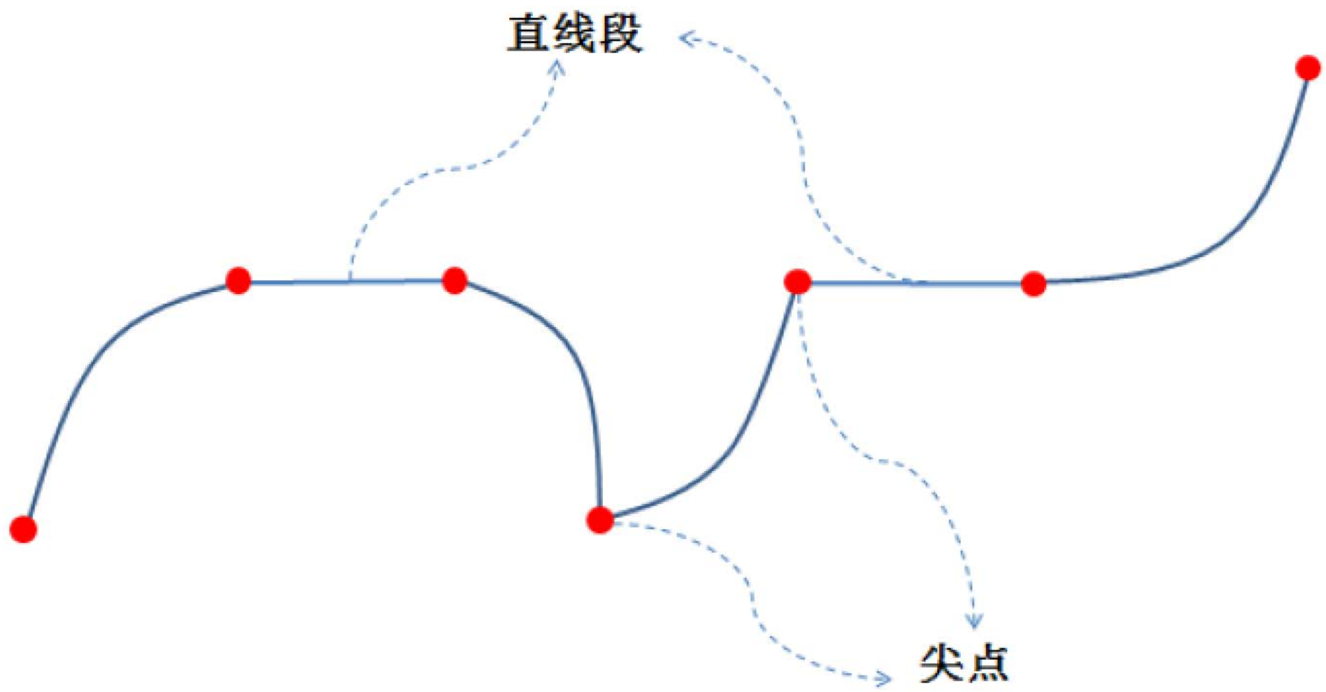
构造3次插值Bezier曲线的几何方法



在作业4中也实现下

广义样条曲线

- 分段的多项式曲线（Bezier曲线）



中国科学技术大学
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谢　谢　！