中国科学技术大学
University of Science and Technology of China

GAMES102

GAMES 102在线课程

几何建模与处理基础

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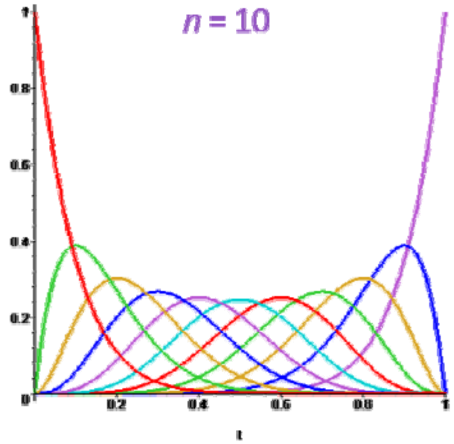
GAMES 102在线课程：几何建模与处理基础

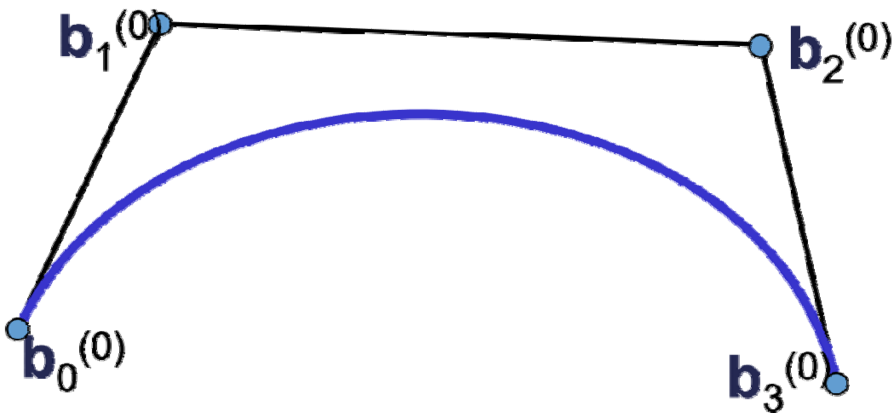
B样条曲线

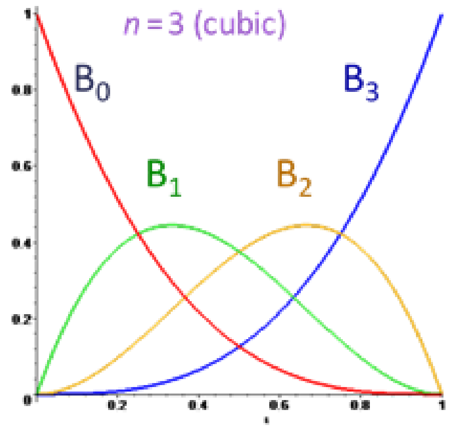
Bezier曲线的不足

- n次Bezier曲线： $n+1$个控制顶点



$$x(t)=\sum_{i=0}^{n} B_i^n(t)\cdot b_i$$



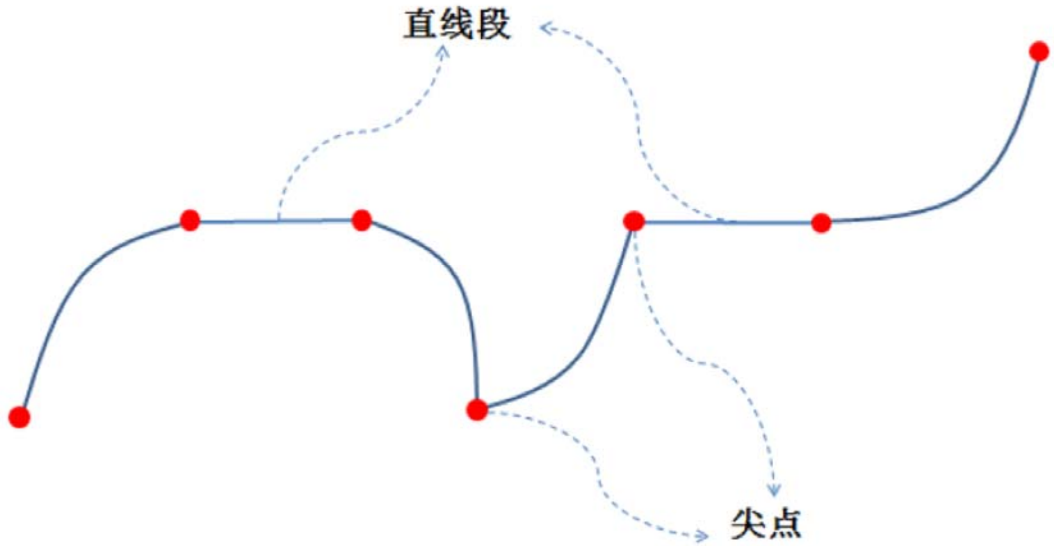


全局性：牵一发而动全身，不利于设计

原因：基函数是全局的

样条曲线

- 分段的多项式曲线（Bezier曲线）
  - 分段表达，具有局部性



有无统一的表达方式？

思考：样条曲线的统一表达

- 形式类比：每个控制顶点用一个基函数进行组合

$$\boldsymbol{x}(t) = \sum_{i=0}^{n} N_{i,k}(t) \cdot \boldsymbol{d}_i$$

- 性质要求：
  - 基函数须局部性（局部支集）
  - 基函数要有正性+权性
  - …

- 如何构造？

B样条的产生

- Early use of splines on computers for data interpolation
  - Ferguson at Boeing, 1963
  - Gordon and de Boor at General Motors
  - B-splines, de Boor 1972

- Free form curve design
  - Gordon and Riesenfeld, 1974 → B-splines as a generalization of Bezier curves

启发：

- Bernstein基函数的递推公式：

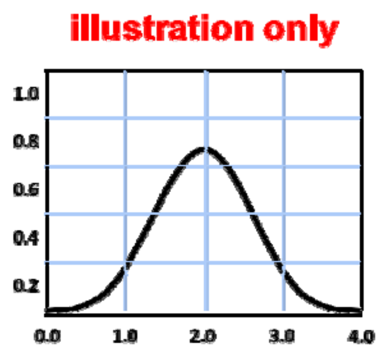
$$B_i^n(t) = (1-t)B_i^{(n-1)}(t) + tB_{i-1}^{(n-1)}(1-t)$$

$$\text{with } B_0^0(t) = 1, B_i^n(t) = 0 \text{ for } i \notin \{0 \dots n\}$$

- 思路：
  - 局部处处类似定义，由一个基函数平移得到
  - 高阶的基函数由2个低阶的基函数“升阶”得到
    - 利于保持一些良好的性质，比如提高光滑性

Key Ideas

- 以三次为例
  - We design one basis function $b(t)$
  - Properties:
    - $b(t)$is C^2 continuous
    - $b(t)$is piecewise polynomial, degree 3 (cubic)
    - $b(t)$has local support
    - Overlaying shifted $b(t+i)$ forms a partition of unity
    - $b(t) \geq 0$ for all t
  - In short:
    - All desirable properties build into the basis
    - Linear combinations will inherit these



Shifted Basis Functions

- 型值点参数化：节点向量

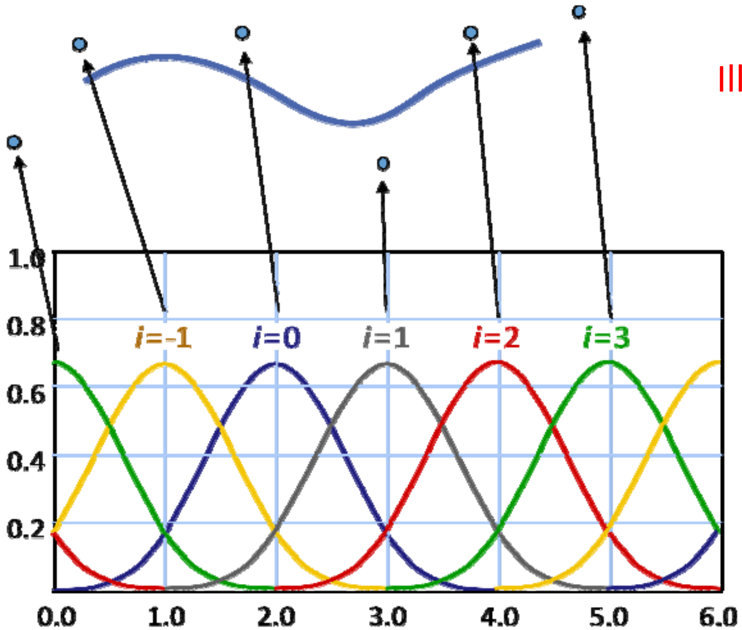


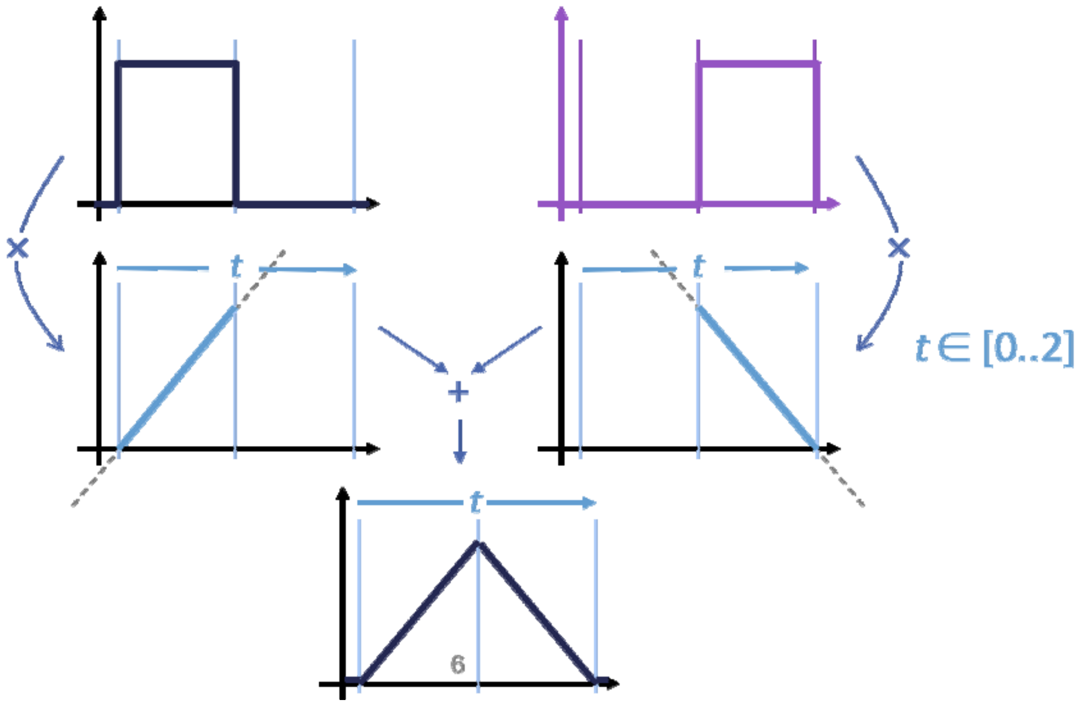
Illustration only

Shifted basis function $b(t)$

Courtesy of Renjie Chen

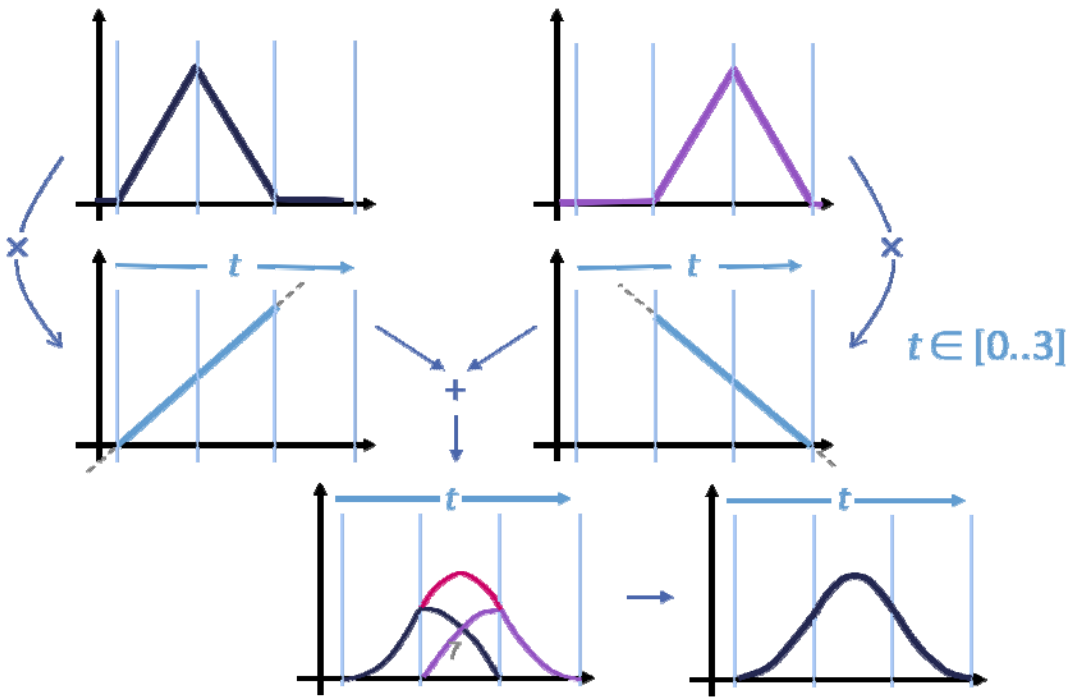
Repeated linear interpolation

- Another way to increase smoothness:



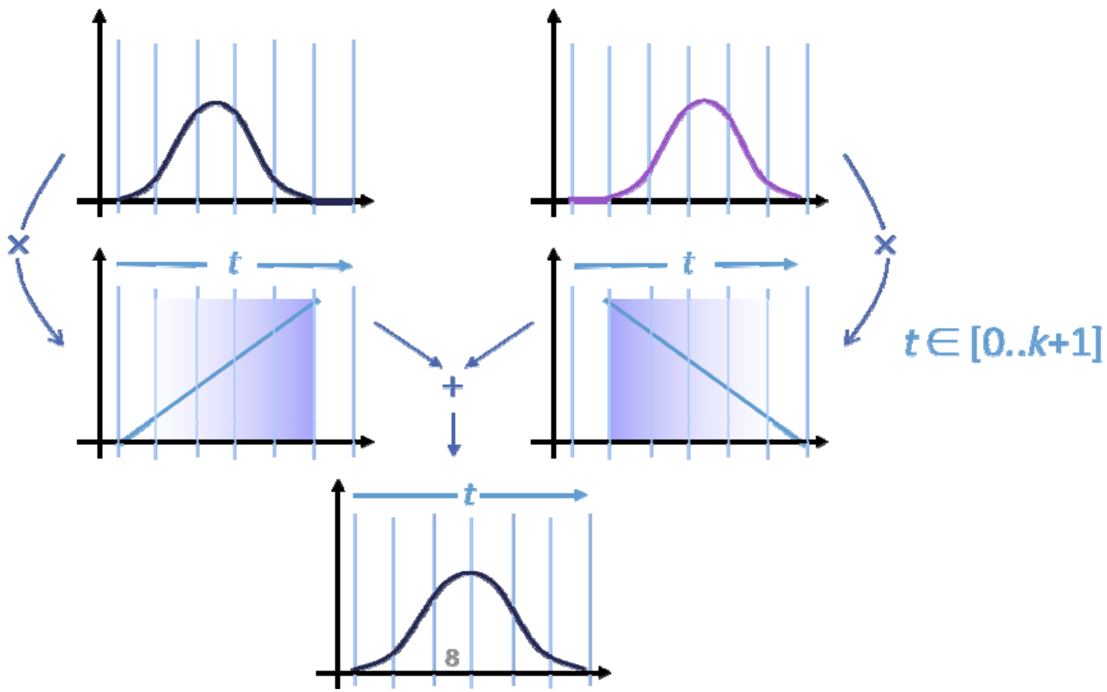
Repeated linear interpolation

- Another way to increase smoothness:



Repeated linear interpolation

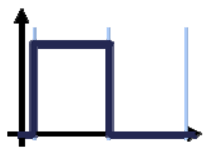
- Another way to increase smoothness



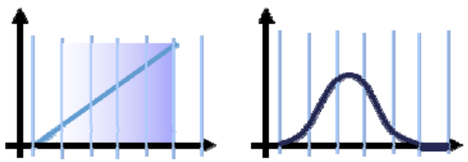
De Boor Recursion: uniform case

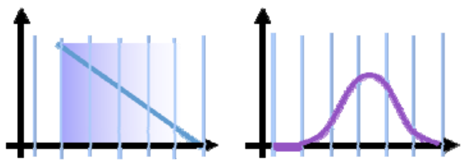
- The uniform B-spline basis of order $\boldsymbol{k}$ (degree $\boldsymbol{k-1}$) is given as

$$N_i^1(t) = \begin{cases} 1, \text{if } i \le t < i+1 \\ 0, \quad \text{otherwise} \end{cases}$$



$$N_i^k(t) = \frac{t-i}{(i+k-1)-i} N_i^{k-1}(t) \quad + \quad \frac{(i+k)-t}{(i+k)-(i+1)} N_{i+1}^{k-1}(t)$$





$$= \frac{t-i}{k-1} N_i^{k-1}(t) + \frac{i+k-t}{k-1} N_{i+1}^{k-1}(t)$$

B-spline curves: general case

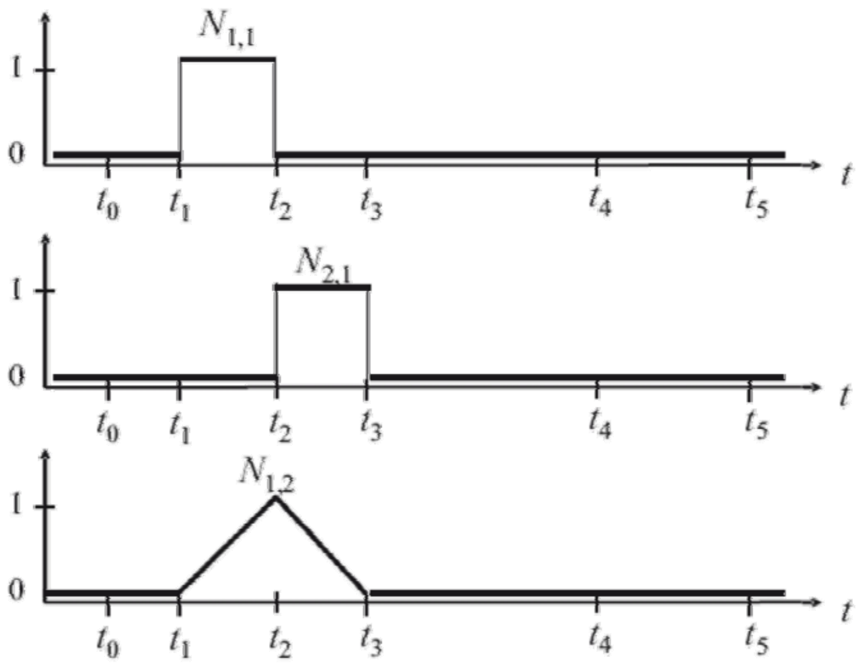
- Given: knot sequence $t_0 < t_1 < \cdots < t_n < \cdots < t_{n+k}$
  ($(t_0, t_1, \dots, t_{n+k})$ is called knot vector)
- Normalized B-spline functions $N_{i,k}$ of the order k (degree $k-1$) are defined as:

$$N_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

for $k > 1$ and $i = 0, \dots, n$

Example



$$N_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

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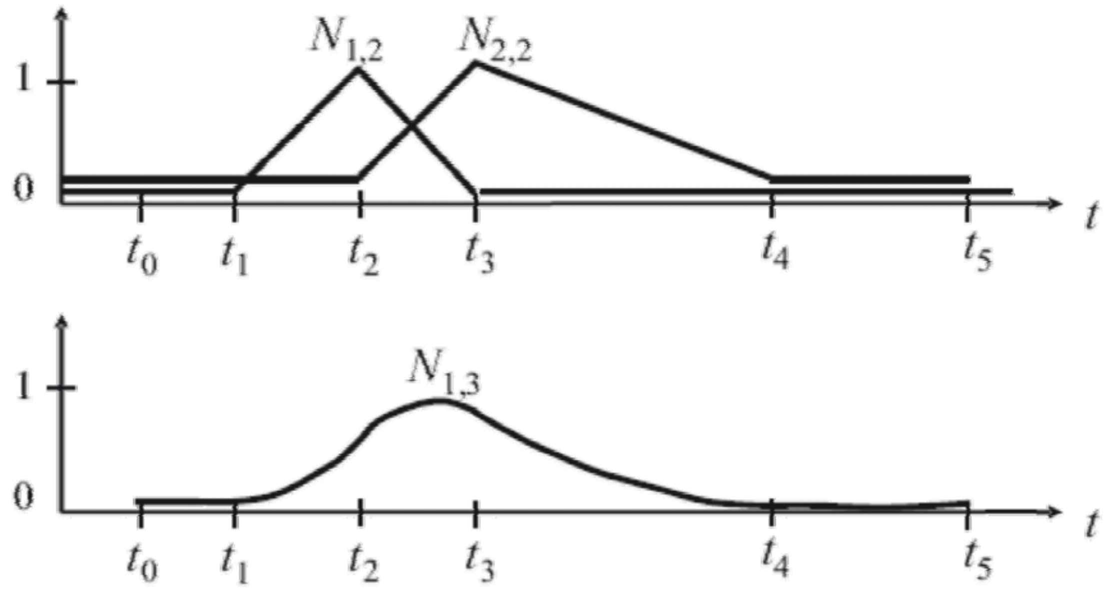
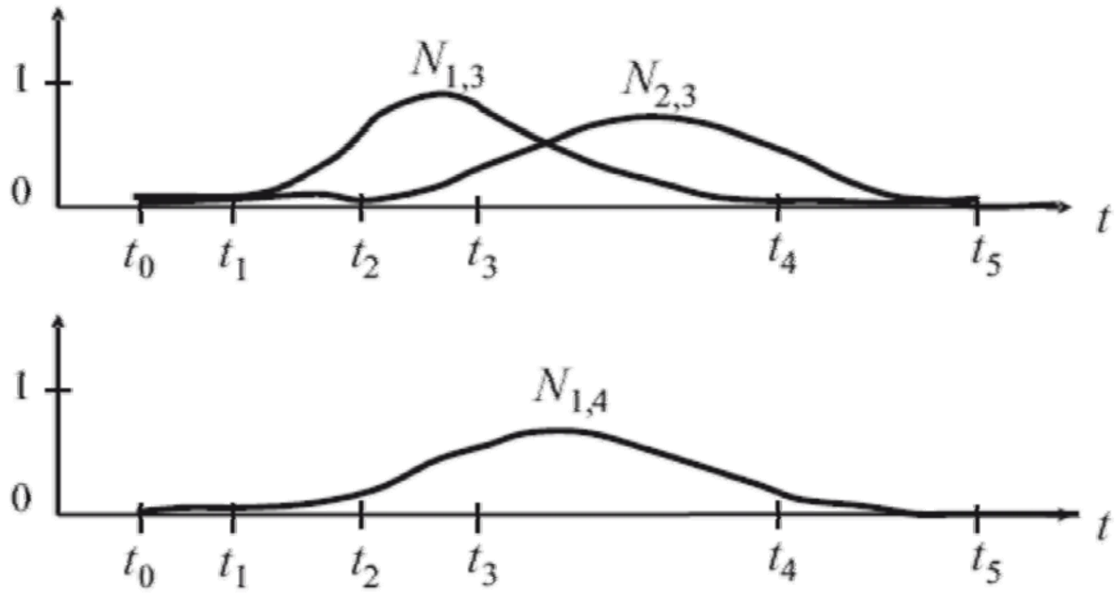


Example



Basis properties

- For the so defined basis functions, the following properties can be shown:
  - $N_{i,k}(t) > 0$ for $t_i < t < t_{i+k}$
  - $N_{i,k}(t) = 0$ for $t_0 < t < t_i$ or $t_{i+k} < t < t_{n+k}$
  - $\sum_{i=0}^{n} N_{i,k}(t) = 1$ for $t_{k-1} \leq t \leq t_{n+1}$

- For $t_i \leq t_j \leq t_{i+k}$, the basis functions $N_{i,k}(t)$ are C^{k-2} at the knots t_j

- The interval $[t_i, t_{i+k}]$ is called support of $N_{i,k}$

B-spline curves

- B-spline curves
  - Given: $n+1$ control points $\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n \in \mathbb{R}^3$
    knot vector $T = (t_0, \ldots, t_n, \ldots t_{n+k})$
  - Then, the B-spline curve $\boldsymbol{x}(t)$ of the order k is defined as

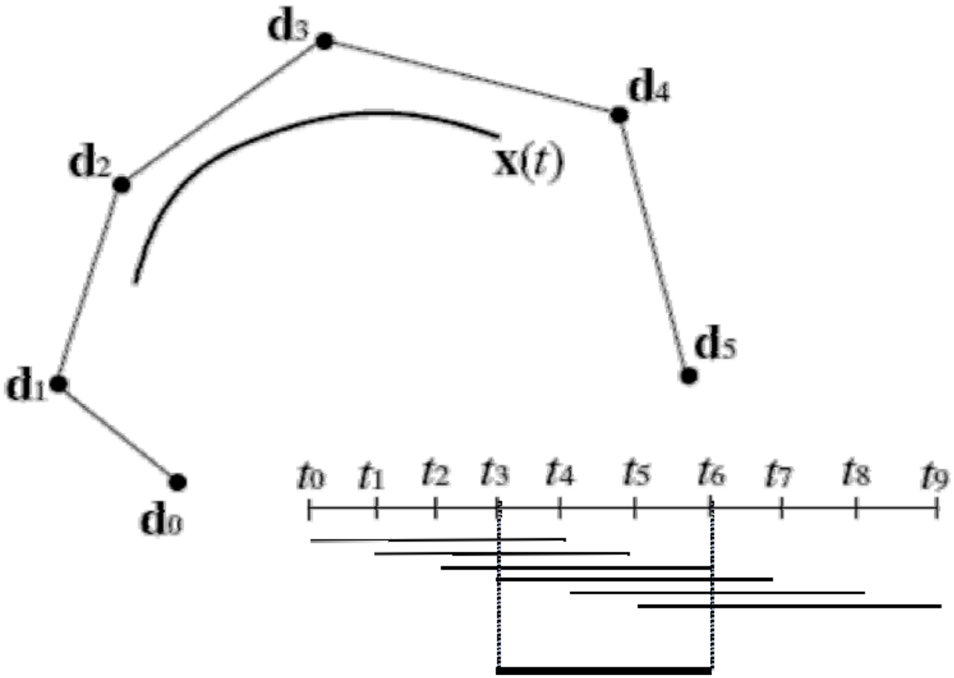
$$\boldsymbol{x}(t) = \sum_{i=0}^{n} N_{i,k}(t) \cdot \boldsymbol{d}_i$$

  - The points $\boldsymbol{d}_i$ are called *de Boor points*

Carl R. de Boor
German-American mathematician
University of Wisconsin-Madison

Example

- $k = 4, n = 5$



Support intervals of $N_{i,k}$

Curve defined in interval $t_3 \leq t \leq t_6$

B-spline curves

- Multiple weighted knot vectors
  - So far: $T = (t_0, \dots, t_n, \dots, t_{n+k})$ with $t_0 < t_1 < \cdots < t_{n+k}$
  - Now: also multiple knots allowed, i.e. with $t_0 \leq t_1 \leq \cdots \leq t_{n+k}$
  - The recursive definition of the B spline function $N_{i,k}$ $(i = 0, \dots, n)$ works nonetheless, as long as no more than k knots coincide

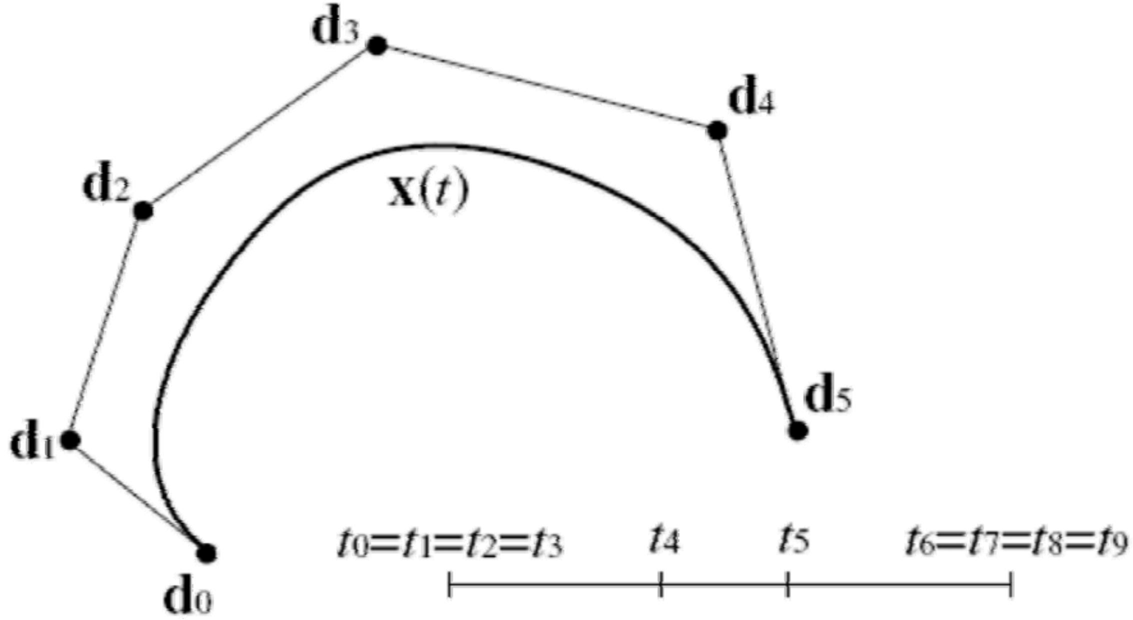
B-spline curves

- Effect of multiple knots:
  - set: $t_0 = t_1 = \cdots = t_{k-1}$
  - and $t_{n+1} = t_{n+2} = \cdots = t_{n+k}$

$\boldsymbol{d}_0$ and $\boldsymbol{d}_n$ are interpolated

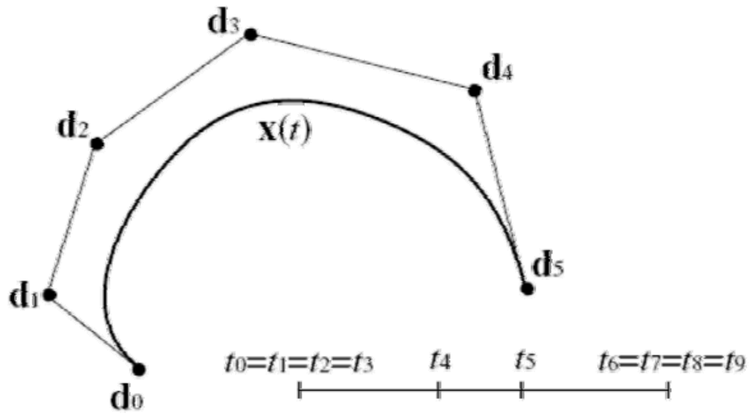
B-spline curves

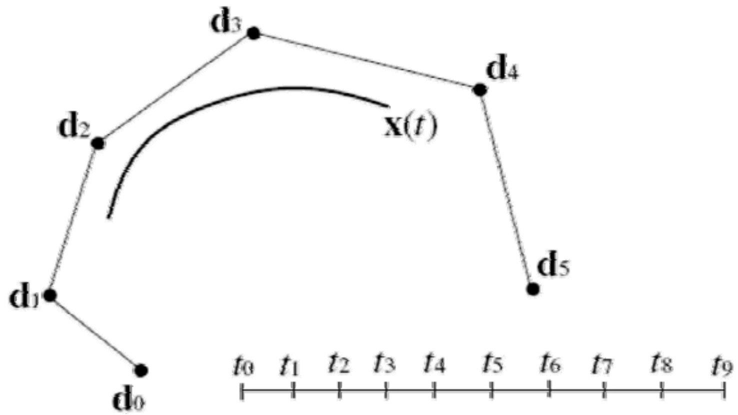
- Example: $k = 4, n = 5$



B-spline curves

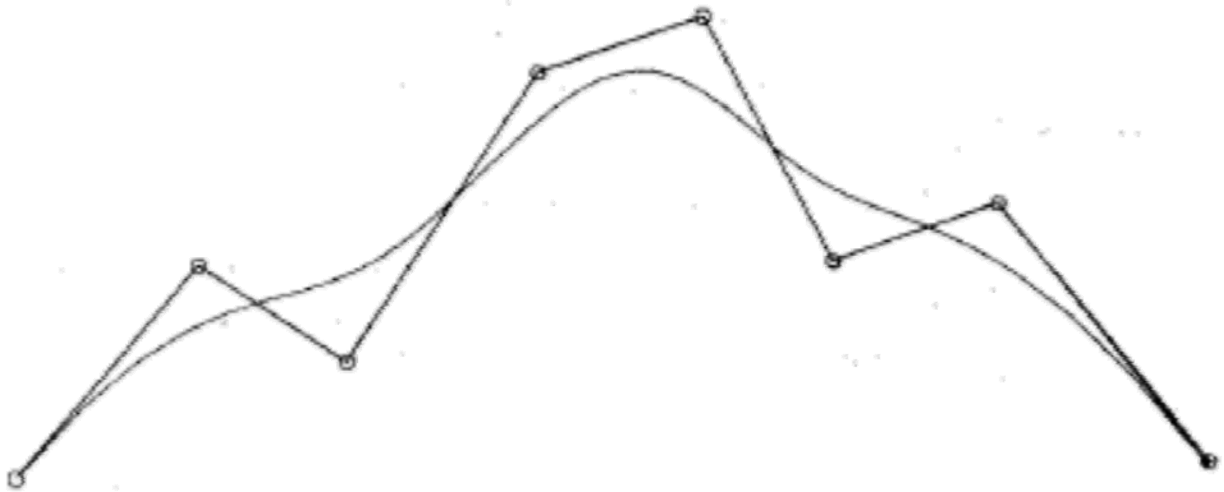
- Example: $k = 4, n = 5$





B-spline curves

- Further example



B-spline curves

- Interesting property:
  - B-spline functions $N_{i,k}$ $(i = 0, \dots, k-1)$ of the order k over the knot vector $T = (t_0, t_1, \dots, t_{2k-1}) = (\underbrace{0, \dots, 0}_{k \text{ times}}, \underbrace{1, \dots, 1}_{k \text{ times}})$

  are Bernstein polynomials B_i^{k-1} of degree $k-1$

B-spline curves properties

- Given:
  - $T = (\underbrace{t_0, \dots, t_0}_{k \text{ times}}, t_k, \dots, t_n, \underbrace{t_{n+1}, \dots, t_{n+1}}_{k \text{ times}})$
  - de Boor polygon $\boldsymbol{d}_0, \dots, \boldsymbol{d}_n$
- Then, the following applies for the related B-spline curve $\boldsymbol{x}(t)$:

B-spline curves properties

- $\boldsymbol{x}(t_0) = \boldsymbol{d}_0, \boldsymbol{x}(t_{n+1}) = \boldsymbol{d}_n$ (end point interpolation)

- $\dot{\boldsymbol{x}}(t_0) = \frac{k-1}{t_k - t_0}(\boldsymbol{d}_1 - \boldsymbol{d}_0)$ (tangent direction at $\boldsymbol{d}_0$, similar in $\boldsymbol{d}_n$)

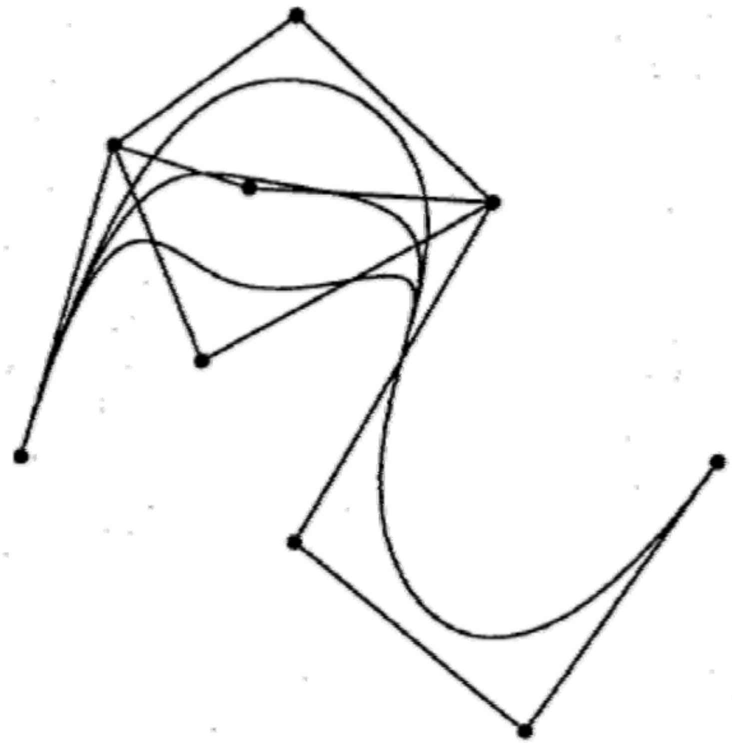
- $\boldsymbol{x}(t)$ consists of $n - k + 2$ polynomial curve segments of degree $k - 1$ (assuming no multiple inner knots)

B-spline curves properties

- Multiple inner knots ⇒ reduction of continuity of $x(t)$.
  l-times inner knot $(1 \leq l < k)$ means
  C^{k-l-1}-continuity
- Local impact of the de Boor points: moving of d_i only changes the curve in the region $[t_i, t_{i+k}]$
- The insertion of new de Boor points does not change the polynomial degree of the curve segments

B-spline curves properties

- Locality of B-spline curves



B-spline curves

- Evaluation of B-spline curves
  - Using B-spline functions

  - Using the de Boor algorithm

  Similar algorithm to the de Casteljau algorithm for Bezier curves;
  consists of a number of linear interpolations on the de Boor polygon

The de Boor algorithm

- Given:

  $\boldsymbol{d}_0, \ldots, \boldsymbol{d}_n$: de Boor points

  $(t_0, \ldots, t_{k-1} = t_0, t_k, t_{k+1}, \ldots, t_n, t_{n+1}, \ldots, t_{n+k} = t_{n+1})$:

  Knot vector

- wanted:

  Curve point $\boldsymbol{x}(t)$ of the B-spline curve of the order k

The de Boor algorithm

1. Search index r with $t_r \leq t < t_{r+1}$
2. for $i = r - k + 1, \dots, r$

   $d_i^0 = d_i$

- for $j = 1, \dots, k - 1$

  for $i = r - k + 1 + j, \dots, r$

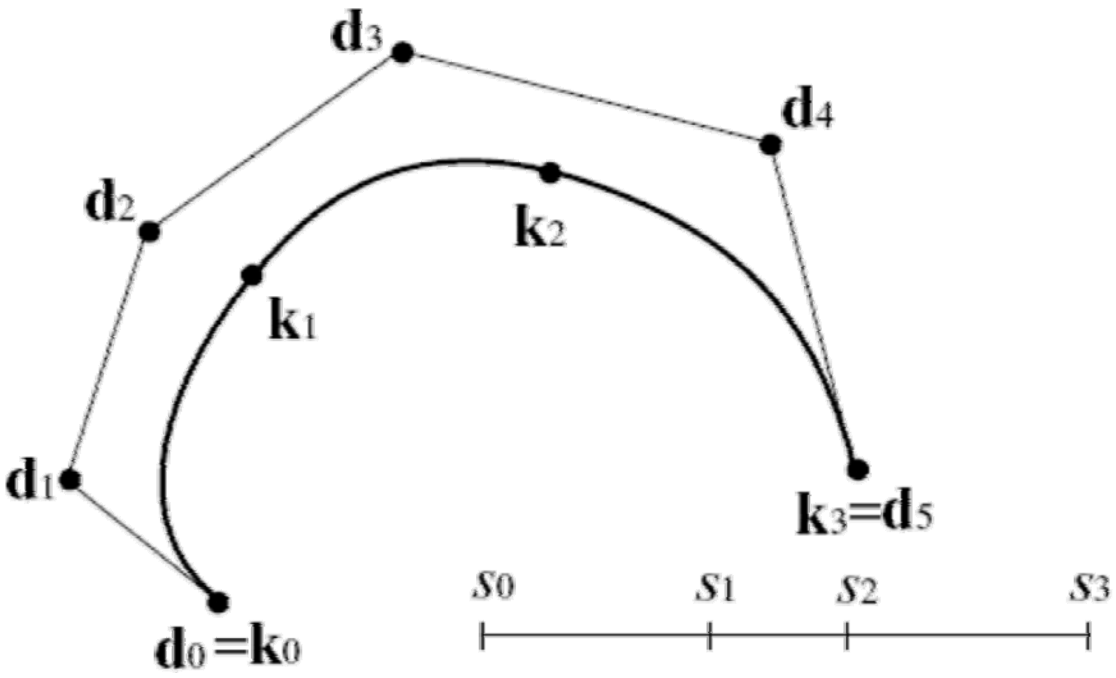
  $$d_i^j = \left(1 - \alpha_i^j\right) \cdot d_{i-1}^{j-1} + \alpha_i^j \cdot d_i^{j-1}$$

  with $\alpha_i^j = \frac{t - t_i}{t_{i+k-j} - t_i}$

Then: $d_r^{k-1} = x(t)$

B样条曲线：分段Bezier曲线

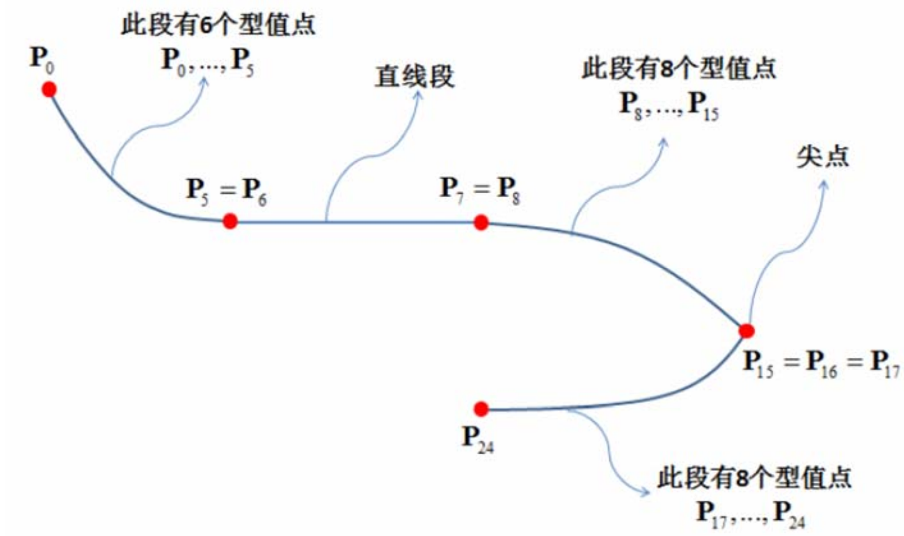
- $n = 3$



B：Basic（亦称“基本样条”）

B样条的其他理论知识

- B样条的许多性质
  - 局部凸包性、变差缩减性、包络性
  - B样条的导数、积分递推式、几何作图
- 重节点的B样条基函数及B样条曲线
- Bezier样条曲线转换为B样条曲线
- B样条插值方法
- …



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