中国科学技术大学
University of Science and Technology of China

GAMES 102在线课程

几何建模与处理基础

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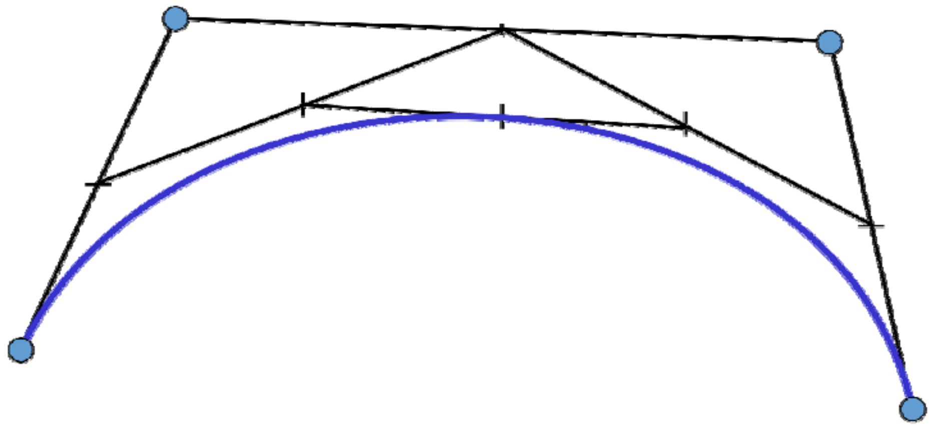
GAMES102

GAMES 102在线课程：几何建模与处理基础

细分曲线

回顾：Bezier曲线的作图法

- de Casteljau作图算法

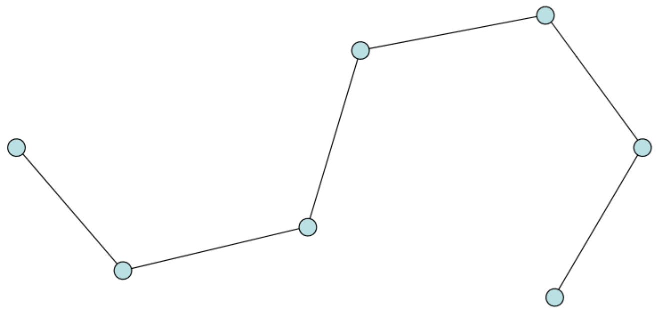


- 几何直观性：逐步割角、磨光
  - 类似于雕塑雕刻过程

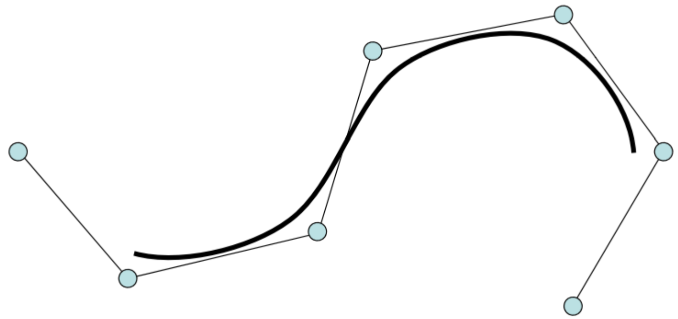
"其实，这座雕塑本来就在那里，我只是将它多余的边边角角去掉而已。"

问题

- 输入：一个简单多边形（控制多边形）

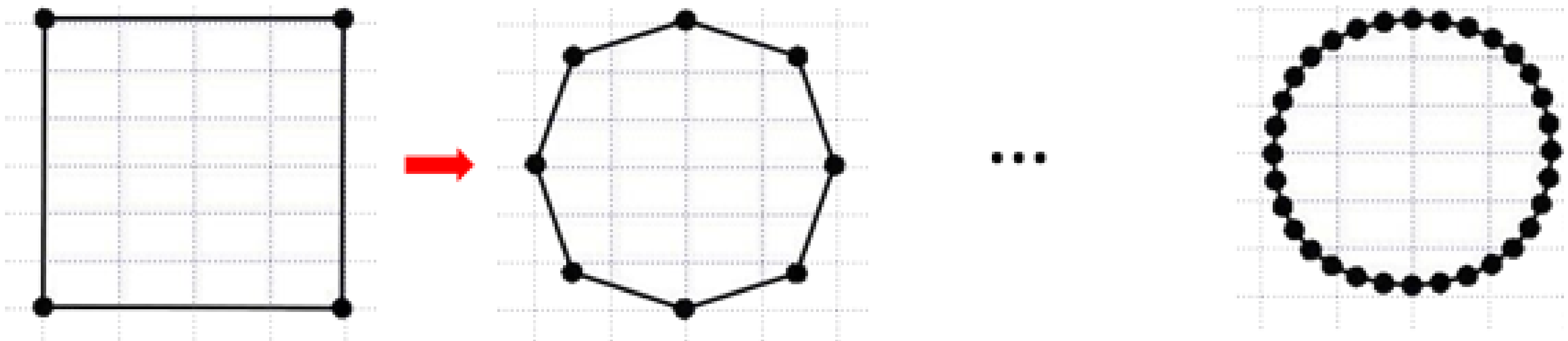


- 输出：一条与之关联的光滑曲线



启发：通过不断“割角”构造曲线？

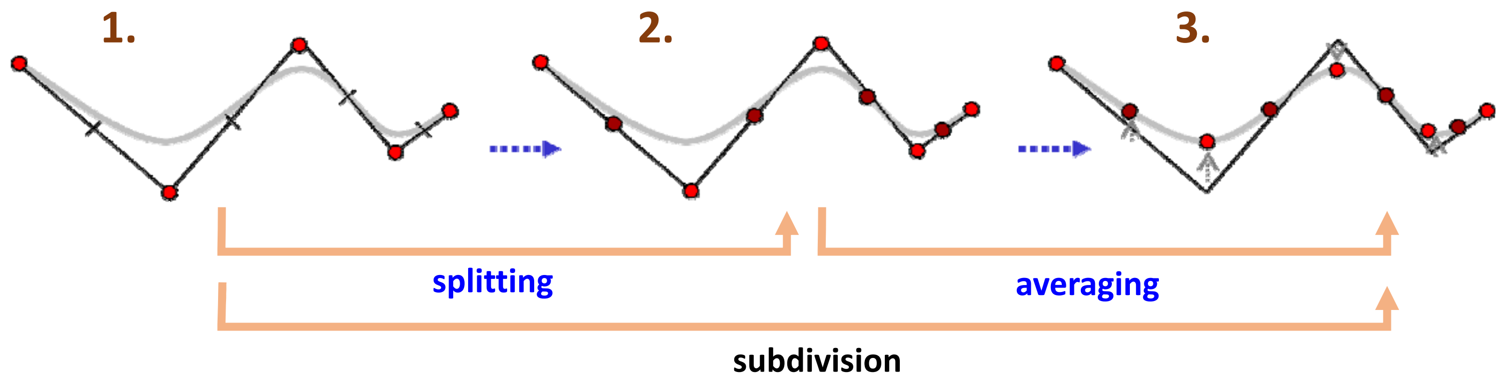
- 给定一个简单多边形
- 通过一定规则，割角磨光，产生更多边的多边形
- 不断迭代操作割角磨光，产生（极限）光滑曲线



细分方法的思想

两个步骤：

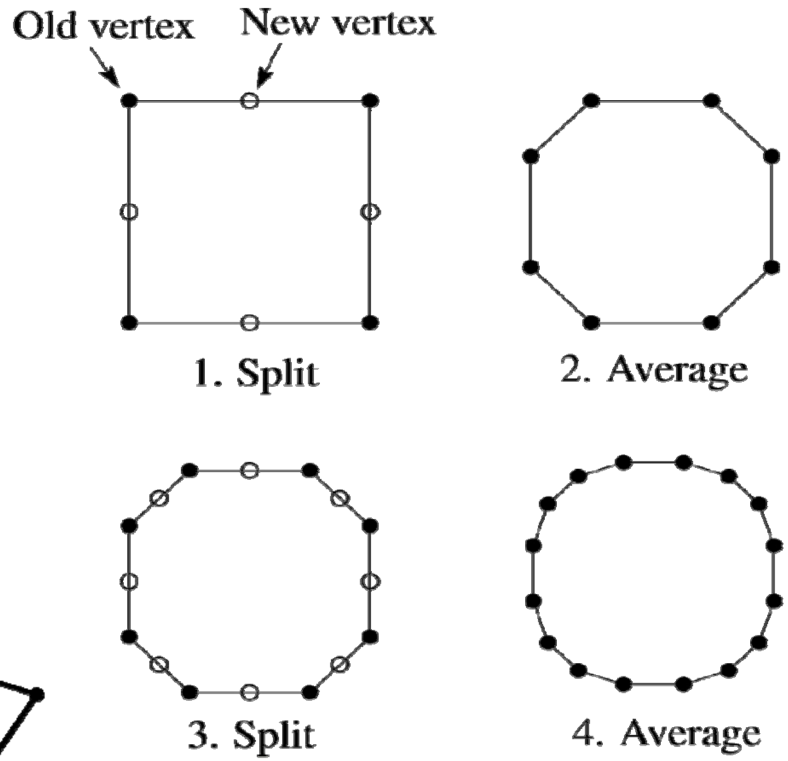
- 拓扑规则：加入新点，组成新多边形 (*splitting*)
- 几何规则：移动顶点，局部加权平均 (*averaging*)
  - 对所有顶点都移动：逼近型
  - 只对新顶点移动：插值型

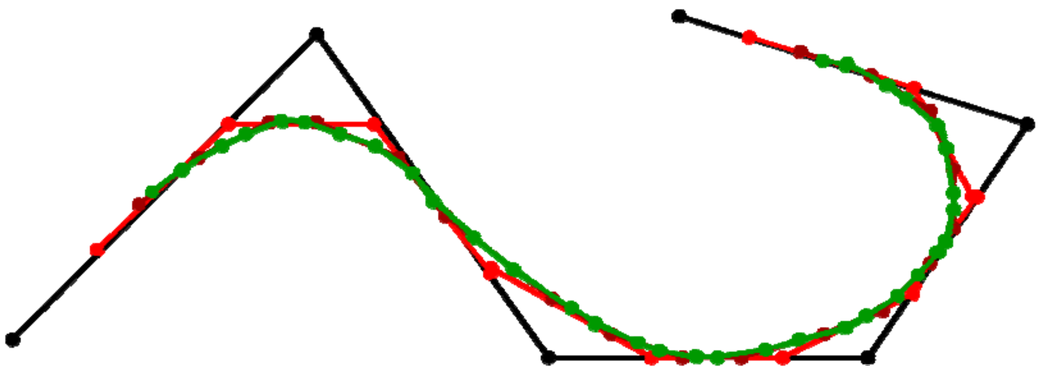


Chaikin细分方法

Chaikin割角法[1974]

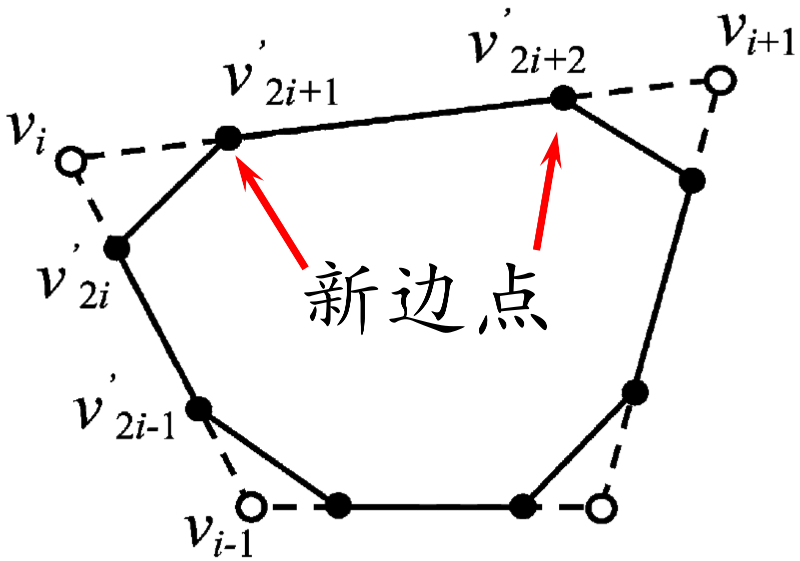
- 每条边取中点，生成新点
- 每个点与其相邻点平均（顺时针）
- 迭代生成曲线





Chaikin割角法[1974]

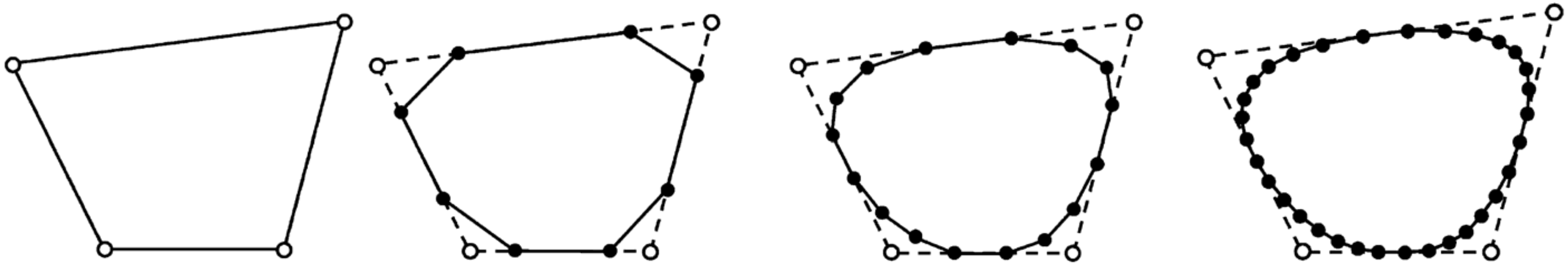
- 拓扑规则：
  - 点分裂成边（割角），老点被抛弃（逼近型）
  - 新点老点重新编号
- 几何规则：新顶点是老顶点的线性组合



$$v'_{2i} = \frac{1}{4} v_{i-1} + \frac{3}{4} v_i$$

$$v'_{2i+1} = \frac{3}{4} v_i + \frac{1}{4} v_{i+1}$$

Chaikin细分曲线



初始多边形　细分一次　细分两次　细分三次

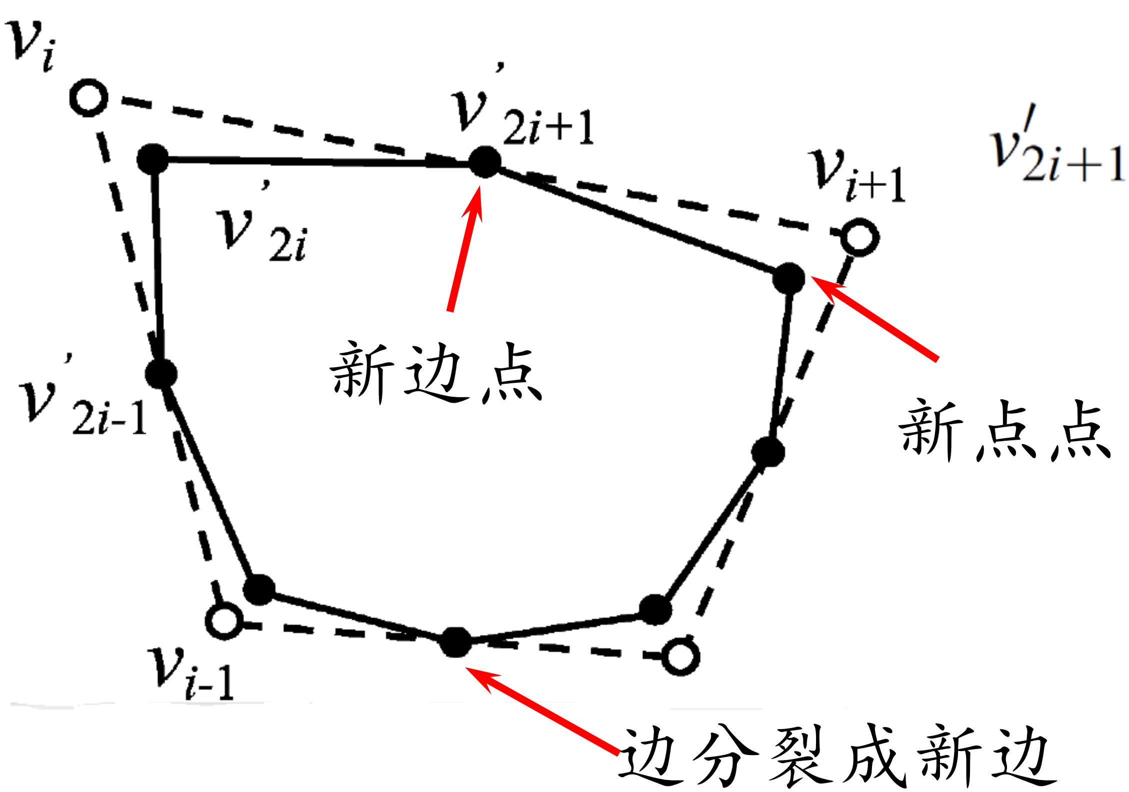
- 可以证明:
  - 极限曲线为二次均匀B样条曲线
  - 节点处C^1，其余点处C^∞

均匀三次B样条曲线细分方法

- 拓扑规则：边分裂成两条新边
- 几何规则：

$$v'_{2i} = \frac{1}{8} v_{i-1} + \frac{3}{4} v_i + \frac{1}{8} v_{i+1}$$

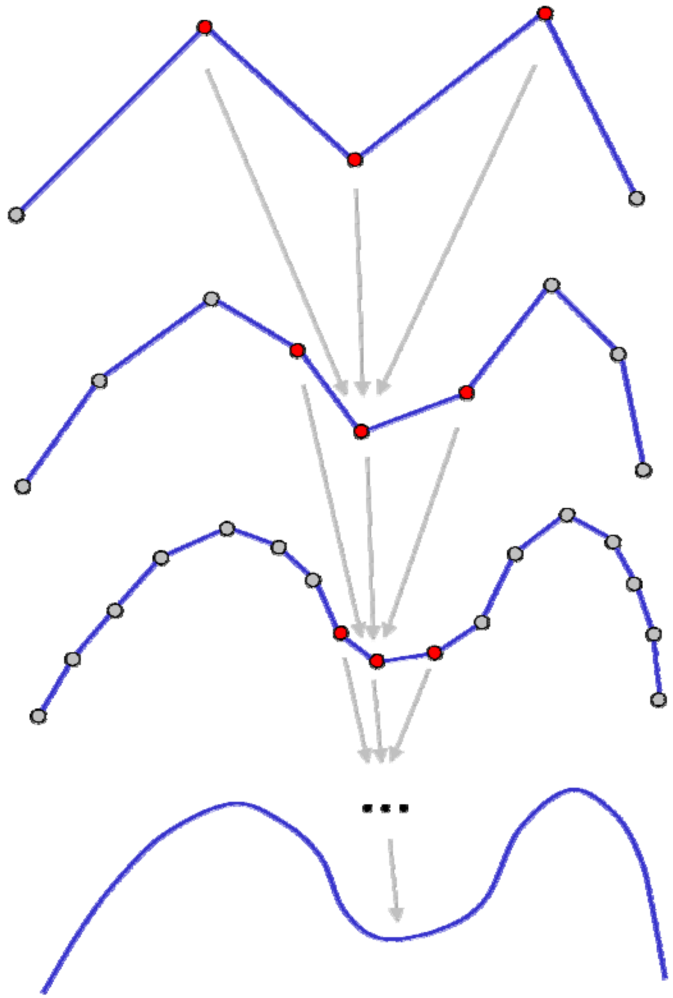
$$v'_{2i+1} = \frac{1}{2} v_i + \frac{1}{2} v_{i+1}$$



细分曲线的性质证明

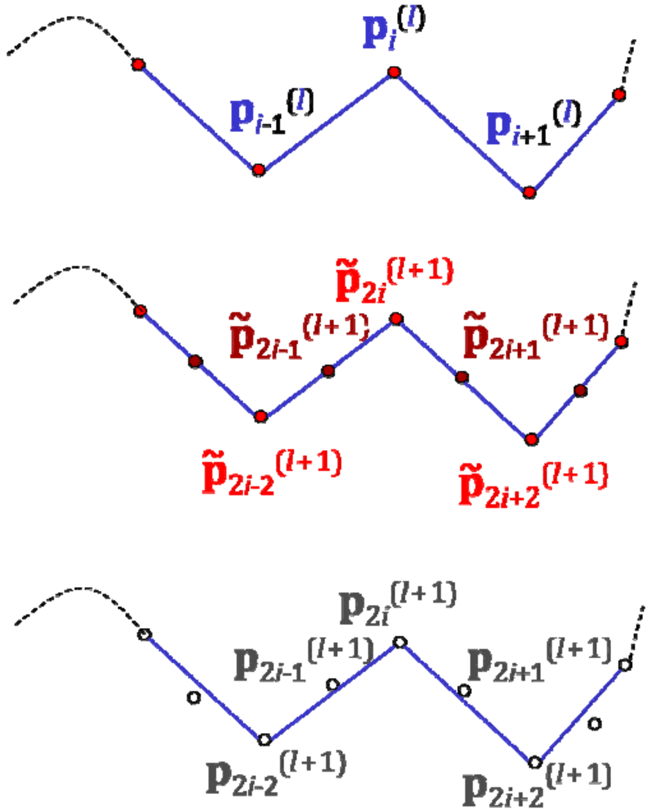
证明的思路

- 将细分过程表达成矩阵形式
  - 新顶点是老顶点的线性组合
- 讨论细分矩阵的谱性质（特征根）



举例： Chaikin细分

矩阵形式:



- Control points at level l: $\boldsymbol{p}_i^{(l)}$
- "Splitted" points at level $l+1$: $\tilde{\boldsymbol{p}}_i^{(l+1)}$
- "Averaged" control points at level $l+1$: $\boldsymbol{p}_i^{(l+1)}$

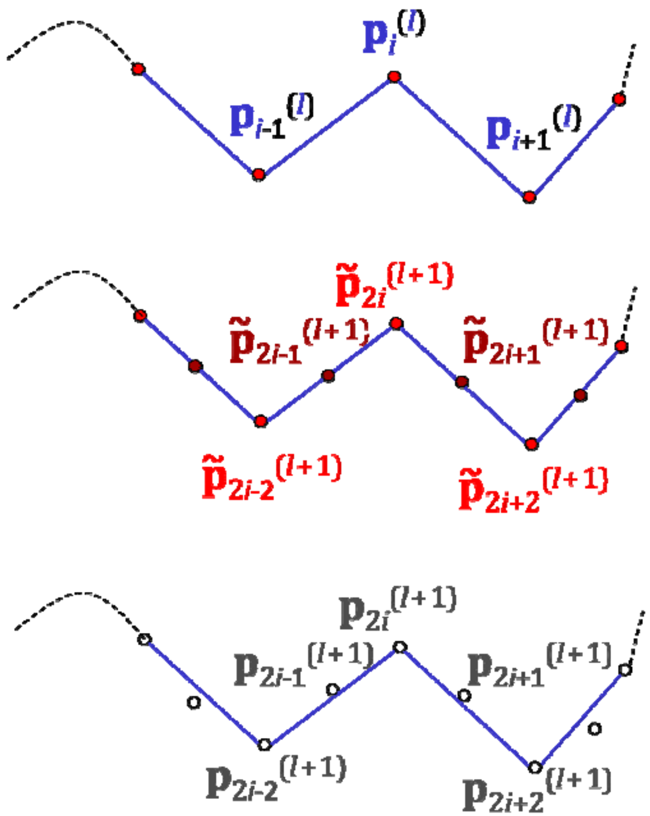
Chaikin细分的矩阵形式

Splitting in matrix notation

$$
2n\left\{\begin{pmatrix} \vdots \\ \tilde{x}_{2i}^{(l+1)} \\ \tilde{x}_{2i+1}^{(l+1)} \\ \vdots \end{pmatrix}\right. = 2n\left\{\underbrace{\begin{pmatrix} \ddots & & & & \\ & 1 & & & \\ & 1/2 & 1/2 & & \\ & & 1 & & \\ & & 1/2 & 1/2 & \\ & & & & \ddots \end{pmatrix}}_{n}\right. \left.\begin{pmatrix} \vdots \\ x_{i}^{(l)} \\ x_{i+1}^{(l)} \\ \vdots \end{pmatrix}\right\} n
$$

Averaging in matrix notation

$$
2n\left\{\begin{pmatrix} \vdots \\ x_{2i}^{(l+1)} \\ x_{2i+1}^{(l+1)} \\ \vdots \end{pmatrix}\right. = 2n\left\{\underbrace{\begin{pmatrix} \ddots & & & \\ & 1/2 & 1/2 & \\ & & 1/2 & 1/2 \\ & & & \ddots \end{pmatrix}}_{2n}\right. \left.\begin{pmatrix} \vdots \\ \tilde{x}_{2i}^{(l+1)} \\ \tilde{x}_{2i+1}^{(l+1)} \\ \vdots \end{pmatrix}\right\} 2n
$$



Chaikin细分的矩阵形式

$$\begin{pmatrix} \vdots \\ P_{2i-2}^{k+1} \\ P_{2i-1}^{k+1} \\ P_{2i}^{k+1} \\ P_{2i+1}^{k+1} \\ P_{2i+2}^{k+1} \\ P_{2i+3}^{k+1} \\ \vdots \end{pmatrix} = \frac{1}{4} \begin{pmatrix} \ddots & & & & & & & \iddots \\ & 0 & 3 & 1 & 0 & 0 & 0 & \\ & 0 & 1 & 3 & 0 & 0 & 0 & \\ & 0 & 0 & 3 & 1 & 0 & 0 & \\ & 0 & 0 & 1 & 3 & 0 & 0 & \\ & 0 & 0 & 0 & 3 & 1 & 0 & \\ & 0 & 0 & 0 & 1 & 3 & 0 & \\ \iddots & & & & & & & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ P_{i-2}^{k} \\ P_{i-1}^{k} \\ P_{i}^{k} \\ P_{i+1}^{k} \\ P_{i+2}^{k} \\ P_{i+3}^{k} \\ \vdots \end{pmatrix}$$

极限情况

极限曲线上的点可由细分矩阵的幂次的极限求得:

$$\begin{pmatrix} x_{-}^{[\infty]} \\ x^{[\infty]} \\ x_{+}^{[\infty]} \end{pmatrix} = \lim_{k\to\infty} \boldsymbol{M}_{subdiv}^{k} \begin{pmatrix} x_{-}^{[l]} \\ x^{[l]} \\ x_{+}^{[l]} \end{pmatrix}$$

极限情况

收敛的必要条件:

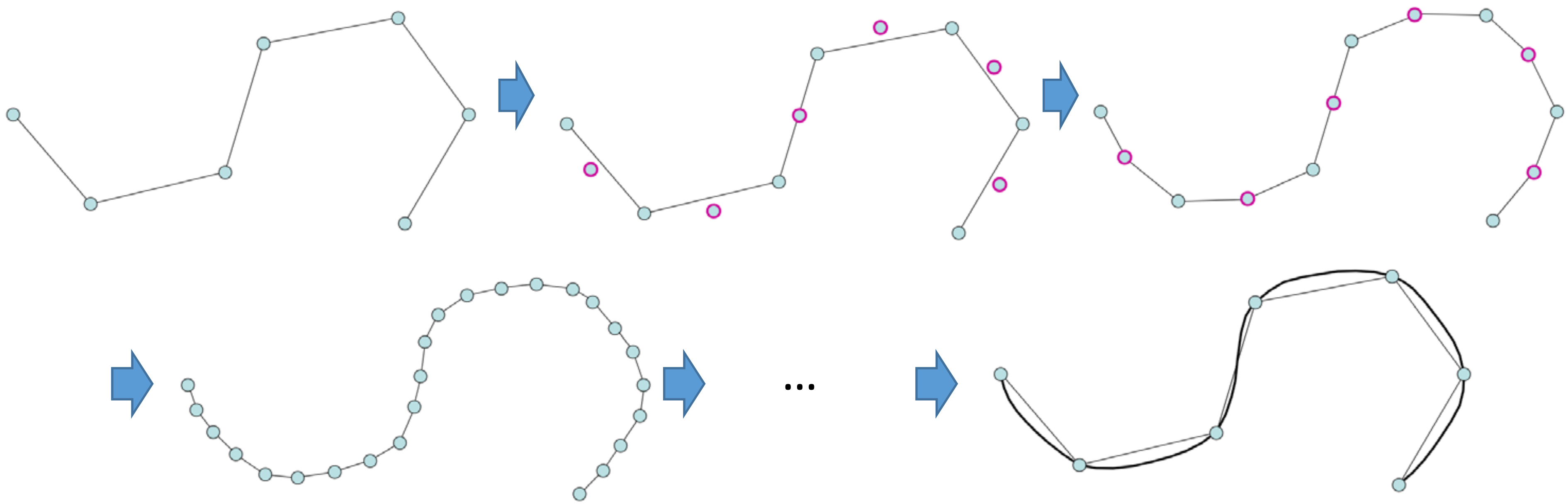
- 细分矩阵的最大特征根为1
- 否则会爆炸 (>1) 或收缩 (<1)

$$\begin{pmatrix} x_{-n}^{[l+k]} \\ \vdots \\ x_{0}^{[l+k]} \\ \vdots \\ x_{+n}^{[l+k]} \end{pmatrix} = \boldsymbol{M}_{subdiv}^{k} \begin{pmatrix} x_{-n}^{[l]} \\ \vdots \\ x_{0}^{[l]} \\ \vdots \\ x_{+n}^{[l]} \end{pmatrix} = \boldsymbol{U}\boldsymbol{D}^{k}\boldsymbol{U}^{-1} \begin{pmatrix} x_{-n}^{[l]} \\ \vdots \\ x_{0}^{[l]} \\ \vdots \\ x_{+n}^{[l]} \end{pmatrix}$$

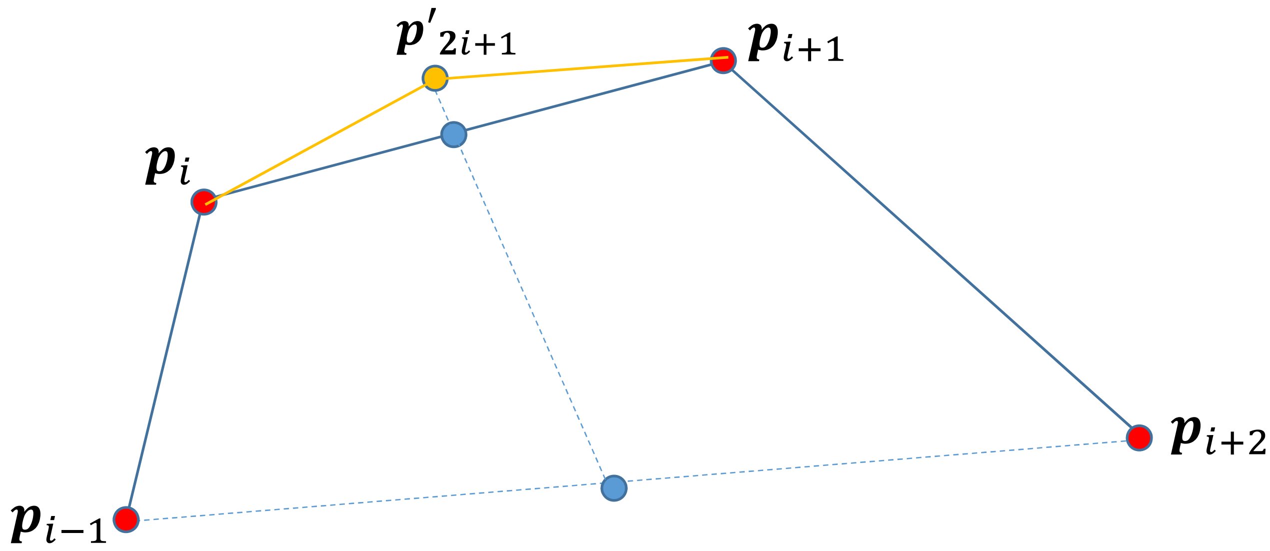
插值型细分方法

插值型细分方法

- 细分方法:
  - 保留原有顶点
  - 对每条边，增加一个新顶点
  - 不断迭代，生成一条曲线
- 可以看成是“补角法”



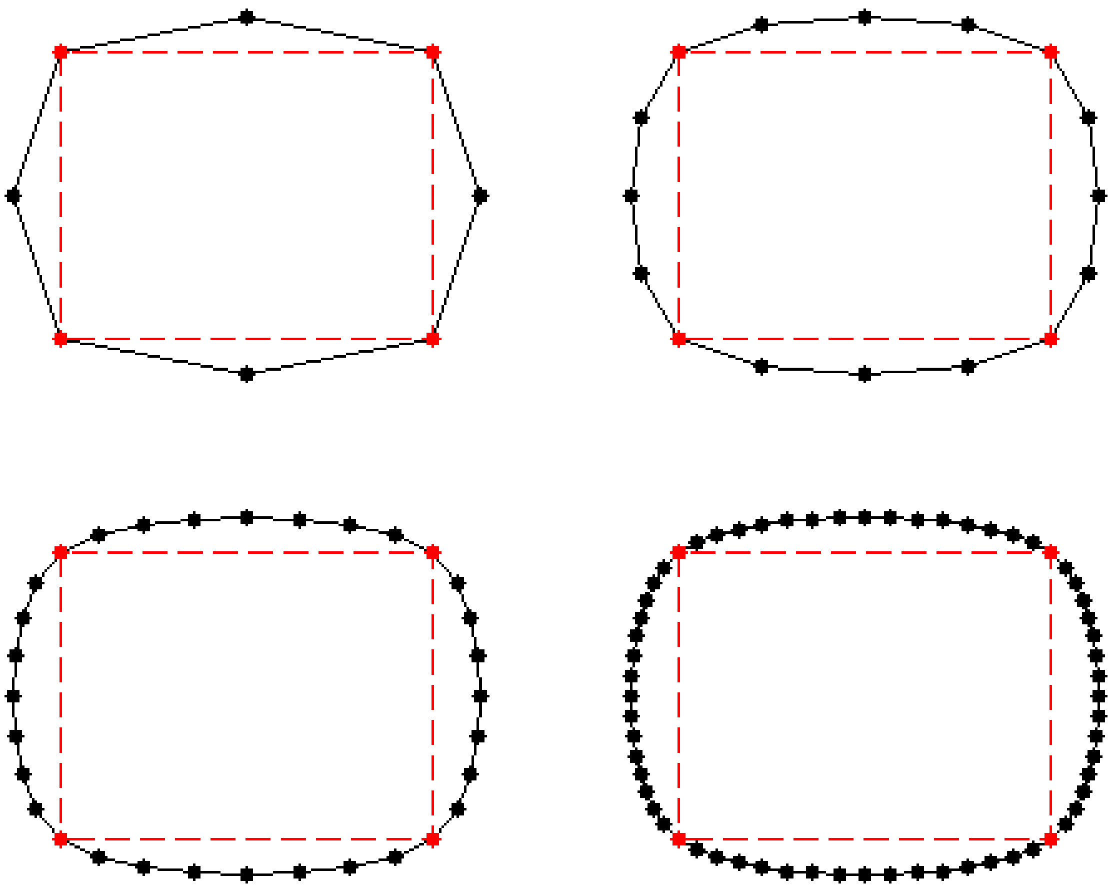
4点插值型细分规则



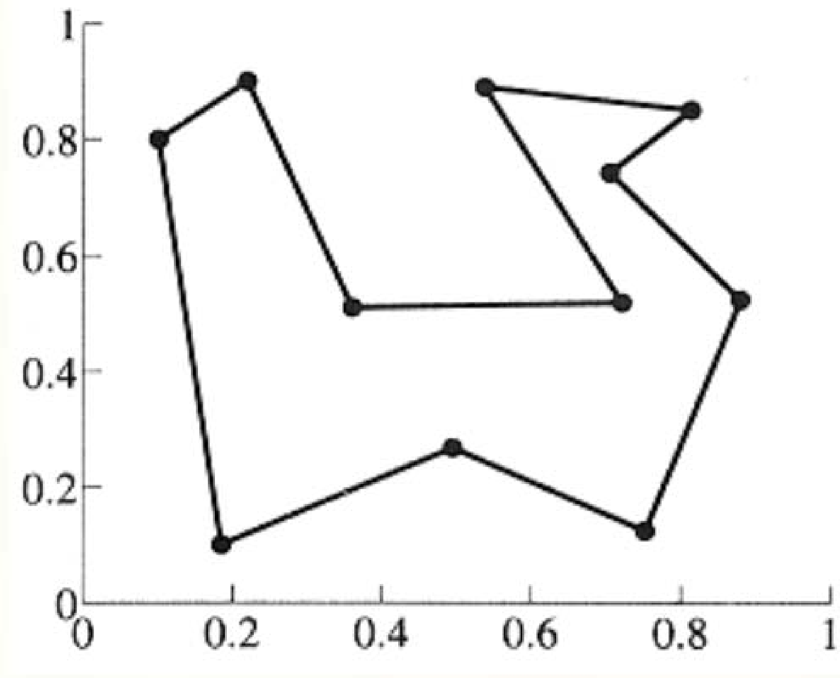
$$\boldsymbol{p}'_{2i+1} = \frac{\boldsymbol{p}_i + \boldsymbol{p}_{i+1}}{2} + \alpha\left(\frac{\boldsymbol{p}_i + \boldsymbol{p}_{i+1}}{2} - \frac{\boldsymbol{p}_{i-1} + \boldsymbol{p}_{i+2}}{2}\right)$$

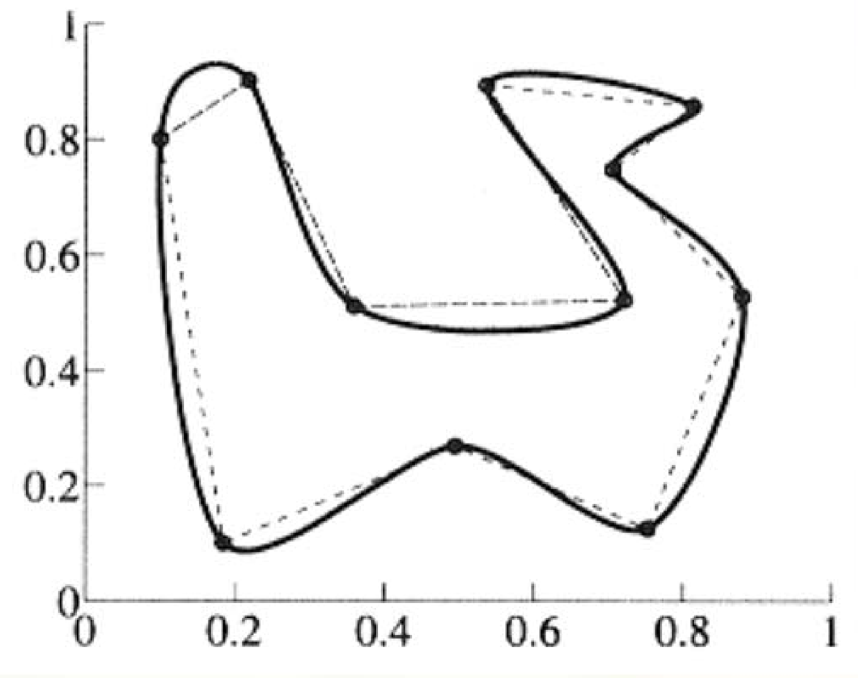
Nira Dyn, David Levin, John A. Gregory. A 4-point interpolatory subdivision scheme for curve design. Computer Aided Geometric Design, 4(4): 257-268, 1987.

4点插值型细分曲线的例子

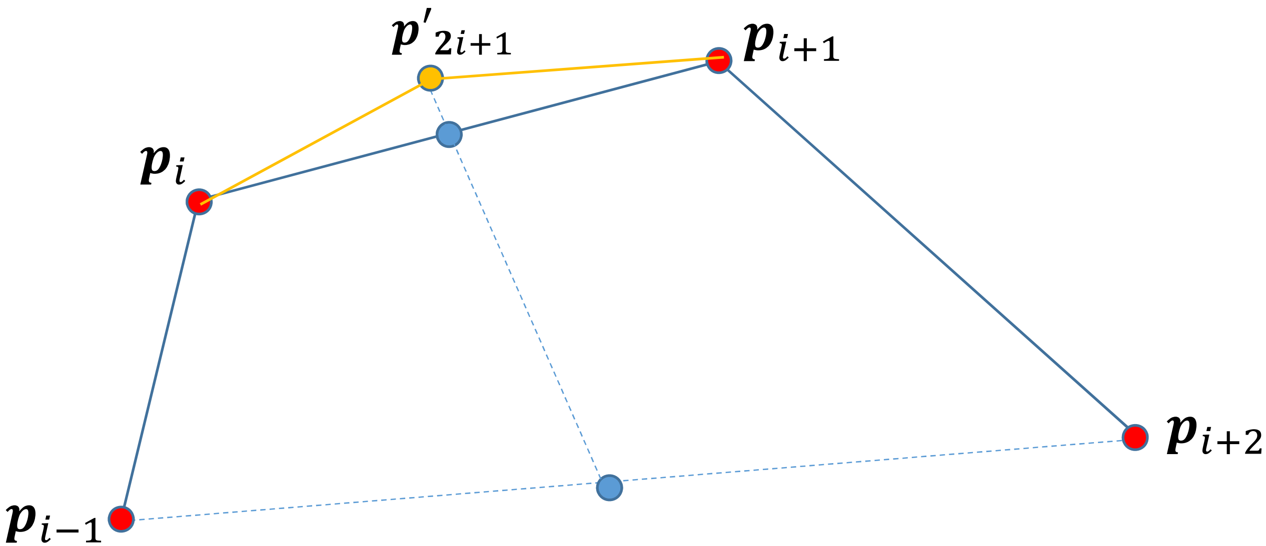


4点插值型细分曲线的例子





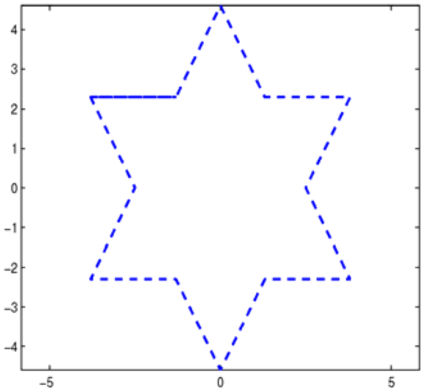
4点插值型细分规则



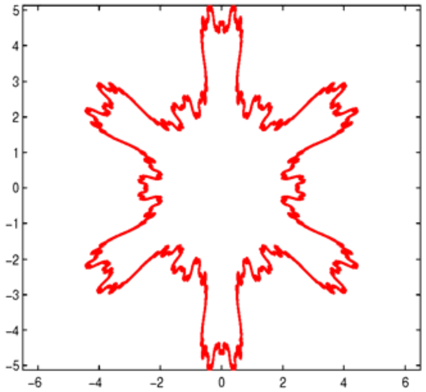
$$\boldsymbol{p'}_{2i+1} = \frac{\boldsymbol{p}_i + \boldsymbol{p}_{i+1}}{2} + \alpha\left(\frac{\boldsymbol{p}_i + \boldsymbol{p}_{i+1}}{2} - \frac{\boldsymbol{p}_{i-1} + \boldsymbol{p}_{i+2}}{2}\right)$$

可以证明：当$\alpha \in (0, \frac{1}{8})$时，生成的细分曲线是光滑的；否则，细分曲线非光滑，生成了分形曲线。

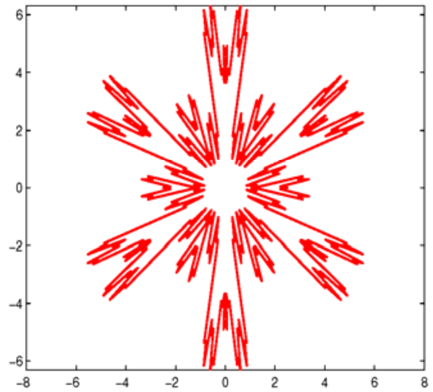
4点细分曲线的例子



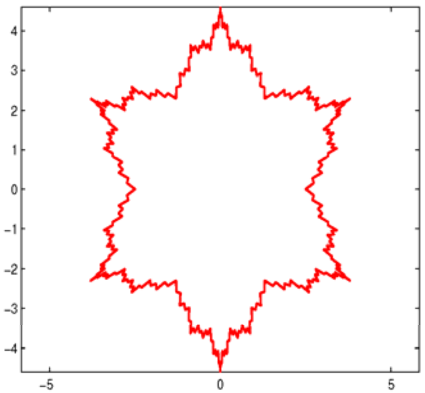
(a)



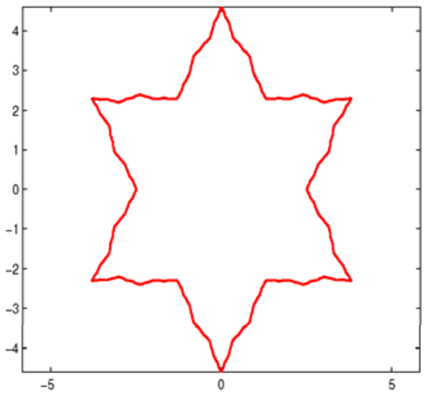
(b)



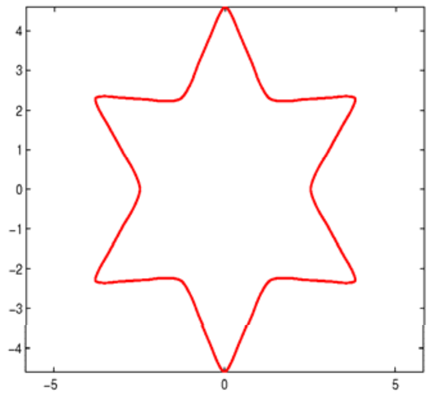
(c)



(d)



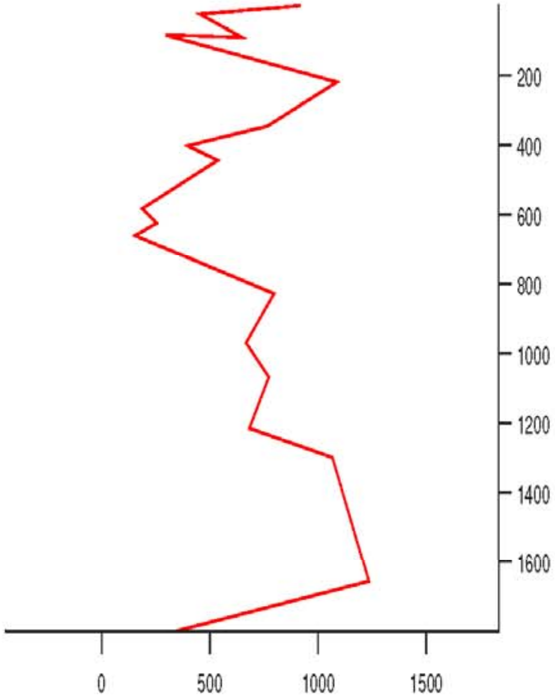
(e)



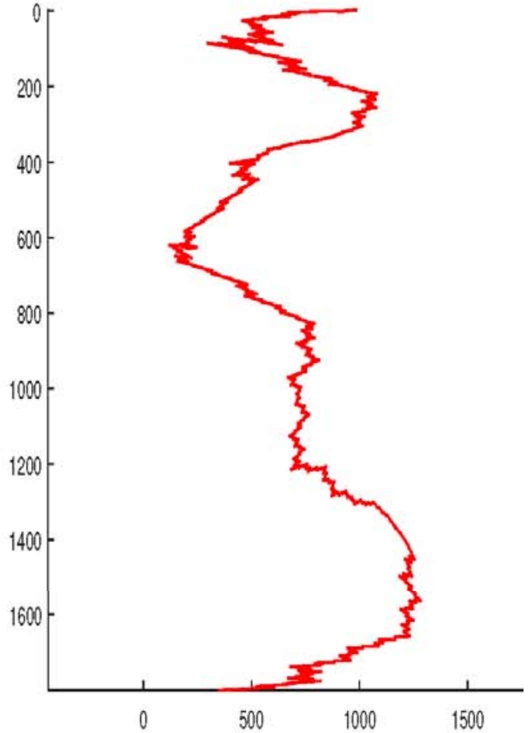
(f)

分形曲线（分数维）：分形几何

(a)



(b)



(c)

一般：2*n*点插值细分方法

- 连续阶随着*n*增大而增加

2点插值细分方法

$$\boldsymbol{P}_{2i+1}^{k+1} = \frac{1}{2}(\boldsymbol{P}_i^k + \boldsymbol{P}_{i+1}^k)$$

4点插值细分方法

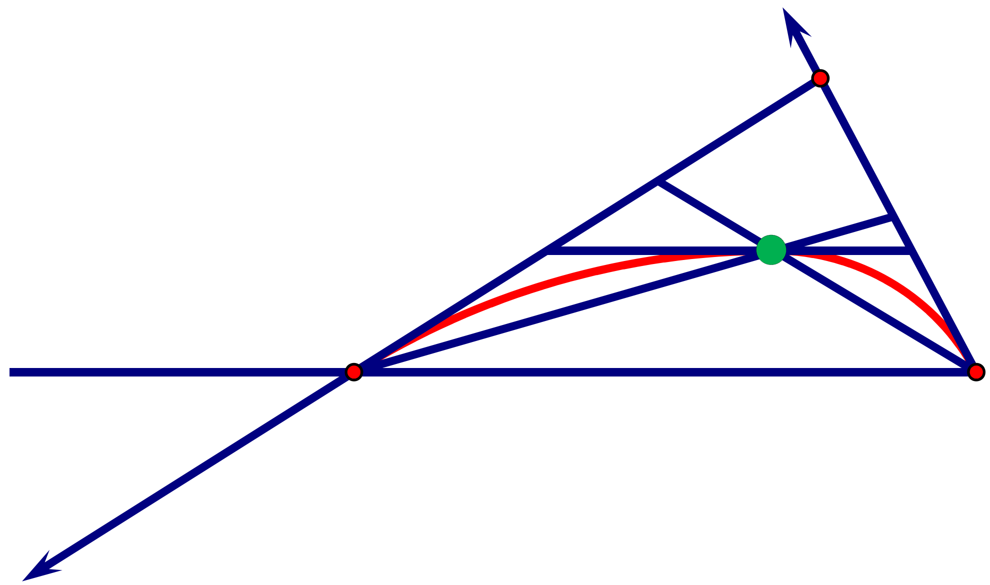
$$\boldsymbol{P}_{2i+1}^{k+1} = -\frac{1}{16}\boldsymbol{P}_{i-1}^k + \frac{9}{16}\boldsymbol{P}_i^k + \frac{9}{16}\boldsymbol{P}_{i+1}^k - \frac{1}{16}\boldsymbol{P}_{i+2}^k$$

6点插值细分方法

$$\begin{aligned}\boldsymbol{P}_{2i+1}^{k+1} \;=\; & \frac{3}{256}\boldsymbol{P}_{i-2}^k - \frac{25}{256}\boldsymbol{P}_{i-1}^k + \frac{150}{256}\boldsymbol{P}_i^k + \frac{150}{256}\boldsymbol{P}_{i+1}^k \\ & -\frac{25}{256}\boldsymbol{P}_{i+2}^k + \frac{3}{256}\boldsymbol{P}_{i+3}^k.\end{aligned}$$

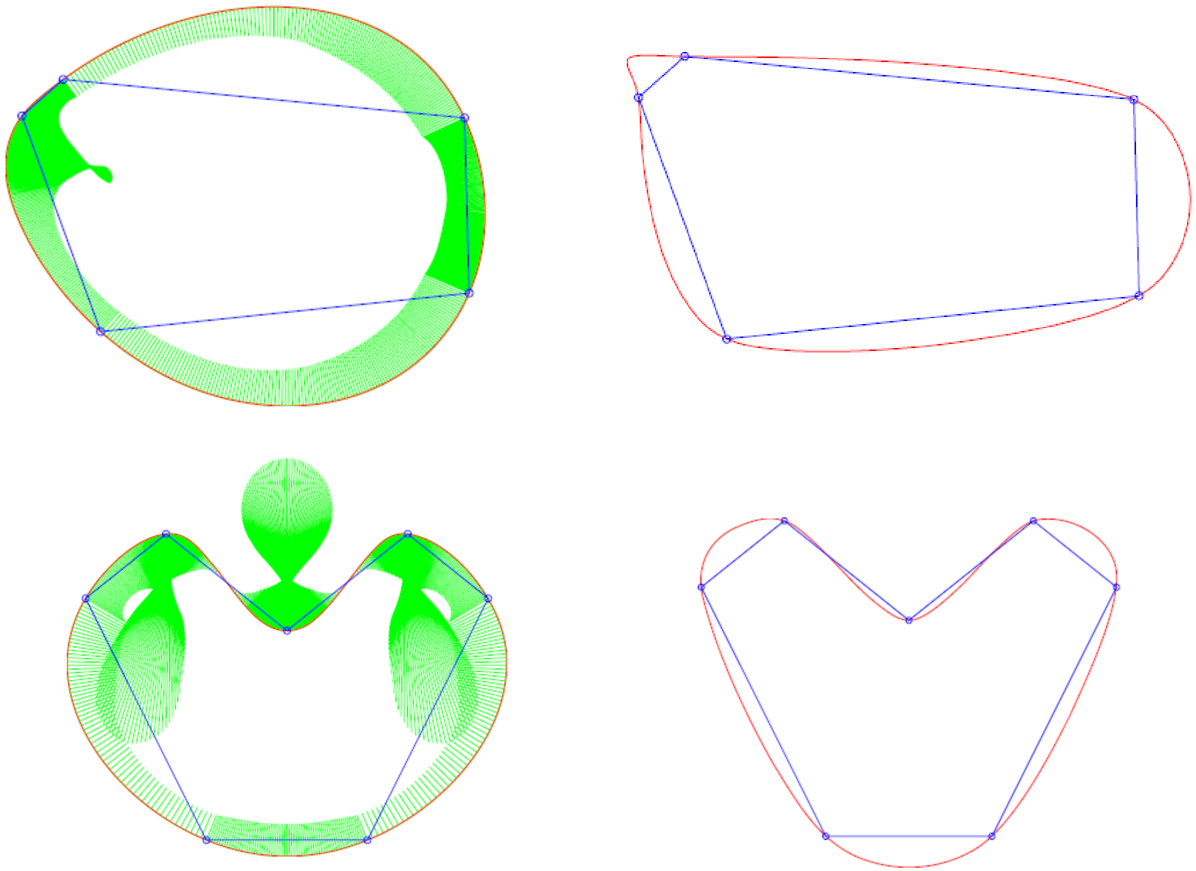
非线性细分方法

- 基于双圆弧插值的曲线细分方法
  - 给定一条边,新点为插值其两端点及两端切向的双圆弧的一个连接点,也是其两端点两端切向的所确定三角形的内心.
  - 每个细分步骤后调整切向.



基于双圆弧插值的曲线细分方法

- 性质（证明稍难）
  - 极限曲线G^2，光顺，保形



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- Warren and Weimer. Subdivision Methods for Geometric Design: A Constructive Approach. Morgan Kaufmann Publishers, 2002
- M.S. Sabin. Recent Progress in Subdivision: a Survey. Advances in Multiresolution for Geometric Modelling, Mathematics and Visualization 2005, 203-230
- Cashman. Beyond Catmull–Clark? A survey of advances in subdivision surface methods. Compute Graphics Forum, 31(1), 2012, 42–61

作业5

- 任务： 实现两种细分曲线的生成方法
  - 逼近型细分：Chaikin方法（二次B样条）、三次B样条细分方法
  - 插值型细分：4点细分方法
- 目的
  - 学习使用细分方法生成曲线的原理和方法
- Deadline： 2020 年11 月21日晚

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谢 谢 ！