中国科学技术大学
University of Science and Technology of China

GAMES102

GAMES 102在线课程

几何建模与处理基础

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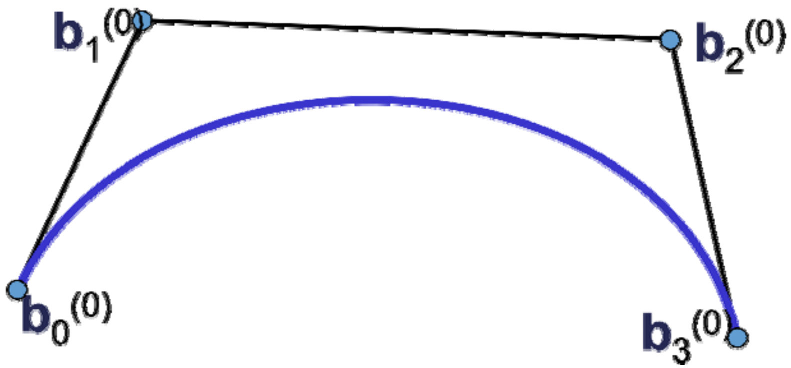
GAMES 102在线课程：几何建模与处理基础

隐式曲线

回顾：参数曲线

- 曲线定义在一个单参数t的区间上，有t上的基函数来线性组合控制顶点来定义

$$x(t) = \sum_{i=0}^{n} B_i^n(t) b_i$$



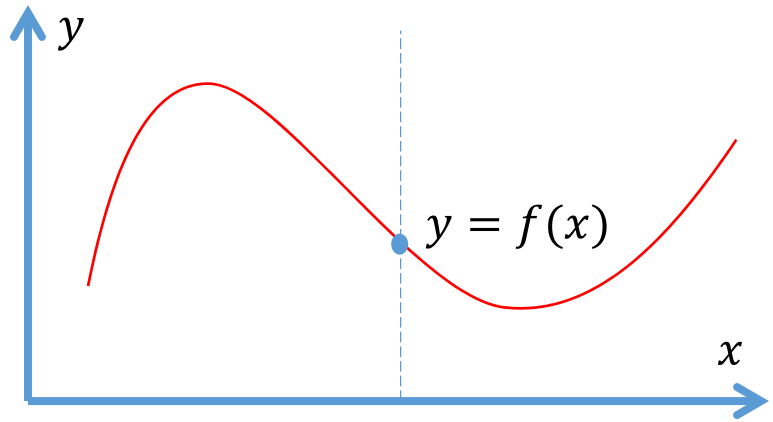
曲线的性质来源于基函数的性质

回顾：平面曲线的定义方法

- 显式函数

$$f: R^1 \to R^1$$
$$y = f(x)$$

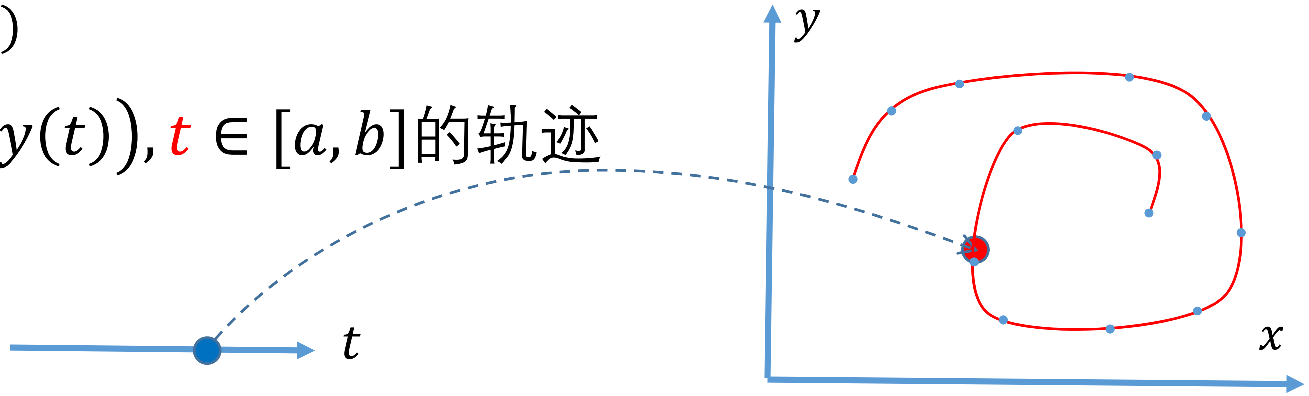
  - 点$(x, f(x)), x \in [a, b]$的轨迹



- 参数曲线

$$\boldsymbol{p}: R^1 \to R^2$$
$$x = x(t)$$
$$y = y(t)$$

  - 点$(x(t), y(t)), t \in [a, b]$的轨迹



隐式函数

- 自变量x和应变量y的关系非显式关系，是一个隐式的关系（代数方程）：
$$f(x, y) = 0$$
- 比如:
  - $ax + by + c = 0$
  - $x^2 + y^2$=1
  - $y^2 = x^3 + ax + b$
  - $xy^2 + \ln(x \sin y - e^{y-\sqrt{x}}) = \cos(x - \sqrt{x^3 - 2y})$

所有满足该代数方程的点的轨迹是条曲线

隐函数定理

Implicit Function Theorem:

- Given a *differentiable* function $f: \mathbb{R}^n \supseteq D \to \mathbb{R}, f\left(\boldsymbol{x}^{(0)}\right) = 0, \frac{\partial}{\partial x_n} f\left(\boldsymbol{x}^{(0)}\right) = \frac{\partial}{\partial x_n} f\left(x_1^{(0)}, \dots, x_n^{(0)}\right) \neq 0$
- Within an ε-neighborhood of $\boldsymbol{x}^{(0)}$ we can represent the zero level set of f completely as a heightfield function g
  $g: \mathbb{R}^{n-1} \to \mathbb{R}$ such that for $\boldsymbol{x} - \boldsymbol{x}^{(0)} < \varepsilon$ we have:
  $f\left(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})\right) = 0$ and
  $f(x_1, \dots, x_n) \neq 0$ everywhere else
- The heightfield is a differentiable $(n-1)$-manifold and its surface normal is the colinear to the gradient of f.

隐式曲线

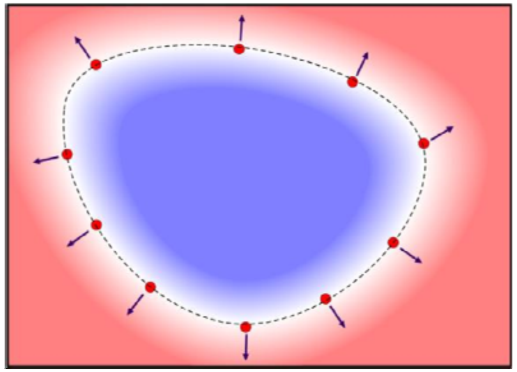
- 将隐函数升高一维，看成是x和y的二元函数

$$z = f(x, y), \qquad x, y \in [a, b] \times [c, d]$$

- 则该隐式曲线为上述二元函数的0等值线（平面$z = 0$ 与$z = f(x, y)$的交线）

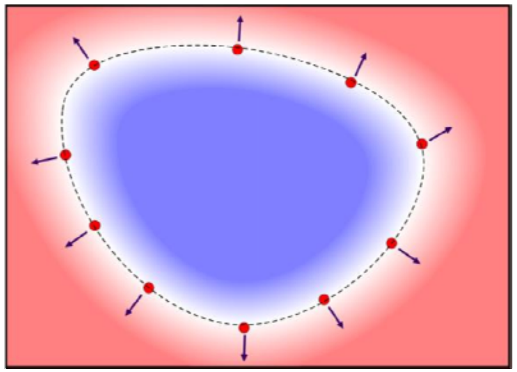
$$f(x, y) = 0$$

- $f(x, y) = 0$，曲线上；
- $f(x, y) < 0$，曲线的左侧（内部）；
- $f(x, y) > 0$，曲线的右侧（外部）；



隐式函数表达

- 已知一条封闭曲线，如何构造隐式函数表达？
  - General case
    - Non-zero gradient at zero crossings
    - Otherwise arbitrary
  - Signed implicit function:
    - sign(f): negative inside and positive outside the object
      (or the other way round, but we assume this orientation here)
  - Signed distance field (SDF)
    - $|f|$ = distance to the surface
    - sign(f): negative inside, positive outside
  - Squared distance function
    - f = (distance to the surface)^2



Differential Properties

Some useful differential properties:

- We look at a surface point $\boldsymbol{x}$, i.e. $f(\boldsymbol{x}) = 0$.
- We assume $\nabla f(\boldsymbol{x}) \neq 0$.
- The unit normal of the implicit surface is given by:

$$n(\boldsymbol{x}) = \frac{\nabla f(\boldsymbol{x})}{\|\nabla f(\boldsymbol{x})\|}$$

  - For signed functions, the normal is pointing outward
  - For signed distance functions, this simplifies to $\boldsymbol{n}(\boldsymbol{x}) = \nabla f(\boldsymbol{x})$

Differential Properties

Some useful differential properties:

- The mean curvature of the surface is proportional to the divergence of the unit normal:

$$-2H(\boldsymbol{x}) = \nabla \cdot \boldsymbol{n}(\boldsymbol{x}) = \frac{\partial}{\partial x} n_x(\boldsymbol{x}) + \frac{\partial}{\partial y} n_y(\boldsymbol{x}) + \frac{\partial}{\partial z} n_z(\boldsymbol{x})$$

$$= \nabla \cdot \frac{\nabla f(\boldsymbol{x})}{\|\nabla f(\boldsymbol{x})\|}$$

- For a signed distance function, the formula simplifies to:

$$-2H(\boldsymbol{x}) = \nabla \cdot \nabla f(x) = \frac{\partial^2}{\partial x^2} f(\boldsymbol{x}) + \frac{\partial^2}{\partial y^2} f(\boldsymbol{x}) + \frac{\partial^2}{\partial z^2} f(\boldsymbol{x}) = \Delta f(\boldsymbol{x})$$

隐式曲线的绘制

如何找隐式函数表达的点的集合?

- 自变量x和应变量y的关系非显式关系，是一个隐式的关系（代数方程）：

$$f(x, y) = 0$$

- 比如:
  - $ax + by + c = 0$
  - $x^2 + y^2$=1
  - $y^2 = x^3 + ax + b$
  - $xy^2 + \ln(x \sin y - e^{y-\sqrt{x}}) = \cos(x - \sqrt{x^3 - 2y})$

等值线抽取

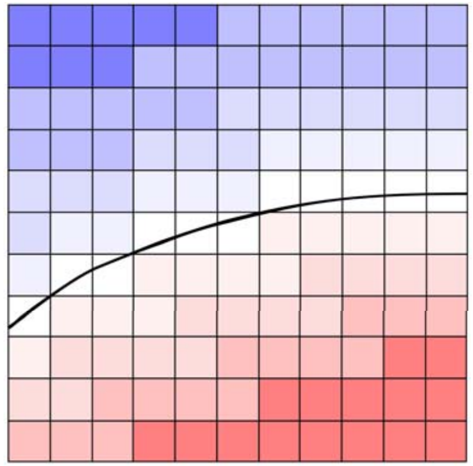
- 输入：一个二元隐式函数$z = f(x, y)$
- 输出：值为0(或a)的等值线$z = 0$（或$z - a = 0$）

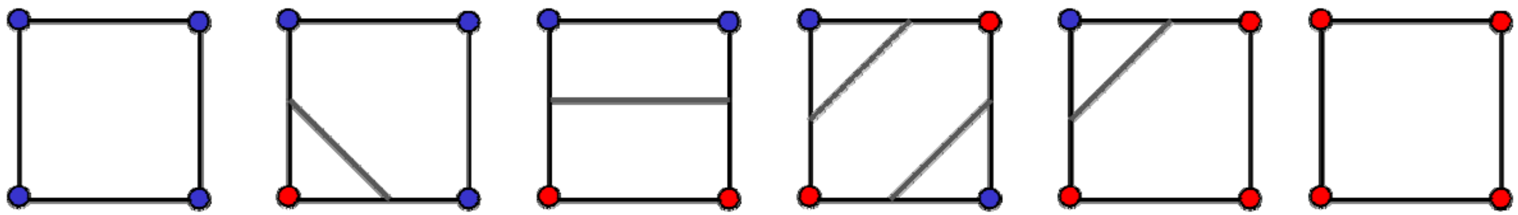
- 目的：
  - 将隐式曲线转化为参数形式、离散曲线（多边形）形式
  - 绘制曲线

Marching Cubes算法[Siggraph1987]

- 隐式曲线绘制的最常用方法
  - 网上能找到很多开源实现代码

- 思想（2D: Marching Squares）
  - 在一些离散格子点上求值
  - 然后利用局部连续性插值出值为0的点
  - 按一定的顺序连接这些点形成离散曲线

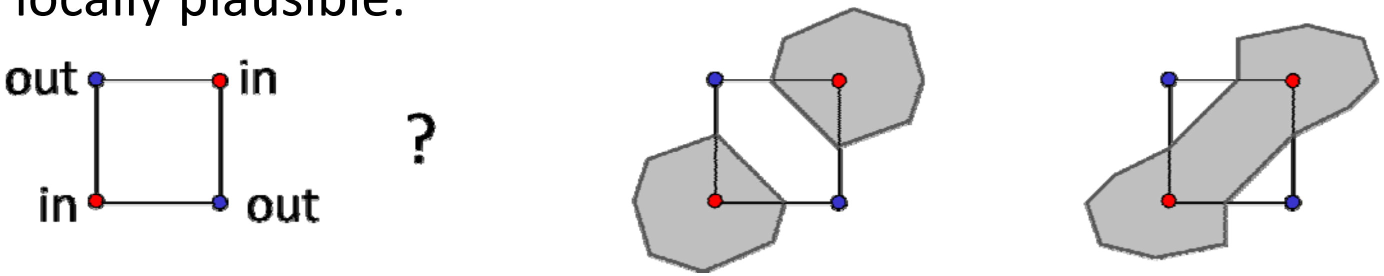




歧义情况

There is a (minor) technical problem remaining:

- The triangulation can be ambiguous
- In some cases, different topologies are possible which are all locally plausible:

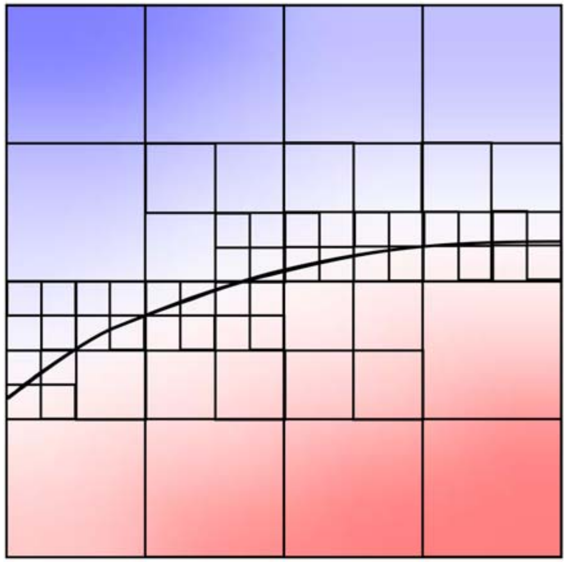


- This is an *undersampling artifact*. At a sufficiently high resolution, this cannot occur.
- Problem: Inconsistent application can lead to holes in the surface (non-manifold solutions)

Adaptive Grids

Adaptive / hierarchical grids:

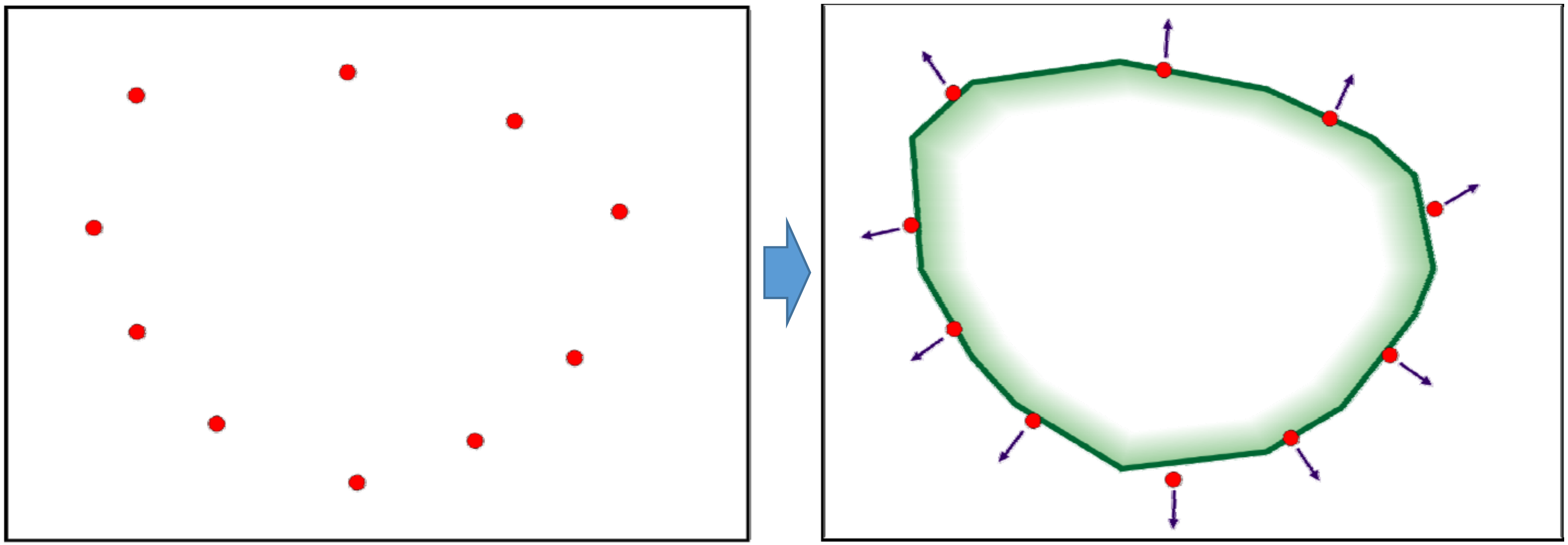
- Perform a quadtree / octree tessellation of the domain (or any other partition into elements)
- Refine where more precision is necessary (near surface, maybe curvature dependent)
- Associate basis functions with each cell (constant or higher order)



隐式曲线拟合

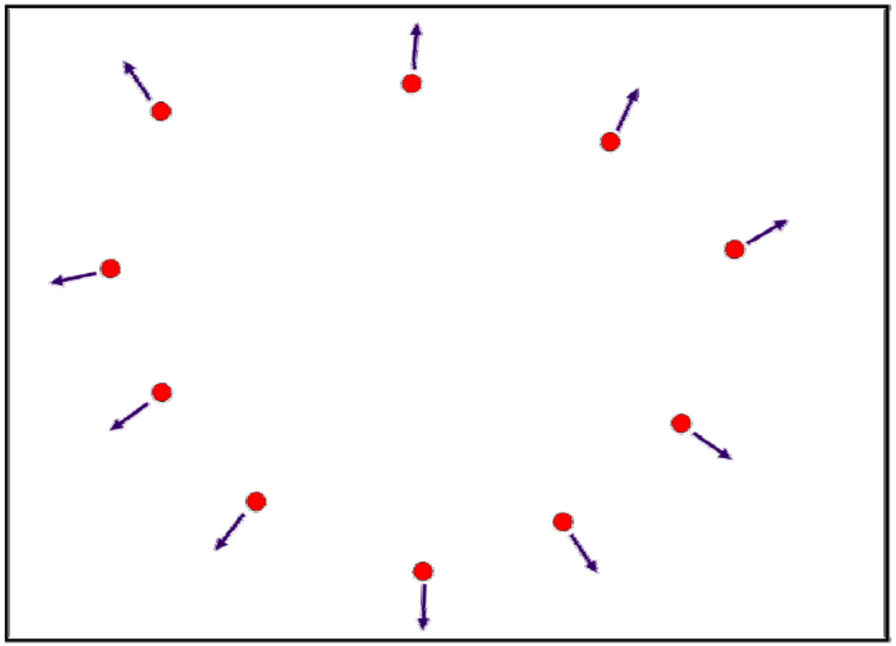
问题

- 输入：平面上的一些点（设采样自封闭曲线）
  - 一般还需给定或估计点的法向信息
- 输出：拟合这些点的一个隐式函数
  - 该隐式函数所表达的曲线就是拟合曲线



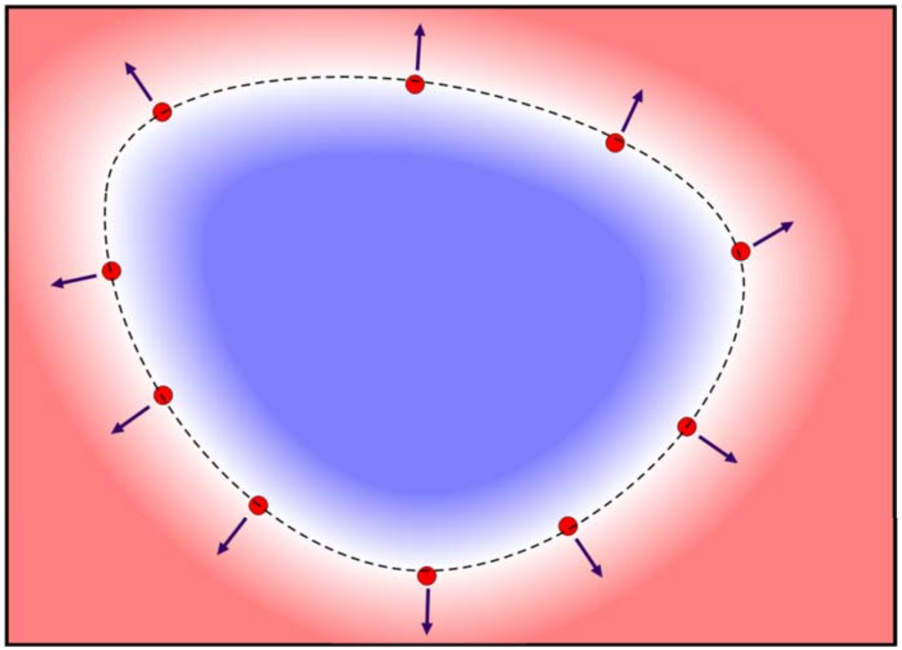
拟合问题的求解步骤

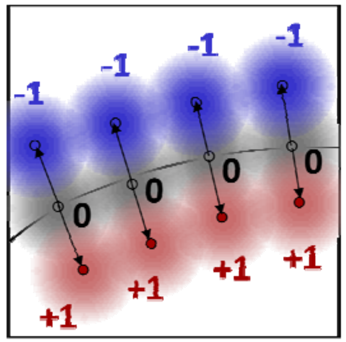
- 1. 估计法向：利用邻近点来估计切平面

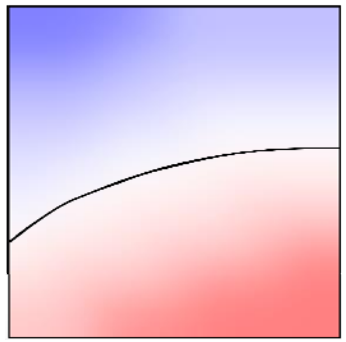


拟合问题的求解步骤

- 1. 估计法向：利用邻近点来估计切平面
- 2. 拟合一个二元函数：在型值点上值为0，外部（法向方向的点）为正，内部为负

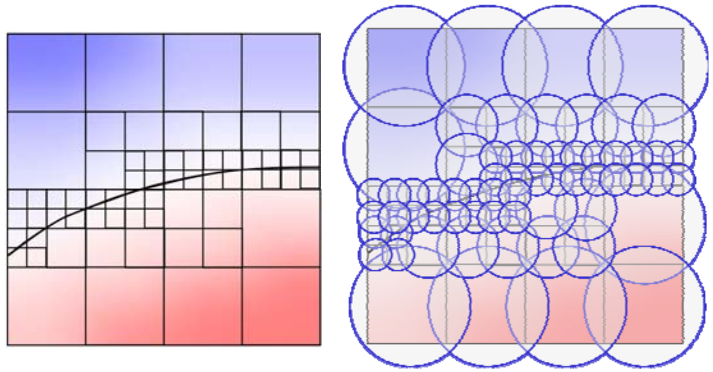






隐式函数构造方法

- Blobby molecules
- Metaball
- RBF based method
- Multi-level partition of unity implicits (MPU)
- Poisson reconstruction method
- Screened Poisson method
- …



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谢 谢 ！