The Pricing of Basket Options: A Weak Convergence Approach

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Abstract

We consider a limit price of basket options in a large portfolio where the dynamics of basket assets is described as a CEV jump diffusion system. The explicit representation of the limiting price is established using weak convergence of empirical measure valued processes generated by the system. As an application, the closed-form formula of the limit price is derived when the price dynamics of basket assets follows a mixed-double exponential jump-diffusion system.

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1 Introduction

A basket option is an option whose payoff is a weighted sum or average of prices of the two or more risky assets that have been grouped together in a basket at maturity. In general, it is difficult to price basket options explicitly since the joint distribution of the underlying basket asset price process are unknown due to multi-dimensionality, in particular for the multi-dimensional jump-diffusion basket asset price dynamics. Instead, some works have focused on the development of fast and accurate approximation techniques and establishment of sharp lower and upper bounds for basket option prices, see [12] therein they review and compare six different methods for valuing basket options in a systematic way, and discuss the influence of model parameters on the performance of the different approximations. Recently [14] apply the asymptotic expansion method to find the approximating value of the lower bound of European style basket call option prices for a local volatility jump-diffusion model.

In this paper, we are concerned with the pricing of a class of basket options in a large portfolio of the underlying basket assets when the price dynamics of basket assets is described as a multi-dimensional CEV-type jump-diffusion process. This is related to the limit characterization problem of the risk-neutral expectation for the basket option payoff at maturity. Differently from [12] and [14] reviewed above, we here mainly focus on the explicit representation of the limiting price of basket options as a pricing problem for the large portfolio of risky assets considered in [2]. In addition, the CEV-type model considered herein can also capture the local volatility, see also [8] and [7]. In practise, our explicit limit price presentation can be also used to valuate basket options when the number of risky assets in the portfolio is relatively large, see also [1] which reviews the VaR method applied to revalue complex options in a large portfolio. The method we used in this paper heavily depends on the technique of weak convergence for empirical measure-valued processes generated by the weighted price of risky assets, parameter set and functions of jumps' size of the price dynamics, see also [10] and [3] which apply the weak convergence method to study default clustering and systemic risk. We will prove that the sequence of the above empirical measure-valued processes is relatively compact on the Skorohod space and identify the limit of the sequence of the empirical

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measure-valued processes explicitly in the distributional sense. We here point out that our weak convergence analysis as a limiting martingale problem is not degenerate and hence the limit is also a measure-valued random process which is not the deterministic limit case as in [10] and [3]. [11] also study the weak convergence of the empirical measure-valued processes. They prove that the density of the limiting measure can be characterized as the solution to a nonlinear SPDE. However, they do not apply this density to the pricing issue. Using the limit as a measure-valued process, we characterize the limit price of basket options by employing Skorohod's representation theorem and Vitali's convergence theorem. Finally we derive the closed-form limit price when the price dynamics of the underlying basket assets is modeled as a multi-dimensional mixed-exponential jump-diffusion (the one-dimensional case is first considered by [6]) via Laplace transform.

The rest of the paper is organized as follows. Section 2 introduces the multi-dimensional CEV-type jump-diffusion model for basket options. Section 3 provides a detailed weak convergence analysis for empirical measure-valued processes generated by the weighted price of risky assets, parameter set and functions of jumps' size of the price dynamics. Section 4 proves the explicit limit of basket option price using the weak convergence obtained in Section 3. A numerical example is presented in Section 5.

2 Basket Options

In this section, we describe the model and price representation for basket options. We consider N underlying risky assets (e.g. stocks) in the financial market. The price of a basket of these N risky assets is then defined as the weighted sum of the prices of N risky assets at time t, i.e.,

$$B_t^N = \sum_{i=1}^N w_N^i S_t^i, \qquad t \geq 0, \tag{1}$$

where the weights w_N^i, $i = 1, \ldots, N$, are assumed to depend on the number of the underlying risky assets. Here we consider the weights with the form $w_N^i = \frac{1}{N} w_i$ for $i = 1, \ldots, N$, where constants $w_i \in \mathbb{R}$ are independent of N. If $w_i = 1$, then B_t^N is the average of time-t prices of N risky assets that have been grouped together in the basket.

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a risk-neutral probability space, under which, the price dynamics of the i-th risky asset is assumed to satisfy the following CEV-type jump-diffusion process, for $i = 1, \ldots, N$,

$$\frac{dS_t^i}{S_{t-}^i} = (r - v_i)dt + \sum_{j=1}^d \sigma_{ij} \left(S_t^i\right)^\beta dW_t^j + \int_{\mathcal{U}} \ell_i(y) \tilde{Q}(dy, ds), \quad S_0^i = S_{0-}^i > 0, \tag{2}$$

where $r > 0$ is the interest rate, $v_i > 0$ is the dividend yield of the i-th risky asset, $\sigma_{ij} > 0$ is the volatility of the i-th risky asset subject to the j-th risky asset, and $W_t = (W_t^i;\ i = 1, \ldots, d)$, $t \geq 0$, is a d-dimensional Brownian motion. Here $\beta \in [-\frac{1}{2}, 0]$ is the constant-elasticity-of-variance (CEV) parameter. Since $1 + \beta \in [\frac{1}{2}, 1]$, the existence and uniqueness of the strong solution of SDE (2) can be guaranteed by Theorem 9.1 on page 231 of [13] under the assumption **(A1)** below. Further, if $\beta = -\frac{1}{2}$, it corresponds to a CIR process with jumps, while if $\beta = 0$, it is an exponential Lévy process. For $\beta \in (-\frac{1}{2}, 0)$, it is called a CEV process with jumps. In particular, if the CEV parameter $\beta = 0$ and the jump function $\ell_i(y) \equiv 0$, then the price dynamics (2) is reduced to the multi-dimensional geometric Brownian motion model considered in [12].

Here $\mathcal{U}$ denotes a topological space and ν is a σ-finite Borel measure on $\mathcal{U}$. Further $Q(dy, ds)$ denotes a Poisson random measure which is independent of d-dimensional Brownian motion $W = (W_t;\ t \geq 0)$. Correspondingly $\tilde{Q}(dy, ds) := Q(dy, ds) - \nu(dy)ds$ denotes the compensated Poisson random measure with compensator $\nu(dy)ds$. The measurable function of jump's size $\ell_i : \mathcal{U} \to \mathbb{R}$ is assumed to satisfy $1 + \ell_i(y) > 0$ for all $(y, i) \in \mathcal{U} \times \mathbb{N}$. This condition can guarantee that the i-th risky asset price is still positive after a common jump due to Poisson random measure.

Let $\mathbb{F} = (\mathcal{F}_t;\ t \geq 0)$ with $\mathcal{F}_t$ being the right-continuous augmented filtration generated jointly by (W, Q) up to time t. Let $T > 0$ be the maturity of the basket option and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be the payoff function. Here we assume that ϕ is continuous and admits the growth condition $|\phi(x)| \leq C\,(1 + |x|^{\kappa})$ with $\kappa > 0$. We consider the payoff associated with our basket option given by $\phi\left(B_T^N\right)$. Here, for a strike price $K > 0$, $\phi(x) = \{\theta(x - K)\}^+$ for $x \in \mathbb{R}$ corresponds to the call basket option if $\theta = 1$, and corresponds to the put basket option if $\theta = -1$. In this case, the payoff function ϕ satisfies the linear growth condition (i.e. $\kappa = 1$). Thus the price of the basket option is given by

$$V^N(\phi) := e^{-rT}\mathbb{E}\left[\phi\left(\frac{1}{N}\sum_{i=1}^{N} w_i S_T^i\right)\right]. \tag{3}$$

Here $\mathbb{E}$ denotes the expectation w.r.t. the risk-neutral measure $\mathbb{Q}$. The main of the paper is to characterize the limit of the price $V^N(\phi)$ analytically as $N \to +\infty$ by adopting the technique of weak convergence which will be implemented in the coming section.

3 Weak Convergence Analysis

In this section, we provide a detailed analysis of weak convergence for the basket option price under the multi-dimensional CEV-type jump-diffusion model described as (2).

We collect the parameters associated with the price dynamics of the i-th risky asset as follows, for $i \in \mathbb{N}$,

$$p_i = (v_i, \sigma_{i1}, \ldots, \sigma_{id}, w_i) \in \mathcal{O}_p := \mathbb{R}_+^{d+1} \times \mathbb{R}. \tag{4}$$

Let $\mathcal{C}$ denote the set of all Borel measurable functions on $\mathcal{U}$ and $\mathcal{O} := \mathcal{O}_p \times \mathcal{C} \times \mathbb{R}$. Then we can define the sequence of empirical measure-valued processes as, for $N \in \mathbb{N}$,

$$\nu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{(p_i, \ell_i, X_t^i)}, \qquad t \geq 0 \tag{5}$$

on the Borel field $\mathcal{B}(\mathcal{O})$. Here δ denotes the Dirac-delta measure on $\mathcal{O}$ and the i-th weighted price process, for $i = 1, \ldots, N$,

$$X_t^i := w_i S_t^i, \qquad t \geq 0. \tag{6}$$

Let $S = \mathcal{P}(\mathcal{O})$, i.e., the set of all Borel probability measures on $\mathcal{O}$. Then $\nu^N = (\nu_t^N;\ t \geq 0)$ can be viewed as an S-valued right continuous with left limits (r.c.l.l.) stochastic process for each $N \in \mathbb{N}$. For any smooth function $f(p, y, x) \in C^\infty(\mathcal{O})$ defined on $(p, y, x) \in \mathcal{O}$ and $\nu \in S$, define the integral w.r.t. the Borel probability measure ν by $\nu(f) := \int_{\mathcal{O}} f(p, y, x)\nu(dp \times dy \times dx)$. Then it holds that for $f \in C^\infty(\mathcal{O})$,

$$\nu_t^N(f) = \frac{1}{N}\sum_{i=1}^{N} f\left(p_i, \ell_i, X_t^i\right), \qquad t \geq 0.$$

Moreover, it follows that from (2) that, for $i = 1, \ldots, N$, X_t, $t \geq 0$, satisfies the following SDE with jumps, $X_0^i = X_{0-}^i = w_i S_0^i$, and

$$\frac{dX_t^i}{X_{t-}^i} = (r - v_i)dt + \sum_{j=1}^{d} \sigma_{ij} w_i^{-\beta}\left(X_t^i\right)^{\beta} dW_t^j + \int_{\mathcal{U}} \ell_i(y)\tilde{Q}(dy, dt). \tag{7}$$

We next estimate the moment of the above weighted price process which will be used to verify the relative compactness of the sequence of empirical measure-valued processes $(\nu^N;\ N \in \mathbb{N})$ given by (5). To this purpose, we impose the following assumption on the model parameters:

(A1) The parameter set $(|p_i|, S_0^i;\ i \in \mathbb{N})$ and $(\sum_{l=1}^4 \int_{\mathcal{U}} |\ell_i(y)|^l \nu(dy);\ i \in \mathbb{N})$ are dominated by a constant $C_{p,\ell} > 0$.

Then we have

Lemma 3.1. *Under the assumption* **(A1)**, *for any $\alpha \in [1,4]$ and any $T > 0$, it holds that*

$$\sup_{N\in\mathbb{N},\ t\in[0,T]} \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\left|X_t^i\right|^\alpha\right] < +\infty. \tag{8}$$

Proof. Itô's formula implies that, for all $t \geq 0$,

$$\begin{aligned}\mathbb{E}\left[\left|X_t^i\right|^\alpha\right] &= \mathbb{E}\left[\left|w_i S_0^i\right|^\alpha\right] + \delta(r - v_i)\mathbb{E}\left[\int_0^t \left|X_s^i\right|^\alpha ds\right] + \frac{\alpha(\alpha-1)}{2}\bar{\sigma}_i^2 w_i^{-2\beta}\mathbb{E}\left[\int_0^t \left|X_s^i\right|^{2\beta+\alpha} ds\right] \\ &\quad + \mathbb{E}\left[\int_0^t \left|X_s^i\right|^\alpha \left(\int_{\mathcal{U}} \left((1+\ell_i(y))^\alpha - 1 - \alpha\ell_i(y)\right)\nu(dy)\right) ds\right].\end{aligned}$$

Here $\bar{\sigma}_i^2 = \sum_{j=1}^d \sigma_{ij}^2$ for $i = 1, \ldots, N$. Notice that $2\beta + \alpha \in [\alpha - 1, \alpha]$. Then using Young's inequality, it follows that

$$\mathbb{E}\left[\int_0^t \left|X_s^i\right|^{2\beta+\alpha} ds\right] \leq \frac{2\beta+\alpha}{\alpha}\mathbb{E}\left[\int_0^t \left|X_s^i\right|^\alpha ds\right] + C_{T,\alpha},$$

for some constant $C_{T,\alpha} > 0$ which depends on (T, α). On the other hand, there exists a constant $C_\alpha > 0$ which only depends on α such that

$$\begin{aligned}\left|\int_{\mathcal{U}} \left((1+\ell_i(y))^\alpha - 1 - \alpha\ell_i(y)\right)\nu(dy)\right| &\leq \alpha\int_{\mathcal{U}} \left[1 + (1+|\ell_i(y)|)^{\alpha-1}\right] |\ell_i(y)|\, \nu(dy). \\ &\leq C_\alpha \int_{\mathcal{U}} \left(|\ell_i(y)| + |\ell_i(y)|^\alpha\right)\nu(dy).\end{aligned}$$

Then it follows from **(A1)** that there exists a constant $C_{p,\ell,T,\alpha} > 0$ which is independent of N so that

$$\sup_{t\in[0,T]} \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\left|X_t^i\right|^\alpha\right] \leq C_{p,\ell,T,\alpha} + C_{p,\ell,T,\alpha}\int_0^T \sup_{s\in[0,t]} \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\left|X_s^i\right|^\alpha\right] dt.$$

Thus the moment estimate (8) follows form Gronwall's lemma. □

We next want to characterize the limiting generator of the sequence of empirical measure-valued processes $(\nu^N;\ N \in \mathbb{N})$ given by (5) as $N \to \infty$. To this purpose, for any smooth function $\varphi \in C^\infty(\mathbb{R}^K)$ with $K \in \mathbb{N}$, and for any Borel probability measure $\nu \in S$, we define $\Phi(\nu) := \varphi(\nu(\mathbf{f}))$. Here $\mathbf{f} = (f_1, \ldots, f_K)$ with each element belonging to $C^\infty(\mathcal{O})$ and the vector $\nu(\mathbf{f}) := (\nu(f_1), \ldots, \nu(f_K)) \in \mathbb{R}^K$. Then we have

Lemma 3.2. *Let the operator $\mathcal{A}$ acting on the function $\Phi(\nu) = \varphi(\nu(\mathbf{f}))$ with $\nu \in S$ be defined as*

$$\begin{aligned}\mathcal{A}\Phi(\nu) := &\sum_{k=1}^K \frac{\partial\varphi(\nu(\mathbf{f}))}{\partial x_k}\left\{r\nu(\mathcal{L}^{c1}f_k) - \nu(\mathcal{L}^{c2}f_k) + \nu(\mathcal{L}^{c3}f_k)\right\} \\ &+ \frac{1}{2}\sum_{k,l=1}^K \frac{\partial^2\varphi(\nu(\mathbf{f}))}{\partial x_k \partial x_l}\sum_{j=1}^d \left(\nu(\mathcal{L}^{dj}f_k)\nu(\mathcal{L}^{dj}f_l)\right) \\ &+ \int_{\mathcal{U}} \left\{\varphi\left(\nu(\mathcal{L}_y^1\mathbf{f})\right) - \varphi\left(\nu(\mathbf{f})\right) - \sum_{k=1}^K \frac{\partial\varphi(\nu(\mathbf{f}))}{\partial x_k}\nu(\mathcal{L}_y^2 f_k)\right\}\nu(dy).\end{aligned} \tag{9}$$

Here the operators are defined by, for $p = (v, \sigma_1, \ldots, \sigma_d, w) \in \mathcal{O}_p$, $\ell \in \mathcal{C}$ and $x \in \mathbb{R}_+$,

$$\begin{aligned}
&\mathcal{L}^{c1} f(p, \ell, x) := x \frac{\partial f(p, \ell, x)}{\partial x}, \quad \mathcal{L}^{c2} f(p, \ell, x) := xv \frac{\partial f(p, \ell, x)}{\partial x}, \\
&\mathcal{L}^{c3} f(p, \ell, x) := \frac{1}{2} x^{2(\beta+1)} w^{-2\beta} \bar{\sigma}^2 \frac{\partial^2 f(p, \ell, x)}{\partial x^2}, \qquad \bar{\sigma}^2 := \sum_{j=1}^{d} \sigma_j^2, \\
&\mathcal{L}_y^1 f(p, \ell, x) := f(p, \ell, x(1 + \ell(y))), \quad \mathcal{L}_y^2 f(p, \ell, x) := x\ell(y) \frac{\partial f(p, \ell, x)}{\partial x}, \quad y \in \mathcal{U},
\end{aligned} \tag{10}$$

and for $j = 1, \ldots, d$, the operator

$$\mathcal{L}^{dj} f(p, \ell, x) := x^{\beta+1} \sigma_j w^{-\beta} \frac{\partial f(p, \ell, x)}{\partial x}. \tag{11}$$

Then for the sequence of measure-valued processes $\nu^N = (\nu_t^N;\ t \geq 0)$ given by (5), it holds that

$$\lim_{N \to \infty} \mathbb{E}\left[\left(\Phi(\nu_{t_{m+1}}^N) - \Phi(\nu_{t_m}^N) - \int_{t_m}^{t_{m+1}} \mathcal{A}\Phi(\nu_s^N) ds\right) \prod_{j=1}^{m} \Psi_j(\nu_{t_j}^N)\right] = 0, \tag{12}$$

where $0 \leq t_1 < \cdots < t_{m+1} < \infty$ and $\Psi_j \in B(S)$ (all bounded measurable functions on S) with $j = 1, \ldots, m \in \mathbb{N}$.

Proof. It follows from Itô's formula that, for all $f \in \mathcal{C}^\infty(\mathcal{O})$,

$$\begin{aligned}
f(p_i, \ell_i, X_t^i) = f(p_i, \ell_i, X_0^i) &+ \int_0^t \frac{\partial f(p_i, \ell_i, X_s^i)}{\partial x} X_s^i (r - v_i) ds \\
&+ \frac{1}{2} \sum_{j=1}^{d} \int_0^t \frac{\partial^2 f(p_i, \ell_i, X_s^i)}{\partial x^2} \sigma_{ij}^2 w_i^{-2\beta} \left(X_s^i\right)^{2(\beta+1)} ds + \mathcal{M}_t^i \\
&+ \int_0^t \int_{\mathcal{U}} \left[f(p_i, \ell_i, X_s^i + X_s^i \ell_i(y)) - f(p_i, \ell_i, X_s^i) - \frac{\partial f(p_i, \ell_i, X_s^i)}{\partial x} X_s^i \ell_i(y)\right] \nu(dy) ds,
\end{aligned}$$

where the $\mathbb{F}$-(local) martingale, for $t \geq 0$,

$$\begin{aligned}
\mathcal{M}_t^i := &\sum_{j=1}^{d} \int_0^t \frac{\partial f(p_i, \ell_i, X_s^i)}{\partial x} \sigma_{ij} w_i^{-\beta} \left(X_s^i\right)^{\beta+1} dW_s^j \\
&+ \int_0^t \int_{\mathcal{U}} \left[f(p_i, \ell_i, X_{s-}^i + X_{s-}^i \ell_i(y)) - f(p_i, \ell_i, X_{s-}^i)\right] \tilde{Q}(dy, ds).
\end{aligned} \tag{13}$$

Thus we obtain

$$\begin{aligned}
\frac{1}{N} \sum_{i=1}^{N} f(p_i, \ell_i, X_t^i) &= \frac{1}{N} \sum_{i=1}^{N} f(p_i, \ell_i, X_0^i) + \frac{1}{N} \sum_{i=1}^{N} \int_0^t \frac{\partial f(p_i, \ell_i, X_s^i)}{\partial x} X_s^i (r - v_i) ds \\
&+ \frac{1}{2} \sum_{j=1}^{d} \int_0^t \frac{1}{N} \sum_{i=1}^{N} \frac{\partial^2 f(p_i, \ell_i, X_s^i)}{\partial x^2} \sigma_{ij}^2 w_i^{-2\beta} \left(X_s^i\right)^{2(\beta+1)} ds + \frac{1}{N} \sum_{i=1}^{N} \mathcal{M}_t^i \\
&+ \int_0^t \int_{\mathcal{U}} \frac{1}{N} \sum_{i=1}^{N} \left[f(p_i, \ell_i, X_s^i + X_s^i \ell_i(y)) - f(p_i, \ell_i, X_s^i) - \frac{\partial f(p_i, \ell_i, X_s^i)}{\partial x} X_s^i \ell_i(y)\right] \nu(dy) ds.
\end{aligned}$$

Then it holds that

$$\nu_t^N(f) = \nu_0^N(f) + \int_0^t \left\{r \nu_s^N(\mathcal{L}^{c1} f) - \nu_s^N(\mathcal{L}^{c2} f) + \nu_s^N(\mathcal{L}^{c3} f)\right\} ds + \frac{1}{N} \sum_{i=1}^{N} \mathcal{M}_t^i$$

$$+\int_0^t\int_{\mathcal{U}}\Big\{\nu_s^N(\mathcal{L}_y^1 f)-\nu_s^N(f)-\nu_s^N(\mathcal{L}_y^2 f)\Big\}\nu(dy)ds, \tag{14}$$

while for $t\geq 0$,

$$\frac{1}{N}\sum_{i=1}^N\mathcal{M}_t^i=\sum_{j=1}^d\int_0^t\nu_s^N(\mathcal{L}^{dj}f)dW_s^j+\int_0^t\int_{\mathcal{U}}\Big\{\nu_{s-}^N(\mathcal{L}_y^1 f)-\nu_{s-}^N(f)\Big\}\tilde{Q}(dy,ds). \tag{15}$$

By virtue of Doob-Meyer decomposition (14), we also have

$$\begin{aligned}\varphi(\nu_t^N(\mathbf{f}))&=\varphi(\nu_0^N(\mathbf{f}))+\sum_{k=1}^K\int_0^t\frac{\partial\varphi(\nu_s^N(\mathbf{f}))}{\partial x_k}\Big\{r\nu_s^N(\mathcal{L}^{c1}f_k)-\nu_s^N(\mathcal{L}^{c2}f_k)+\nu_s^N(\mathcal{L}^{c3}f_k)\Big\}ds\\&\quad+\sum_{k=1}^K\int_0^t\int_{\mathcal{U}}\frac{\partial\varphi(\nu_s^N(\mathbf{f}))}{\partial x_k}\Big\{\nu_s^N(\mathcal{L}_y^1 f_k)-\nu_s^N(f_k)-\nu_s^N(\mathcal{L}_y^2 f_k)\Big\}\nu(dy)ds\\&\quad+\frac{1}{2}\sum_{k,l=1}^K\int_0^t\frac{\partial^2\varphi(\nu_s^N(\mathbf{f}))}{\partial x_k\partial x_l}\sum_{j=1}^d\left(\nu_s^N(\mathcal{L}^{dj}f_k)\nu_s^N(\mathcal{L}^{dj}f_l)\right)ds\\&\quad+\int_0^t\int_{\mathcal{U}}\left[\varphi\left(\nu_{s-}^N(\mathcal{L}_y^1\mathbf{f})\right)-\varphi\left(\nu_{s-}^N(\mathbf{f})\right)\right]\tilde{Q}(dy,ds)+\sum_{j=1}^d\left(\sum_{k=1}^K\int_0^t\frac{\partial\varphi(\nu_s^N(\mathbf{f}))}{\partial x_k}\nu_s^N(\mathcal{L}^{dj}f_k)\right)dW_s^j\\&\quad+\int_0^t\int_{\mathcal{U}}\left[\varphi\left(\nu_s^N(\mathcal{L}_y^1\mathbf{f})\right)-\varphi\left(\nu_s^N(\mathbf{f})\right)-\sum_{k=1}^K\frac{\partial\varphi(\nu_s^N(\mathbf{f}))}{\partial x_k}\left(\nu_s^N(\mathcal{L}_y^1 f_k)-\nu_s^N(f_k)\right)\right]\nu(dy)ds.\end{aligned}$$

Then it holds that

$$\begin{aligned}\varphi(\nu_t^N(\mathbf{f}))&=\varphi(\nu_0^N(\mathbf{f}))+\sum_{k=1}^K\int_0^t\frac{\partial\varphi(\nu_s^N(\mathbf{f}))}{\partial x_k}\Big\{r\nu_s^N(\mathcal{L}^{c1}f_k)-\nu_s^N(\mathcal{L}^{c2}f_k)+\nu_s^N(\mathcal{L}^{c3}f_k)\Big\}ds\\&\quad+\frac{1}{2}\sum_{k,l=1}^K\int_0^t\frac{\partial^2\varphi(\nu_s^N(\mathbf{f}))}{\partial x_k\partial x_l}\sum_{j=1}^d\left(\nu_s^N(\mathcal{L}^{dj}f_k)\nu_s^N(\mathcal{L}^{dj}f_l)\right)ds+M_t^N\\&\quad+\int_0^t\int_{\mathcal{U}}\left[\varphi\left(\nu_s^N(\mathcal{L}_y^1\mathbf{f})\right)-\varphi\left(\nu_s^N(\mathbf{f})\right)-\sum_{k=1}^K\frac{\partial\varphi(\nu_s^N(\mathbf{f}))}{\partial x_k}\nu_s^N(\mathcal{L}_y^2 f_k)\right]\nu(dy)ds.\end{aligned}\tag{16}$$

Here M_t^N, $t\geq 0$, is an $\mathbb{F}$-(local) martingale. This implies that

$$\Phi(\nu_t^N)-\Phi(\nu_0^N)-\int_0^t\mathcal{A}\Phi(\nu_s^N)ds,\qquad t\geq 0$$

is an $\mathbb{F}$-(local) martingale with initial mean-zero. Notice that $\prod_{j=1}^m\Psi_j(\nu_{t_j}^N)$ is bounded and $\mathcal{F}_{t_m}$-measurable. Then the desired result follows from a local localization. $\square$.

We next prove that the sequence of measure-valued processes $(\nu^N;\ N\in\mathbb{N})$ defined by (5) are relatively compact, when viewed as a sequence of random processes on the Skorokhod space $D_S([0,\infty))$. This can be implied by the following two lemmas.

Lemma 3.3. *Suppose that the assumption* **(A1)** *holds. Then for every $T>0$, it holds that for any smooth function $f\in C^\infty(\mathcal{O})$,*

$$\lim_{R\to+\infty}\sup_{N\in\mathbb{N}}\mathbb{P}\left(\sup_{t\in[0,T]}\left|\nu_t^N(f)\right|\geq R\right)=0. \tag{17}$$

Proof. In light of Doob-Meyer decomposition (14), we have

$$\nu_t^N(f) = \nu_0^N(f) + A_t^N + B_t^N + \frac{1}{N}\sum_{i=1}^N \mathcal{M}_t^i, \qquad t \geq 0,$$

where the terms for $t \geq 0$,

$$A_t^N := \frac{1}{N}\sum_{i=1}^N \int_0^t \frac{\partial f(p_i,\ell_i,X_s^i)}{\partial x} X_s^i(r-v_i)ds + \frac{1}{2}\sum_{j=1}^d \int_0^t \frac{1}{N}\sum_{i=1}^N \frac{\partial^2 f(p_i,\ell_i,X_s^i)}{\partial x^2}\sigma_{ij}^2 w_i^{-2\beta}\left(X_s^i\right)^{2(\beta+1)}ds,$$

$$B_t^N := \int_0^t \int_{\mathcal{U}} \frac{1}{N}\sum_{i=1}^N \Big[f(p_i,\ell_i,X_s^i + X_s^i\ell_i(y)) - f(p_i,\ell_i,X_s^i) - \frac{\partial f(p_i,\ell_i,X_s^i)}{\partial x}X_s^i\ell_i(y)\Big]\nu(dy)ds.$$

First we have, for any $T > 0$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|A_t^N\right|\right] &\leq \frac{1}{N}\sum_{i=1}^N \int_0^T \mathbb{E}\left[\left|\frac{\partial f(p_i,\ell_i,X_s^i)}{\partial x}X_s^i(r-v_i)\right|\right]ds \\
&\quad + \frac{1}{2N}\sum_{i=1}^N \int_0^T \mathbb{E}\left[\left|\frac{\partial^2 f(p_i,\ell_i,X_s^i)}{\partial x^2}\bar{\sigma}_i^2 w_i^{-2\beta}\left(X_s^i\right)^{2(\beta+1)}\right|\right]ds \\
&\leq C_{p,\ell}\left\|\frac{\partial f}{\partial x}\right\| \int_0^T \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\left|X_s^i\right|\right]ds + C_{p,\ell}\left\|\frac{\partial^2 f}{\partial x^2}\right\| \int_0^T \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\left|X_s^i\right|^{2(\beta+1)}\right]ds.
\end{aligned}$$

Notice that $2(\beta+1) \in [1,2]$. Then by virtue of Lemma 3.1, it follows that $\mathbb{E}\left[\sup_{t\in[0,T]}\left|A_t^N\right|\right] \leq C_{p,T,f}$ for some finite constant $C_{p,T,f} > 0$ which is independent of N. Using the mean-value theorem and Lemma 3.1, we also have

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|B_t^N\right|\right] &\leq \left\|\frac{\partial f}{\partial x}\right\| \mathbb{E}\left[\int_0^T \frac{1}{N}\sum_{i=1}^N \left|X_s^i\right|\left(\int_{\mathcal{U}}\left|\ell_i(y)\right|\nu(dy)\right)ds\right] \\
&\leq C_{p,\ell}\left\|\frac{\partial f}{\partial x}\right\| \mathbb{E}\left[\int_0^T \frac{1}{N}\sum_{i=1}^N \left|X_s^i\right|ds\right] \\
&\leq C_{p,\ell,T,f},
\end{aligned}$$

for some finite constant $C_{p,\ell,T,f} > 0$ which is independent of N. Using B-D-G inequality and the (local) martingale (15), it follows that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\frac{1}{N}\sum_{i=1}^N \mathcal{M}_t^i\right|\right] &\leq C_T\mathbb{E}\left\{\left[\sum_{j=1}^d \int_0^T \left|\nu_s^N(\mathcal{L}^{dj}f)\right|^2 ds\right]^{\frac{1}{2}}\right\} \\
&\quad + C_T\mathbb{E}\left\{\left[\int_0^T \int_{\mathcal{U}} \left|\nu_s^N(\mathcal{L}_y^1 f) - \nu_s^N(f)\right|^2 Q(dy,ds)\right]^{\frac{1}{2}}\right\}.
\end{aligned}$$

In terms of the operator (11), we first have

$$\begin{aligned}
\mathbb{E}\left[\sum_{j=1}^d \int_0^T \left|\nu_s^N(\mathcal{L}^{dj}f)\right|^2 ds\right] &\leq C_{p,\ell}\left\|\frac{\partial f}{\partial x}\right\|^2 \mathbb{E}\left[\int_0^T \left(\frac{1}{N}\sum_{i=1}^N \left(X_s^i\right)^{\beta+1}\right)^2 ds\right] \\
&\leq \frac{C_{p,\ell,f}}{N^2}\mathbb{E}\left[\int_0^T \left(\sum_{i,j=1}^N \left(X_s^i\right)^{\beta+1}\left(X_s^j\right)^{\beta+1}\right)ds\right]
\end{aligned}$$

$$
\leq \frac{C_{p,\ell,f}}{2N^2}\mathbb{E}\left[\int_0^T \sum_{i,j=1}^N \left|X_s^i\right|^{2(\beta+1)} ds + \int_0^T \sum_{i,j=1}^N \left|X_s^j\right|^{2(\beta+1)} ds\right]
$$

$$
= C_{p,\ell,f}\int_0^T \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\left|X_s^i\right|^{2(\beta+1)}\right] ds,
$$

and by virtue of (10), the mean-value theorem and the assumption **(A1)**, it follows that

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T\int_{\mathcal{U}} \left|\nu_s^N(\mathcal{L}_y^1 f)-\nu_s^N(f)\right|^2 Q(dy,ds)\right] &= \mathbb{E}\left[\int_0^T\int_{\mathcal{U}} \left|\nu_s^N(\mathcal{L}_y^1 f)-\nu_s^N(f)\right|^2 \nu(dy)ds\right]\\
&\leq \left\|\frac{\partial f}{\partial x}\right\|^2 \mathbb{E}\left[\int_0^T\int_{\mathcal{U}}\left|\frac{1}{N}\sum_{i=1}^N X_s^i\ell_i(y)\right|^2 \nu(dy)ds\right]\\
&\leq \left\|\frac{\partial f}{\partial x}\right\|^2 \mathbb{E}\left[\int_0^T \frac{1}{N}\sum_{i=1}^N \left(X_s^i\right)^2\left(\int_{\mathcal{U}}(\ell_i(y))^2\nu(dy)\right) ds\right]\\
&\leq C_{p,\ell,f}\int_0^T\frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\left|X_s^i\right|^2\right] ds.
\end{aligned}
$$

Thus from Cauchy's inequality and Lemma 3.1, it follows that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\frac{1}{N}\sum_{i=1}^N \mathcal{M}_t^i\right|\right] &\leq C_{p,\ell,T,f}\mathbb{E}\left[\sum_{j=1}^d\int_0^T\left|\nu_s^N(\mathcal{L}^{dj}f)\right|^2 ds\right] + C_{p,\ell,T,f}\\
&\quad + C_{p,\ell,T,f}\mathbb{E}\left[\int_0^T\int_{\mathcal{U}}\left|\nu_s^N(\mathcal{L}_y^1 f)-\nu_s^N(f)\right|^2 Q(dy,ds)\right]\\
&\leq C_{p,\ell,T,f},
\end{aligned}
$$

for some finite constant $C_{p,\ell,T,f}>0$ which is independent of N. Hence the desired limit (17) follows form Markov inequality. □

The following lemma verifies the time regularity of the sequence of measure-valued processes $(\nu^N;\ N\in\mathbb{N})$.

Lemma 3.4. *Let the assumption* **(A1)** *hold and $h(x,y):=|x-y|\wedge 1$ for $x,y\in\mathbb{R}$. Then there exists a positive random variable $H^N(\gamma)$ with $\gamma>0$ and $\lim_{\gamma\to 0}\sup_{N\in\mathbb{N}}\mathbb{E}\left[H^N(\gamma)\right]=0$ such that for all $t\in[0,T]$, $u\in[0,\gamma]$, and $v\in[0,\gamma\wedge 1]$, it holds that*

$$
\mathbb{E}_t\left[h^2(\nu_{t+u}^N(f),\nu_t^N(f))h^2(\nu_t^N(f),\nu_{t-v}^N(f))\right]\leq \mathbb{E}_t\left[H^N(\gamma)\right],\qquad N\in\mathbb{N}. \tag{18}
$$

Here $\mathbb{E}_t[\cdot]:=\mathbb{E}[\cdot|\mathcal{F}_t]$ with $t\in[0,T]$.

Proof. Using Doob-Meyer decomposition (14), for all $t\in[0,T]$ and $u\in[0,\gamma]$,

$$
\nu_{t+u}^N(f)-\nu_t^N(f) = A_{t+u}^N - A_t^N + B_{t+u}^N - B_t^N + \frac{1}{N}\sum_{i=1}^N\left(\mathcal{M}_{t+u}^i-\mathcal{M}_t^i\right).
$$

First we have for all $t\in[0,T]$ and $u\in[0,\gamma]$,

$$
\begin{aligned}
\left|A_{t+u}^N - A_t^N\right| &\leq C_{p,\ell}\left\|\frac{\partial f}{\partial x}\right\|\int_t^{t+u}\frac{1}{N}\sum_{i=1}^N\left|X_s^i\right| ds + \frac{C_{p,\ell}}{2}\left\|\frac{\partial^2 f}{\partial x^2}\right\|\int_t^{t+u}\frac{1}{N}\sum_{i=1}^N\left|X_s^i\right|^{2(\beta+1)} ds\\
&\leq C_{p,\ell,T,f}u^{\frac{1}{4}}\left(2+\int_0^T\frac{1}{N}\sum_{i=1}^N\left|X_s^i\right|^2 ds + \int_0^T\frac{1}{N}\sum_{i=1}^N\left|X_s^i\right|^{4(\beta+1)} ds\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_{p,\ell,T,f}\gamma^{\frac{1}{4}}\left(2+\int_0^T \frac{1}{N}\sum_{i=1}^N \left|X_s^i\right|^2 ds + \int_0^T \frac{1}{N}\sum_{i=1}^N \left|X_s^i\right|^{4(\beta+1)} ds\right)\\
&=: H_1^N(\gamma),
\end{aligned}
$$

and it holds that

$$
\begin{aligned}
\left|B_{t+u}^N - B_t^N\right| &\leq 2\left\|\frac{\partial f}{\partial x}\right\| \int_t^{t+u} \frac{1}{N}\sum_{i=1}^N \left|X_s^i\right| \left(\int_{\mathcal{U}} |\ell_i(y)|\,\nu(dy)\right) ds\\
&\leq C_{p,\ell,T,f} \int_t^{t+u} \frac{1}{N}\sum_{i=1}^N \left|X_s^i\right| ds\\
&\leq C_{p,\ell,T,f}\gamma^{\frac{1}{4}}\left(1+\int_0^T \frac{1}{N}\sum_{i=1}^N \left|X_s^i\right|^2 ds\right) =: H_2^N(\gamma).
\end{aligned}
$$

Finally we obtain using (15) that

$$
\begin{aligned}
\mathbb{E}_t\left[\left|\frac{1}{N}\sum_{i=1}^N \left(\mathcal{M}_{t+u}^i - \mathcal{M}_t^i\right)\right|^2\right] &= \mathbb{E}_t\left[\sum_{j=1}^d \int_t^{t+u} \left|\nu_s^N(\mathcal{L}^{dj} f)\right|^2 ds\right]\\
&\quad + \mathbb{E}_t\left[\int_t^{t+u}\int_{\mathcal{U}} \left|\nu_s^N(\mathcal{L}_y^1 f) - \nu_s^N(f)\right|^2 \nu(dy)ds\right].
\end{aligned}
$$

Here we first get

$$
\begin{aligned}
\mathbb{E}_t\left[\sum_{j=1}^d \int_t^{t+u} \left|\nu_s^N(\mathcal{L}^{dj} f)\right|^2 ds\right] &\leq C_{p,\ell}\left\|\frac{\partial f}{\partial x}\right\|^2 \mathbb{E}_t\left[\int_t^{t+u} \frac{1}{N}\sum_{i=1}^N \left|X_s^i\right|^{2(\beta+1)} ds\right]\\
&\leq \mathbb{E}_t\left[C_{p,\ell,T,f}\gamma^{\frac{1}{4}}\left(1+\int_0^T \frac{1}{N}\sum_{i=1}^N \left|X_s^i\right|^{4(\beta+1)} ds\right)\right] =: \mathbb{E}_t\left[H_3^N(\gamma)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\left[\int_t^{t+u}\int_{\mathcal{U}} \left|\nu_s^N(\mathcal{L}_y^1 f) - \nu_s^N(f)\right|^2 \nu(dy)ds\right] &\leq \left\|\frac{\partial f}{\partial x}\right\|^2 \mathbb{E}_t\left[\int_t^{t+u}\int_{\mathcal{U}} \left|\frac{1}{N}\sum_{i=1}^N X_s^i \ell_i(y)\right|^2 \nu(dy)ds\right]\\
&\leq \left\|\frac{\partial f}{\partial x}\right\|^2 \mathbb{E}_t\left[\int_t^{t+u} \frac{1}{N}\sum_{i=1}^N \left(X_s^i\right)^2 \left(\int_{\mathcal{U}} (\ell_i(y))^2\,\nu(dy)\right) ds\right]\\
&\leq \mathbb{E}_t\left[C_{p,\ell,f}\gamma^{\frac{1}{4}}\left(1+\int_0^T \frac{1}{N}\sum_{i=1}^N \left|X_s^i\right|^4 ds\right)\right] =: \mathbb{E}_t\left[H_4^N(\gamma)\right].
\end{aligned}
$$

Let $H^N(\gamma) := \left(H_1^N(\gamma)\right)^2 + \left(H_2^N(\gamma)\right)^2 + H_3^N(\gamma) + H_4^N(\gamma)$ with $\gamma > 0$. Notice that $4(1+\beta) \in [2,4]$. Then we have $\lim_{\gamma\to 0}\sup_{N\in\mathbb{N}} \mathbb{E}\left[H^N(\gamma)\right] = 0$ using Lemma 3.1, and for all $N \in \mathbb{N}$, it holds that

$$
\mathbb{E}_t\left[h^2(\nu_{t+u}^N(f), \nu_t^N(f)) h^2(\nu_t^N(f), \nu_{t-v}^N(f))\right] \leq 16\mathbb{E}_t\left[H^N(\gamma)\right],
$$

where we used $h^2(\nu_t^N(f), \nu_{t-v}^N(f)) \leq 1$. This proves the estimate (18). □

Next we will characterize the weak limit of the sequence of measure-valued processes $(\nu^N;\ N \in \mathbb{N})$ defined by (5). To this purpose, we define the empirical measures $q^N := \frac{1}{N}\sum_{i=1}^N \delta_{p_i}$, $\eta^N := \frac{1}{N}\sum_{i=1}^N \delta_{\ell_i}$ and $\psi^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_0^i}$ which are respectively related to the parameter set, jump functions and initial weighted price. Assume that

(A2) The limiting empirical measures $q = \lim_{N\to\infty} q^N$, $\eta = \lim_{N\to\infty} \eta^N$ and $\psi = \lim_{N\to\infty} \psi^N$ exist in $\mathcal{P}(\mathcal{O}_p)$, $\mathcal{P}(\mathcal{C})$ and $\mathcal{P}(\mathbb{R})$ respectively.

For $p = (v, \sigma_1, \ldots, \sigma_d, w) \in \mathcal{O}_p$, $\ell \in \mathcal{C}$, and $x \in \mathbb{R}$, define the following measure-valued process by

$$\nu_t(B \times C \times D) := \int_{\mathcal{O}} \mathbf{1}_{B\times C\times D}\big(p, \ell, X_t((p,\ell,x))\big) q(dp)\eta(d\ell)\psi(dx), \qquad t \geq 0, \tag{19}$$

where $B \in \mathcal{B}(\mathcal{O}_p)$, $C \in \mathcal{B}(\mathcal{C})$ and $D \in \mathcal{B}(\mathbb{R})$. Here the underlying parameterized process is given by the unique strong solution of the following SDE with jumps:

$$\begin{aligned} X_t((p,\ell,x)) = x &+ \int_0^t (r-v)X_s((p,\ell,x))ds + \int_0^t \bar{\sigma} w^{-\beta} X_s^{\beta+1}((p,\ell,x))d\bar{W}_s \\ &+ \int_0^t \int_{\mathcal{U}} X_s((p,\ell,x))\ell(y)\tilde{Q}(dy,ds), \end{aligned} \tag{20}$$

where $\bar{W} = (\bar{W}_t;\ t \geq 0)$ is a 1-dimensional standard Brownian motion which is independent of the Poisson random measure $\tilde{Q}(dy,ds)$ and $\bar{\sigma} := \sqrt{\sum_{j=1}^d \sigma_j^2}$.

Then we have

Theorem 3.5. *Let assumptions* **(A1)** *and* **(A2)** *hold. Then the sequence of measure-valued processes* $(\nu^N;\ N \in \mathbb{N})$ *defined by* (5) *weakly converges to the above measure-valued process* $\nu = (\nu_t;\ t \geq 0)$ *given by* (19) *as* $N \to \infty$.

Proof. By virtue of the weak convergence of martingale problem as in Chapter 3 of [9], from Lemma 3.2, Lemma 3.3 and Lemma 3.4, it follows that ν^N weakly converges to ν as $N \to \infty$. We next show that the limit measure-valued process ν is indeed given by (19) in terms of the uniqueness of martingale problems as in Chapter 3 of [9]. In fact, using (19), we have for $f \in C^\infty(\mathcal{O})$,

$$\nu_t(f) = \int_{\mathcal{O}} f\big(p, \ell, X_t((p,\ell,x))\big) q(dp)\eta(d\ell)\psi(dx). \tag{21}$$

Fix the parameters $(p,\ell,x) \in \mathcal{O}$ and applying Itô's formula, we have

$$\begin{aligned} f\big(p,\ell,X_t((p,\ell,x))\big) &= f\big(p,\ell,X_0((p,\ell,x))\big) + \int_0^t (r-v)X_s((p,\ell,x))\frac{\partial f\big(p,\ell,X_s((p,\ell,x))\big)}{\partial x}ds \\ &+ \frac{1}{2}\sum_{j=1}^d \int_0^t \sigma_j^2 w^{-2\beta} X_s^{2(\beta+1)}((p,\ell,x))\frac{\partial^2 f\big(p,\ell,X_s((p,\ell,x))\big)}{\partial x^2}ds \\ &+ \int_0^t \bar{\sigma} w^{-\beta} X_s^{\beta+1}((p,\ell,x))\frac{\partial f\big(p,\ell,X_s((p,\ell,x))\big)}{\partial x}d\bar{W}_s \\ &+ \int_0^t \int_{\mathcal{U}} \big[f\big(p,\ell,X_{s-}((p,\ell,x))(1+\ell(y))\big) - f\big(p,\ell,X_{s-}((p,\ell,x))\big)\big]\tilde{Q}(dy,ds) \\ &+ \int_0^t \int_{\mathcal{U}} \Big[f\big(p,\ell,X_s((p,\ell,x))(1+\ell(y))\big) - f\big(p,\ell,X_s((p,\ell,x))\big) \\ &\qquad - X_s((p,\ell,x))\ell(y)\frac{\partial f\big(p,\ell,X_s((p,\ell,x))\big)}{\partial x}\Big]\nu(dy)ds. \end{aligned}$$

Take the integral on the both sides of the above display w.r.t. $q(dp)\eta(d\ell)\psi(dx)$. Then it follows from (21) that

$$\nu_t(f) = \nu_0(f) + \int_0^t \Big\{r\nu_s(\mathcal{L}^{c1}f) - \nu_s(\mathcal{L}^{c2}f) + \nu_s(\mathcal{L}^{c3}f)\Big\}ds$$

$$
+\sum_{j=1}^{d}\int_0^t \nu_s(\mathcal{L}^{dj}f)dW_s^j+\int_0^t\int_{\mathcal{U}}\Big\{\nu_{s-}(\mathcal{L}_y^1 f)-\nu_{s-}(f)\Big\}\tilde{Q}(dy,ds)
$$

$$
+\int_0^t\int_{\mathcal{U}}\Big\{\nu_s(\mathcal{L}_y^1 f)-\nu_s(f)-\nu_s(\mathcal{L}_y^2 f)\Big\}\nu(dy)ds. \tag{22}
$$

This results in using Itô's formula that

$$
\begin{aligned}
\varphi(\nu_t(\mathbf{f}))&=\varphi(\nu_0(\mathbf{f}))+\sum_{k=1}^{K}\int_0^t\frac{\partial\varphi(\nu_s(\mathbf{f}))}{\partial x_k}\Big\{r\nu_s(\mathcal{L}^{c1}f_k)-\nu_s(\mathcal{L}^{c2}f_k)+\nu_s(\mathcal{L}^{c3}f_k)\Big\}ds\\
&\quad+\frac{1}{2}\sum_{k,l=1}^{K}\int_0^t\frac{\partial^2\varphi(\nu_s(\mathbf{f}))}{\partial x_k\partial x_l}\sum_{j=1}^{d}\Big(\nu_s(\mathcal{L}^{dj}f_k)\nu_s(\mathcal{L}^{dj}f_l)\Big)ds+\mathcal{M}_t\\
&\quad+\int_0^t\int_{\mathcal{U}}\left[\varphi\left(\nu_s(\mathcal{L}_y^1\mathbf{f})\right)-\varphi\left(\nu_s(\mathbf{f})\right)-\sum_{k=1}^{K}\frac{\partial\varphi(\nu_s(\mathbf{f}))}{\partial x_k}\left(\nu_s(\mathcal{L}_y^1 f_k)-\nu_s(f_k)\right)\right]\nu(dy)ds\\
&\quad+\sum_{k=1}^{K}\int_0^t\int_{\mathcal{U}}\frac{\partial\varphi(\nu_s(\mathbf{f}))}{\partial x_k}\Big\{\nu_s(\mathcal{L}_y^1 f_k)-\nu_s(f_k)-\nu_s(\mathcal{L}_y^2 f_k)\Big\}\nu(dy)ds\\
&=\varphi(\nu_0(\mathbf{f}))+\sum_{k=1}^{K}\int_0^t\frac{\partial\varphi(\nu_s(\mathbf{f}))}{\partial x_k}\Big\{r\nu_s(\mathcal{L}^{c1}f_k)-\nu_s(\mathcal{L}^{c2}f_k)+\nu_s(\mathcal{L}^{c3}f_k)\Big\}ds\\
&\quad+\frac{1}{2}\sum_{k,l=1}^{K}\int_0^t\frac{\partial^2\varphi(\nu_s(\mathbf{f}))}{\partial x_k\partial x_l}\sum_{j=1}^{d}\Big(\nu_s(\mathcal{L}^{dj}f_k)\nu_s(\mathcal{L}^{dj}f_l)\Big)ds+\mathcal{M}_t\\
&\quad+\int_0^t\int_{\mathcal{U}}\Big[\varphi\left(\nu_s(\mathcal{L}_y^1\mathbf{f})\right)-\varphi\left(\nu_s(\mathbf{f})\right)-\sum_{k=1}^{K}\frac{\partial\varphi(\nu_s(\mathbf{f}))}{\partial x_k}\nu_s(\mathcal{L}_y^2 f_k)\Big]\nu(dy)ds\\
&=\varphi(\nu_0(\mathbf{f}))+\int_0^t\mathcal{A}\Phi(\nu_s)ds+\mathcal{M}_t,
\end{aligned}
$$

where $\mathcal{M}_t$, $t\geq 0$, is an $\mathbb{F}$-(local) martingale. This yields that

$$
\Phi(\nu_t)-\Phi(\nu_0)-\int_0^t\mathcal{A}\Phi(\nu_s)ds,\qquad t\geq 0 \tag{23}
$$

is an $\mathbb{F}$-(local) martingale. By virtue of the uniqueness of martingale problems as in Chapter 3 of [9] and (23), the limit measure-valued process ν is indeed given by (19). □

4 Pricing Basket Options

In this section, we turn to the main objective of the paper on the pricing of basket options in a large portfolio (i.e., as $N\to\infty$) by employing Theorem 3.5 in the above section.

Recall that the price of the basket option is given by (3), i.e.,

$$
V^N(\phi)=e^{-rT}\mathbb{E}\left[\phi\left(\frac{1}{N}\sum_{i=1}^{N}w_iS_T^i\right)\right]=e^{-rT}\mathbb{E}\left[\phi\left(\nu_T^N(I)\right)\right],
$$

where $I(p,\ell,x)=x$ for $(p,\ell,x)\in\mathcal{O}$ and the empirical measure ν_T^N is defined by (5) with $t=T$ therein. Here recall that, for a strike price $K>0$, $\phi(x)=\{\theta(x-K)\}^+$ for $x\in\mathbb{R}$ corresponds to the call basket option if $\theta=1$, and corresponds to the put basket option if $\theta=-1$. Then we have the following main limit result on the basket option price given by

Theorem 4.1. *Let assumptions* **(A1)** *and* **(A2)** *be satisfied. Then the price of the basket option admits the following limit as $N \to \infty$,*

$$\lim_{N\to+\infty} V^N(\phi) = e^{-rT}\mathbb{E}\left[\phi\left(\int_{\mathcal{O}} X_T((p,\ell,x))q(dp)\eta(d\ell)\psi(dx)\right)\right], \tag{24}$$

where for $(p,\ell,x)\in\mathcal{O}$, the parameterized state process $X_t((p,\ell,x))$, $t\geq 0$, is given by (20). Here q, η and ψ are the limit empirical measures associated with the parameter set, jump functions and initial weighted price prices given in the assumption **(A2)**. *Moreover, if $p_i\to p^*$, $\ell_i\to\ell^*$ pointwisely and $X_0^i\to x^*$ as $i\to\infty$ for some $(p^*,\ell^*,x^*)\in\mathcal{O}$, then we have*

$$\lim_{N\to+\infty} V^N(\phi) = e^{-rT}\mathbb{E}\big[\phi\left(X_T^*\right)\big], \tag{25}$$

where the state process $X_t^* := X_t((p^*,\ell^*,x^*))$, $t\geq 0$, satisfies the following SDE with jumps given by, $X_0^*=X_{0-}^*=x^*$, and

$$\frac{dX_t^*}{X_{t-}^*} = (r-v^*)dt + \bar{\sigma}^*\left(w^*\right)^{-\beta}\left(X_t^*\right)^{\beta} d\bar{W}_t + \int_{\mathcal{U}} \ell^*(y)\tilde{Q}(dy,dt). \tag{26}$$

Here $\bar{\sigma}^* := \sqrt{\sum_{j=1}^d \left(\sigma_j^*\right)^2}$.

Proof. Recall $I(p,\ell,x)=x$ for $(p,\ell,x)\in\mathcal{O}$. For $n\in\mathbb{N}$, define $g_n(x):=\max\{-n,\min\{x,n\}\}$ on $x\in\mathbb{R}$. Then $g_n\in C_b(\mathbb{R})$ and for all $x\in\mathbb{R}$, $g_n(x)\to I(p,\ell,x)$ as $n\to\infty$. Further $|g_n(x)|\leq|x|$ for all $n\in\mathbb{N}$. Since $g_n\in C_b(\mathbb{R})$ for each fixed $n\in\mathbb{N}$, using Theorem 3.5, it has, for fixed $n\in\mathbb{N}$, $\nu_T^N(g_n)$ weakly converges to $\nu_T(g_n)$ as $N\to\infty$. Notice that the payoff function ϕ is continuous. Then by employing the continuous mapping theorem, for fixed $n\in\mathbb{N}$, $\phi\left(\nu_T^N(g_n)\right)$ weakly converges to $\phi\left(\nu_T(g_n)\right)$ as $N\to\infty$. We next prove the uniform estimate of the moment of $(\phi\left(\nu_T^N(g_n)\right);\ N\in\mathbb{N})$. In fact, notice that the payoff function ϕ admits the linear growth condition. Then there exists a constant $C_K>0$ which may depend on the strike price K such that

$$\begin{aligned}
\mathbb{E}\left[\left|\phi\left(\nu_T^N(g_n)\right)\right|^2\right] &\leq C_K\left(1+\mathbb{E}\left[\left|\nu_T^N(g_n)\right|^2\right]\right) \leq C_K\left(1+\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N g_n(X_T^i)\right|^2\right]\right)\\
&\leq C_K\left(1+\mathbb{E}\left[\frac{1}{N^2}\sum_{i,j=1}^N g_n(X_T^i)g_n(X_T^j)\right]\right)\\
&\leq C_K\left(1+\frac{1}{2}\mathbb{E}\left[\frac{1}{N^2}\sum_{i,j=1}^N\left|g_n(X_T^i)\right|^2\right]+\frac{1}{2}\mathbb{E}\left[\frac{1}{N^2}\sum_{i,j=1}^N\left|g_n(X_T^j)\right|^2\right]\right)\\
&\leq C_K\left(1+\sup_{N\in\mathbb{N}}\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[\left|X_T^i\right|^2\right]\right).
\end{aligned}$$

By virtue of Lemma 3.1, we have that $\sup_{N\in\mathbb{N}}\frac{1}{N}\sum_{i=1}^N\mathbb{E}\left[\left|X_T^i\right|^2\right]<+\infty$ and which is also independent of (N,n). This immediately yields that

$$\sup_{(N,n)\in\mathbb{N}^2}\mathbb{E}\left[\left|\phi\left(\nu_T^N(g_n)\right)\right|^2\right]<+\infty. \tag{27}$$

Since for fixed $n\in\mathbb{N}$, $\phi\left(\nu_T^N(g_n)\right)$ weakly converges to $\phi\left(\nu_T(g_n)\right)$ as $N\to\infty$, by Skorohod representation theorem, there exists a probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ and a sequence of random variables Y, Y^N, $N\in\mathbb{N}$, on it such that $Y\overset{d}{=}\phi\left(\nu_T(g_n)\right)$, $Y^N\overset{d}{=}\phi\left(\nu_T^N(g_n)\right)$, and $Y^N\to Y$ $\tilde{\mathbb{P}}$-a.s. as $N\to\infty$. Let $\tilde{\mathbb{E}}$ denote the expectation w.r.t. $\tilde{\mathbb{P}}$. From $Y^N\overset{d}{=}\phi\left(\nu_T^N(g_n)\right)$ and the above uniform estimate (27) under $\mathbb{P}$, it follows that

$$\sup_{N\in\mathbb{N}}\tilde{\mathbb{E}}\left[\left|\phi\left(Y^N\right)\right|^2\right]<+\infty.$$

This yields that $(Y^N;\ N\in\mathbb{N})$ is uniformly integrable under $\tilde{\mathbb{P}}$. Recall $Y^N\to Y$ $\tilde{\mathbb{P}}$-a.s. as $N\to\infty$, and hence $Y^N\overset{\tilde{\mathbb{P}}}{\to}Y$ as $N\to\infty$. Then using Vitali's convergence theorem, it follows that

$$Y^N{\to}Y \text{ in } L^1(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}}),\quad \text{as } N\to\infty.$$

This implies that $\tilde{\mathbb{E}}\left[Y^N\right]\to\tilde{\mathbb{E}}\left[Y\right]$ as $N\to\infty$. Since $\mathbb{E}\left[\phi\left(\nu_T^N(g_n)\right)\right]=\tilde{\mathbb{E}}\left[Y^N\right]$ and $\mathbb{E}\left[\phi\left(\nu_T(g_n)\right)\right]=\tilde{\mathbb{E}}\left[Y\right]$, we obtain for fixed $n\in\mathbb{N}$, it holds that

$$\lim_{N\to+\infty}\mathbb{E}\left[\phi\left(\nu_T^N(g_n)\right)\right]=\mathbb{E}\left[\phi\left(\nu_T(g_n)\right)\right].$$

Also by the monotone convergence theorem, for fixed $N\in\mathbb{N}$, we have $\mathbb{E}\left[\phi\left(\nu_T^N(g_n)\right)\right]\to\mathbb{E}\left[\phi\left(\nu_T^N(I)\right)\right]$ and $\mathbb{E}\left[\phi\left(\nu_T(g_n)\right)\right]\to\mathbb{E}\left[\phi\left(\nu_T(I)\right)\right]$ as $n\to+\infty$. Together with the following triangle inequality

$$\begin{aligned}\left|\mathbb{E}\left[\phi\left(\nu_T^N(I)\right)\right]-\mathbb{E}\left[\phi\left(\nu_T(I)\right)\right]\right|&\leq\left|\mathbb{E}\left[\phi\left(\nu_T^N(g_n)\right)\right]-\mathbb{E}\left[\phi\left(\nu_T^N(I)\right)\right]\right|\\&\quad+\left|\mathbb{E}\left[\phi\left(\nu_T^N(g_n)\right)\right]-\mathbb{E}\left[\phi\left(\nu_T(g_n)\right)\right]\right|+\left|\mathbb{E}\left[\phi\left(\nu_T(g_n)\right)\right]-\mathbb{E}\left[\phi\left(\nu_T(I)\right)\right]\right|,\end{aligned}$$

we obtain $\mathbb{E}\left[\phi\left(\nu_T^N(I)\right)\right]\to\mathbb{E}\left[\phi\left(\nu_T(I)\right)\right]$ as $N\to\infty$. Notice that it holds that

$$\mathbb{E}\left[\phi\left(\nu_T^N(I)\right)\right]=e^{rT}V^N(\phi),\ \text{and}\ \ \mathbb{E}\left[\phi\left(\nu_T(I)\right)\right]=\mathbb{E}\left[\phi\left(\int_{\mathcal{O}}X_T((p,\ell,x))q(dp)\eta(d\ell)\psi(dx)\right)\right].$$

This proves the limit (24). If $p_i\to p^*$, $\ell_i\to\ell^*$ pointwisely and $X_0^i\to x^*$ as $i\to\infty$ for some $(p^*,\ell^*,x^*)\in\mathcal{O}$, then the assumption **(A2)** is satisfied with the limiting empirical measures $q=\delta_{p^*}$, $\ell=\delta_{\ell^*}$ and $\psi=\delta_{x^*}$. Thus we have $\int_{\mathcal{O}}X_t((p,\ell,x))q(dp)\eta(d\ell)\psi(dx)=X_t((p^*,\ell^*,x^*))=X_t^*$ for $t\geq 0$. Then the limit (24) is reduced to the limit (25). Thus we complete the proof of the theorem. □

As an application, we here consider a basket option where the underlying N risky assets are described as a multi-dimensional mixed-exponential jump-diffusion model (where the CEV parameter $\beta=0$). Namely, under the risk-neutral probability measure $\mathbb{Q}$, the i-th risky asset price satisfies the following SDE given by, for $i=1,\ldots,N$,

$$\frac{dS_t^i}{S_{t-}^i}=\left\{r-v_i-\lambda\mathbb{E}\left[\left(e^{c_iY_1}-1\right)\right]\right\}dt+\sum_{j=1}^d\sigma_{ij}dW_t^j+d\left(\sum_{k=1}^{N_t}\left(e^{c_iY_k}-1\right)\right),\tag{28}$$

where N_t, $t\geq 0$, a Poisson process with intensity parameter $\lambda>0$, and Y_k, $k\in\mathbb{N}$, is a sequence of i.i.d. random variables which have a common probability density function given by

$$h_Y(y):=l_u\sum_{i=1}^m l_i\lambda_i e^{-\lambda_i y}\mathbf{1}_{y\geq 0}+\bar{l}_d\sum_{j=1}^n\bar{l}_j\bar{\lambda}_j e^{\bar{\lambda}_j y}\mathbf{1}_{y<0},\quad y\in\mathbb{R}.\tag{29}$$

Here $l_u\geq 0$, $\bar{l}_d=1-l_u\geq 0$, $l_i,\bar{l}_j\in\mathbb{R}$ which also satisfy $\sum_{i=1}^m l_i=\sum_{j=1}^n\bar{l}_j=1$, and $\lambda_i>1$, $\bar{\lambda}_i>0$ for all $i=1,\ldots,m$ and $j=1,\ldots,n$. If $N=d=1$, $v_i=0$ and $c_i=1$, then the above model (28) is reduced to the single asset price model proposed by [6].

We next are concerned with the explicit limit representation of the following basket option price as $N\to\infty$ using Theorem 4.1:

$$V^N(\phi)=e^{-rT}\mathbb{E}\left[\phi\left(\frac{1}{N}\sum_{i=1}^N w_iS_T^i\right)\right]=e^{-rT}\mathbb{E}\left[\phi\left(\nu_T^N(I)\right)\right].$$

To this purpose, we rewrite (28) as the form given by (2), i.e., for $i=1,\ldots,N$,

$$\frac{dS_t^i}{S_{t-}^i}=(r-v_i)dt+\sum_{j=1}^d\sigma_{ij}dW_t^j+\int_{\mathcal{U}}\ell_i(y)\tilde{Q}(dy,dt).$$

Here $\ell_i(y) = e^{c_i y} - 1$ for $y \in \mathcal{U} := \mathbb{R} \setminus \{0\}$ and hence we have $1 + \ell_i(y) > 0$ for all $(i, y) \in \mathbb{N} \times \mathcal{U}$. The characteristic measure of Poisson random measure $Q(dy, dt)$ is given by $\nu(dy) = \lambda h_Y(y)dy$. It is easy to verify that $\nu(dy)$ satisfies the assumption (**A2**).

We consider the case where the extended parameter set is given by

$$\bar{p}_i = (v_i, \sigma_{i1}, \dots, \sigma_{id}, w_i, c_i, S_0^i) \to \bar{p}^* = (v^*, \sigma_1^*, \dots, \sigma_d^*, w^*, c^*, s^*), \quad \text{as } i \to \infty,$$

and $(|\bar{p}_i|;\ i \in \mathbb{N})$ is dominated by a finite positive constant. Then it is clear to have $\ell_i(y) \to \ell^*(y) := e^{c^* y} - 1$ for all $y \in \mathcal{U}$ as $i \to \infty$. Thus we can apply Theorem 4.1 to conclude that

$$\lim_{N \to +\infty} V^N(\phi) = e^{-rT} \mathbb{E}\big[\phi\left(X_T^*\right)\big], \tag{30}$$

where the state process $X_t^* := X_t((p^*, \ell^*, x^*))$, $t \geq 0$, satisfies the following SDE given by, $X_0^* = X_{0-}^* = w^* s^*$, and

$$\frac{dX_t^*}{X_{t-}^*} = \Big\{ r - v^* - \lambda \mathbb{E}\left[\left(e^{c^* Y_1} - 1\right)\right] \Big\} dt + \sigma^* d\bar{W}_t + d\left(\sum_{k=1}^{N_t} \left(e^{c^* Y_k} - 1\right)\right). \tag{31}$$

Here $\sigma^* := \bar{\sigma}^* \left(w^*\right)^{-\beta}$ with $\bar{\sigma}^* := \sqrt{\sum_{j=1}^d \left(\sigma_j^*\right)^2}$.

Notice that the limiting state process X_t^*, $t \geq 0$, has a similar structure as to Eq. (1) in [6]. Then the quantity on the right hand side of the equality (30) can be characterized by its Laplace transform explicitly. We next consider the call basket option, i.e. $\theta = 1$ (for the put basket option case, it is similar). Hence $e^{-rT}\mathbb{E}\big[\phi\left(X_T^*\right)\big] = e^{-rT}\mathbb{E}\left[(X_T^* - K)^+\right]$. By introducing a scaling factor $\varpi > K$, the price $e^{-rT}\mathbb{E}\left[(X_T^* - K)^+\right]$ is equal to

$$\Gamma_T(z) := \varpi e^{-rT} \mathbb{E}\left[\left(\frac{X_T^*}{\varpi} - e^{-z}\right)^+\right], \tag{32}$$

where $z := \log(\frac{\varpi}{K}) \in \mathbb{R}$. Thus Theorem 3.4 in [6] gives the Laplace transform of $\Gamma_T(z)$ w.r.t. z in the following closed-form (here we set $c^* = 1$ for convenience), which is given by, for all $\gamma \in (0, \lambda_1 - 1)$,

$$\hat{\Gamma}_T(\gamma) := \int_{-\infty}^{\infty} e^{-\gamma z} \Gamma_T(z) dz = \frac{(w^* s^*)^{\gamma+1}}{\gamma(\gamma+1)\varpi^\gamma} e^{(r + G(\gamma+1))T}. \tag{33}$$

Here the function $G(x)$, $x \in (-\bar{\lambda}_1, \lambda_1)$, is given by

$$G(x) := \frac{(\sigma^*)^2}{2} x^2 + \Big\{ r - v^* - \lambda \mathbb{E}\left[\left(e^{Y_1} - 1\right)\right] \Big\} x + \lambda \left(l_u \sum_{i=1}^m \frac{l_i \lambda_i}{\lambda_i - x} + \bar{l}_d \sum_{j=1}^n \frac{\bar{l}_j \bar{\lambda}_j}{\bar{\lambda}_j + x} - 1 \right). \tag{34}$$

Thus we in fact have the Laplace transform of the limit of the basket option price which can be explicitly expressed as (33). Then the corresponding limit of the price can be obtained by applying two-sided Euler inversion (EI) algorithm, see also [4, 5].

If $\lambda = 0$ in (28) (i.e., there is no common jumps in the price dynamics), it is reduced to the multi-dimensional geometric Brownian motion model considered in [12]. Thus in this case, the underlying limiting state process (31) is simplified to the geometric Brownian motion given by

$$\frac{dX_t^*}{X_t^*} = (r - v^*)dt + \sigma^* d\bar{W}_t. \tag{35}$$

The function $G(x)$ given by (34) is reduced to $G(x) = \frac{(\sigma^*)^2}{2} x^2 + (r - v^*)x$ for $x \in \mathbb{R}$. Hence the limit price $V^N(\phi)$ of the basket option as $N \to \infty$ is equal to the following Black-Scholes formula given by

$$e^{-rT}\mathbb{E}\left[(X_T^* - K)^+\right] = \mathcal{N}(d_1^*) x^* e^{-v^* T} - \mathcal{N}(d_2^*) K e^{-rT}, \tag{36}$$

where $x^* = w^* s^*$, $\mathcal{N}(d)$, $d \in \mathbb{R}$, denotes the distribution function of the standard normal random variable, and

$$d_1^* := \frac{1}{\sigma^* \sqrt{T}} \left[\log \left(\frac{x^*}{K} \right) + \left(r - v^* + \frac{|\sigma^*|^2}{2} \right) T \right],$$
$$d_2^* := d_1^* - \sigma^* \sqrt{T}.$$

5 Numerical Analysis

In this section, we present a numerical analysis for the example (31) introduced in the above section and test the quality of our approximation for the call basket option price. More precisely, we compare our analytic limiting price given by (30) to the exact values estimated through Monte-Carlo simulations.

We first set the parameter $\bar{p}_i = (v_i, \sigma_{i1}, \ldots, \sigma_{id}, w_i, c_i, S_0^i)$, $i \geq 1$, as

$$v_i = 0, \quad \sigma_{ij} = \sigma_j^* \left(1 + \frac{1}{i^k}\right), \; j = 1, \ldots, d, \quad w_i = 1, \quad c_i = c^* \left(1 + \frac{1}{i^k}\right), \quad S_0^i = s^* \left(1 + \frac{1}{i^k}\right),$$

where $v_i = 0$ and $w_i = 1$ indicate that we here consider the price of the average of values of N risky assets without dividend yield that have been grouped together in the basket. Here $\sigma_j^* > 0$, $j = 1, \ldots, d$, $c^* \in \mathbb{R}$ and $s^* > 0$. The convergence order of the limiting parameters is given by $k \in \mathbb{N}$. Thus we have

$$(\sigma_{i1}, \ldots, \sigma_{id}, c_i, S_0^i) \to (\sigma_1^*, \ldots, \sigma_d^*, c^*, s^*), \quad \text{as } i \to \infty.$$

Morevoer in light of Theorem 4.1, we define the limiting volatility of the model as $\bar{\sigma}^* := \sqrt{\sum_{j=1}^d \left(\sigma_j^*\right)^2}$, and the limiting initial asset value is given by $x^* = w^* s^* = s^*$ using the above setting of parameters. In addition, in the following numerical analysis, we take the common jump parameters (see also (29)) as $l_u = 0.6$, $\bar{l}_d = 0.4$, $m = n = 1$, $\lambda = \lambda_1 = 0.1$, $\bar{\lambda}_1 = 0.2$ and $l_1 = \bar{l}_1 = 1$.

We next compare our approximating price for the call basket option given by

$$V^N(\phi) = e^{-rT} \mathbb{E} \left[\left(\frac{1}{N} \sum_{i=1}^N S_T^i \right)^+ \right] = e^{-rT} \mathbb{E} \left[\left(\nu_T^N(I) \right)^+ \right]$$

using the exact limit price formula given by (30) to the exact values estimated through Monte-Carlo simulations. As in [12], we will be implementing this comparison by considering the two prices with respect to the strike price $K > 0$, the limiting initial asset value $x^* > 0$, and the limiting volatility $\bar{\sigma}^* > 0$ respectively. We start analyzing the comparison of the two prices with respect to the strike price $K > 0$. Here we fix the number of the risk assets to $N = 300$. From Figure 1, we see that the call basket option price using the analytic limit formula and the estimating price using the Monte-Carlo simulations (we run the $M = 10^3$ simulations) are decreasing with respect to the strike price, which is consistent with the finding on the call basket option price in [12]. Figure 1 shows good agreement between the analytical limit price formula in Theorem 4.1 and the Monte-Carlo estimate. Table 1 compares the numerical values obtained using the analytical limit price formula in Theorem 4.1 against the corresponding Monte-Carlo estimates as we vary the strike price of the call basket option. The results indicate that analytical limit price tracks closely the corresponding Monte-Carlo simulation price, hence confirming the efficiency and accuracy of the analytic limit price formula in Theorem 4.1. The values in parenthesis in Table 1 give the 95% Monte-Carlo confidence interval, i.e. for the sufficiently large number of simulation M this interval will contain the true option value for 95 simulations out of every 100.

We next analyze the comparison of the two prices with respect to the limiting initial risky asset value x^* and the limiting volatility $\bar{\sigma}^*$ respectively. As the limiting initial asset value increases, so

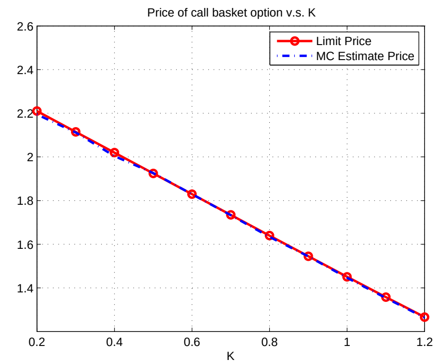

Figure 1: The dependence of the call basket option prices on the strike price K. We fix the other parameters to $\sigma^* = 0.4$, $T = 1$, $k = 6$ and $x^* = 2.4$.

K	Limit price $\lim_{N\to\infty} V^N(\phi)$	Monte-Carlo estimate
0.2	2.2098	2.1947 (2.1360, 2.25337)
0.3	2.1146	2.1119 (2.0489, 2.1749)
0.4	2.0195	2.0039 (1.9397, 2.0681)
0.5	1.9244	1.9260 (1.8635, 1.9886)
0.6	1.8293	1.8292 (1.7692, 1.8893)
0.7	1.7342	1.7330 (1.6725, 1.7934)
0.8	1.6393	1.6335 (1.5705, 1.6964)
0.9	1.5448	1.5444 (1.4819, 1.6069)
1.0	1.4509	1.4466 (1.3835, 1.5098)
1.1	1.3580	1.3536 (1.2903, 1.4169)
1.2	1.2666	1.2637 (1.2031, 1.3243)

Table 1: Numerical values of the limiting price of the call basket option using Theorem 4.1 and the Monte-Carlo estimate for different strike prices. We fix the parameters to $\bar{\sigma}^* = 0.4$, $T = 2$, $k = 6$ and $x^* = 2.4$. The values in parenthesis represent the 95% Monte-Carlo confidence interval.

does the price of the call basket option (see the left graph of Figure 2). This is similar to the case of the call basket option with respect to the limiting volatility from the right graph in Figure 2. Figure 2 shows these results when the number of risky assets is fixed to $N = 300$, where we have considered the symmetrical situation for the limiting volatility, namely σ_j^* is set to the same value $\bar{\sigma}^*$ for $j = 1, \ldots, d$ as in [12]. It is clear that the limit price given by $\lim_{N\to\infty} V^N(\phi)$ with $\phi(x) = x^+$ tracks closely to the Monte-Carlo estimate values under the above different scenarios. Tables 2 and 3 list the numerical comparison results of the same maturity $T = 1$, the strike price $K = 0.8$, and the same convergence order $k = 6$ of model parameters. As in Table 1, the values in parenthesis in Tables 2 and 3 also correspond to the 95% Monte-Carlo confidence interval, where the number of simulations is set to $M = 10^3$ in this numerical test.

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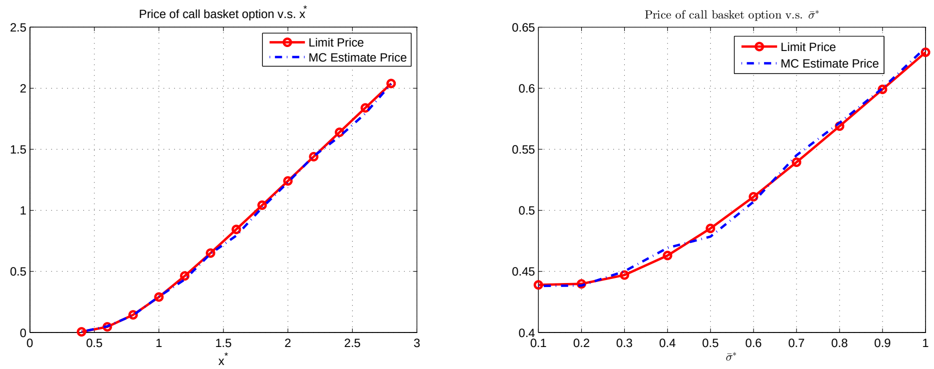

Figure 2: We fix the parameters to $K = 0.8$, $T = 1$ and $k = 6$. The left graph shows the dependence of the call basket option price on the limiting initial asset value x^*. The right graph shows the dependence of the call basket option price on the limiting volatility $\bar{\sigma}^*$.

x^*	Limit price $\lim_{N\to\infty} V^N(\phi)$	Monte-Carlo estimate
0.4	0.0050	0.0053 (0.0032, 0.0073)
0.6	0.0455	0.0504 (0.0413, 0.0596)
0.8	0.1442	0.1406 (0.1264, 0.1548)
1.0	0.2898	0.2936 (0.2710, 0.3162)
1.2	0.4630	0.4336 (0.4057, 0.4615)
1.4	0.6503	0.6448 (0.6081, 0.6815)
1.6	0.8444	0.7930 (0.7531, 0.8329)
1.8	1.0416	1.0225 (0.9752, 1.0699)
2.0	1.2403	1.2295 (1.1785, 1.2805)
2.2	1.4396	1.4446 (1.3860, 1.5031)
2.4	1.6393	1.6044 (1.5458, 1.6629)
2.6	1.8392	1.7933 (1.7306, 1.8559)
2.8	2.0391	2.0344 (1.9634, 2.1053)

Table 2: Numerical values of the limiting price of the call basket option using Theorem 4.1 and the Monte-Carlo estimate for different limiting initial asset values. We fix the parameters to $\bar{\sigma}^* = 0.4$, $T = 1$, $k = 6$ and $K = 0.8$. The values in parenthesis represent the 95% Monte-Carlo confidence interval.

$\bar{\sigma}^*$	Limit price $\lim_{N\to\infty} V^N(\phi)$	Monte-Carlo estimate
0.1	0.4390	0.4381 (0.4306, 0.4456)
0.2	0.4398	0.4384 (0.4231, 0.4537)
0.3	0.4470	0.4501 (0.4283, 0.4718)
0.4	0.4630	0.4692 (0.4387, 0.4997)
0.5	0.4852	0.4785 (0.4412, 0.5159)
0.6	0.5111	0.5070 (0.4620, 0.5519)
0.7	0.5394	0.5451 (0.4837, 0.6065)
0.8	0.5689	0.5717 (0.4992, 0.6443)
0.9	0.5991	0.6000 (0.5303, 0.6698)
1.0	0.6294	0.6333 (0.5193, 0.7474)

Table 3: Numerical values of the limiting price of the call basket option using Theorem 4.1 and the Monte-Carlo estimate for different limiting volatilities. We fix the parameters to $x^* = 1.2$, $T = 1$, $k = 6$ and $K = 0.8$. The values in parenthesis represent the 95% Monte-Carlo confidence interval.

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